Some Structural Considerations on the Theory of Gravitational Field

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Abstract

When the gravitational field is regarded as the time-sequence of space-slices (i.e., the evolution space), the gravitational field itself is treated by means of the differential geometry of total space of the vector bundle whose base manifold is the one-dimensional time-axis and fibre at each time is the three-dimensional space. From this vector bundle-like standpoint, new field equations and new conservation laws are proposed.

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1 Introduction

In this paper, the gravitational field is regarded as the ensemble of space-time events and then, as the time-sequence of space-slices. That is to say, the four-dimensional gravitational field itself is decomposed into the one-dimensional time-part and the three-dimensional space-part and the three-dimensional space evolves along the time-axis. This idea arises from the concept of evolution space in the theory of dynamical systems (cf. [1]). Therefore, from the vector bundle-like standpoint, the gravitational field can be adapted to the total space of the vector bundle whose base manifold is the one-dimensional time-axis ($x^0 = t$: time) and the fibre at each time is the three-dimensional space spanned by points \( \{ x^i \} \ (i = 1, 2, 3) \).

Therefore, the gravitational field can be treated by means of the differential geometry of total space of the vector bundle [2], [3]. From this standpoint, the metrical and connection structures will be introduced and then, new field equations and new conservation laws will be proposed in the following.

2 On the vector bundle-like structures - I

Now, in the total space mentioned above, the so-called adapted frame is set as follows:
\[ dX^A \equiv (dx^0, \delta x^i = dx^i + N_0^i dx^0), \]
\[
\frac{\partial}{\partial X^A} \equiv \left( \frac{\delta}{\delta x^0} = \frac{\partial}{\partial x^0} - N_0^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i} \right),
\]
where \( X^A = (x^0, x^i)(A = (0, i); \ i = 1, 2, 3) \) and \( x^0 = t \) (time). The quantity \( N_0^i \) denotes the nonlinear connection playing physically the role of velocity.

On the basis of (2.1), the connection structures is introduced by

\[
\nabla \frac{\partial}{\partial X^B} \equiv \Gamma^A_{BC} \frac{\partial}{\partial X^A};
\]

\[
\Gamma^A_{BC} \equiv (L^0_{00}, L^i_{j0}, C^0_{0k}, C^i_{jk}),
\]
where
\[
\nabla \frac{\delta}{\delta x^0} = L^0_{00} \frac{\partial}{\partial x^0}; \quad \nabla \frac{\partial}{\partial x^k} = C^j_{jk} \frac{\partial}{\partial x^j}, \text{ etc.}
\]
Namely, the following four kinds of covariant derivatives can be defined, for an arbitrary vector \( V^A = (V^0, V^i) \)

\[
\begin{cases}
V^0|_k = \frac{\delta V^0}{\delta x^0} + L^0_{00} V^0, \\
V^0|_k = \frac{\partial V^0}{\partial x^k} + C^0_{0k} V^0, \\
V^i|_k = \frac{\delta V^i}{\delta x^0} + L^i_{j0} V^j, \\
V^i|_k = \frac{\partial V^i}{\partial x^k} + C^i_{jk} V^j.
\end{cases}
\]

On the other hand, the metrical structure is introduced by

\[
G \equiv G_{AB} dX^A dX^B = g_{00} dx^0 \otimes dx^0 + g_{ij} dx^i \otimes dx^j,
\]
where \( g_{00} \) and \( g_{ij} \) are metric tensors depending on \((x^0, x^i)\). The connection (2.2) can be made metrical by imposing the metrical conditions such as \( g_{00}|_0 = 0, g_{00}|_k = 0, g_{ij}|_0 = 0 \) and \( g_{ij}|_k = 0. \) In the metrical case, the canonical connection coefficients can be determined as follows [2, 3]

\[
\begin{cases}
L^0_{00} = \frac{1}{2} g_{00} \frac{\delta g_{00}}{\delta x^0}, \\
L^i_{j0} = \frac{1}{2} g_{00} \frac{\delta g_{00}}{\delta x^0}, \\
C^0_{0k} = \frac{1}{2} g_{00} \frac{\partial g_{00}}{\partial x^k}, \\
C^i_{jk} = \frac{1}{2} g_{00} \left( \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j} \right),
\end{cases}
\]
where \( g^{00} \) and \( g^{ij} \) are the inverse of \( g_{00} \) and \( g_{ij} \) respectively and the torsion tensor \( S^i_{jk}(= C^i_{jk} - C^i_{kj}) \) is assumed to vanish (see (3.5)). \( C^i_{jk} \) is nothing but the three-dimensional Christoffel symbol formed with \( g_{ij} \). And also, in (2.5), if \( g_{00} = \text{constant} \), then \( L^0_{00} = 0 \) and \( C^0_{0k} = 0. \)
3 On the vector bundle-like structures - II

Now, the torsion and curvature tensors are defined by [2], [3], respectively,

\begin{equation}
T_{BC}^A = \Gamma_{BC}^A - \Gamma_{CB}^A + \omega_{BC}^A, \tag{3.1}
\end{equation}

\begin{equation}
R_{BCD}^A = \frac{\partial}{\partial x^B} \Gamma_{CD}^A - \frac{\partial}{\partial x^C} \Gamma_{BD}^A + \Gamma_{BC}^E \Gamma_{ED}^A - \Gamma_{BD}^E \Gamma_{EC}^A + \Gamma_{DE}^A \omega_{CD}^E, \tag{3.2}
\end{equation}

where \( \omega_{BC}^A \) means the non-holonomic object defined by

\begin{equation}
\left[ \frac{\partial}{\partial x^B} , \frac{\partial}{\partial x^C} \right] = \omega_{BC}^A \frac{\partial}{\partial x^A}. \tag{3.3}
\end{equation}

In our case, only one component \( \omega_{0j} = -\omega_{j0} = \frac{\partial N_{ij}^0}{\partial x^j} \) appears (because \( \omega_{00} = R_{00}^i = 0 \), see below).

Five components of the torsion tensor

\[ T_{BC}^A \equiv (T_{00}^0 = 0, \ R_{00}^i = 0, \ C_{0_{ij}}^0, \ P_{i0}^j, \ S_{jk}^i) \]

and six components of the curvature tensor

\[ R_{BCD}^A \equiv (R_{00}^i = \tau \ R_{10}^i = \tau \ P_{i0}^j, \ P_{i1}^j, \ S_{ij}^k, \ S_{i1}^j) \]

appear explicitly in the following Ricci-identities:

\begin{equation}
\begin{aligned}
V^0_{|0|0} - V^0_{|0|0} &= R_{000}^0 V^0 - T_{00}^0 V^0_{|0} - R_{00}^i V^0_{|i}(= 0), \\
V^0_{|0|1} - V^0_{|0|0} &= P_{00}^0 V^0 - C_{0i}^0 V^0_{|0} - P_{i0}^j V^0_{|j}, \\
V^0_{|1|j} - V^0_{|0|1} &= S_{0j}^0 V^0 - S_{ij}^0 V^0_{|j}, \\
V^i_{|0|0} - V^i_{|0|0} &= R_{j00}^i V^j - T_{j00}^i V^j_{|0} - R_{i00}^j V^j_{|j}(= 0), \\
V^i_{|0|1} - V^i_{|0|0} &= P_{j0}^i V^j - C_{0j}^i V^j_{|0} - P_{j0}^k V^k_{|j}, \\
V^i_{|1|k} - V^i_{|k|1} &= S_{ijk}^i V^j - S_{jk}^i V^i_{|j}.
\end{aligned} \tag{3.4}
\end{equation}

The concrete definitions of these components are given as follows

\begin{equation}
\begin{aligned}
T_{00}^0 &= L_{00}^0 - L_{00}^0 = 0, \\
R_{00}^i &= \frac{\partial N_{ij}^0}{\partial x^i} - \frac{\partial N_{ij}^0}{\partial x^j} = 0, \\
P_{i0}^j &= \frac{\partial N_{ij}^0}{\partial x^j} - L_{i0}^j, \\
S_{jk}^i &= C_{j}^i - C_{k}^i, \tag{3.5}
\end{aligned}
\end{equation}
\[
\begin{align*}
R_{000}^0 &= \frac{\delta L_0^0}{\delta x^0} - \frac{\delta L_0^0}{\delta x^0} + L_0^0 L_0^0 - L_0^0 L_0^0 + C_0^0 R_{00}^0 = 0, \\
R_{i00}^0 &= \frac{\delta L_i^0}{\delta x^0} - \frac{\delta L_j^0}{\delta x^0} + L_{i0}^0 L_{j0}^0 - L_{j0}^0 L_{i0}^0 + C_{i0}^0 R_{i0}^0 = 0, \\
P_{i0}^0 &= \frac{\delta L_i^0}{\delta x^i} - C_{j0}^i P_{0k}^0, \\
P_{00k}^k &= \frac{\delta L_0^0}{\delta x^k} - C_{0k}^0 P_{0k}^0, \\
S_{0k}^i &= \frac{\partial C_{0k}^i}{\partial x^k} - C_{0k}^i C_{0k}^0, \\
S_{j0}^i &= \frac{\partial C_{j0}^i}{\partial x^0} - C_{j0}^i C_{j0}^0.
\end{align*}
\]

(3.6)

In the canonical case of (2.5), the tensor \( S_{j0}^i \) is just the three-dimensional Riemann-Christoffel curvature tensor and also, if \( g_{00} = \text{constant} \), then \( P_{00k}^k = 0 \) and \( S_{0k}^0 = 0 \) in (3.6).

From \( R_{000}^0 \), the Ricci-tensor is given by

\[
(3.7) \quad R_{00} = \frac{R_{00}^0}{R_{00}^0} = \cdots = \frac{R_{00}^0}{R_{00}^0}, \quad \frac{R_{00}^0}{R_{00}^0} = \frac{R_{00}^0}{R_{00}^0}, \quad \frac{R_{00}^0}{R_{00}^0} = \frac{R_{00}^0}{R_{00}^0}, \quad S_{ij} = \frac{S_{ij}^0}{S_{ij}^0}.
\]

Namely, three non-vanishing components appear. And the total scalar is given by

\[
(3.8) \quad R = R_{000}^0 G^{00} = R_{00}^0 + S_{ij}^0 = S_{ij}^0 + S_{ij}^0.
\]

(3.7) and (3.8) will be used in the next Section.

## 4 On the field equations and the conservation laws

As the field equation for the total space (i.e., the gravitational field), we shall put it in the form [2], [3], with use of the Ricci-tensor \( R_{000}^0 \) (3.7) and the total scalar \( R \) (3.8),

\[
(4.1) \quad R_{000}^0 = \begin{array}{c}
\infty \\
\infty
\end{array}
R_{000}^0 = \tau_{AB}.
\]

where \( \tau_{AB} \) represents the energy-momentum tensor with four components \( \tau_{AB} \equiv (\tau_{00}, \tau_{01}, \tau_{02}, \tau_{ij}) \). Then, by use of the components of (3.7) and (3.8), we can obtain the following four kinds of field equations

\[
(4.2) \quad \begin{cases}
-\frac{1}{2} S_{00} = \tau_{00} \\
\frac{1}{2} \tau_{01} \\
\frac{2}{2} P_{0i} = -\tau_{0i} \\
S_{ij} - \frac{1}{2} S g_{ij} = \tau_{ij}.
\end{cases}
\]
These are new equations, different from those obtained in [2], [3]. In the case of the canonical connection (2.5), the last equation is just the (three-dimensional) Einstein’s field equation.

As to the conservative law, we can formulate it in the form [2], [3] (i.e., the divergence-zero of (4.1))

\[ \frac{\nabla_0}{\partial X^A} \left( \mathcal{R}^A_{\mu} - \frac{\infty}{\epsilon} \mathcal{R}^A_{\mu} \right) = 0, \]

where

\[ \mathcal{R}^A_{\mu} \equiv \mathcal{R}_{BC} G^A \equiv (\mathcal{R}_i \equiv \mathcal{R}_{\mu})'' = t, \mathcal{P}^i_{\mu} = \mathcal{P}^{\mu i} \bigg|_{t=0}, \]

\[ - P^0_j = - P^{00} - \frac{1}{2} S_{ij}, \quad S_{ij} = S_{\beta} g^{i}. \]

Therefore, we can obtain the following two kinds of conservation laws

\[ \begin{cases} P^0_0 |_{t} - \frac{1}{2} S_{00} = 0, \\ (S^i_j - \frac{1}{2} S \delta^i_j) |_{t} - P^0_j = 0. \end{cases} \]

These are new conservation laws, different from those obtained in [2], [3]. In some special cases where \( P^0_0 = 0 \) or \( P^0_j |_{t=0} = 0 \), we can obtain the pure conservation law

\[ \left( S^i_j - \frac{1}{2} S \delta^i_j \right) |_{t} = 0, \]

which is the same as the Einstein’s one. In those special cases, the conditions such as

\[ (g_{\mu \nu} = g_{\mu \nu}(x^i) \& g_{i \mu} = g_{i \mu}(x^i)) \text{ or } (g_{\mu \nu} = g_{\mu \nu}(x^i) \& g_{i \mu} = g_{i \mu}(x^i)), \]

etc. must be taken into account.

5 Conclusion

Thus, we can treat the gravitational field by means of the differential geometry of total space of the vector bundle whose base manifold is the time-axis and fibre at each time is the space-slice. And we can propose new field equations (4.2) and new conservation laws (4.5). (Some physical aspects of these subjects are referred to [4]).

References


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