A Convex Polygon as a Discrete Plane Curve

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Abstract

In this paper we examine a convex polygon as a discrete substitute of a plane curve. We introduce a polygon with constant length of a diagonal as a counterpart of an oval with constant width. Moreover we define a convex polygon with constant perimeter of a special class circumscribed polygons.

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1 Introduction

In papers [1,3,4,6] applications of Fourier series to plane curves are presented . Plane curves examined in these papers are expressed by the following formulas:

\[
\begin{align*}
  t \mapsto z(t) &= \int_0^t f(s) e^{is}ds, \\
  z(t) &= \int_0^t k(s) f(s) e^{iK(s)} ds,
\end{align*}
\]

where \( f \) is a periodic function. The representation of considered curves is associated with the integral. Therefore we search for a geometrical domain associated with a finite sum instead of an integral. The geometrical domain is included in the class of all convex polygons in the plane. To define a representation of a convex polygon we imitate formula (1.1). Therefore we consider a periodic sequence instead of a periodic function. Next we introduce a discrete Fourier series for a periodic sequence as follows:

Let \( x_1, x_2, x_3, \ldots \) be a periodic sequence of real numbers with the period \( n \), i.e.:

\[ x_{v+n} = x_v, v = 0, 1, 2, \ldots \]

Then we apply a known trigonometrical interpolative polynomial

\[
y(t) = a_0 + \sum_{j=1}^{n-1} \left[ a_j \cos \frac{2\pi j t}{n} + b_j \sin \frac{2\pi j t}{n} \right],
\]

where

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\[ a_0 = \frac{1}{n} \sum_{\mu=0}^{n-1} x_\mu, a_j = \frac{1}{n} \sum_{\mu=0}^{n-1} x_\mu \cos \frac{2\pi j\mu}{n}, \quad b_j = \frac{1}{n} \sum_{\mu=0}^{n-1} x_\mu \sin \frac{2\pi j\mu}{n}. \]

The trigonometrical interpolative polynomial satisfies the condition:

\[ y(v) = x_v, \quad v = 0, 1, \ldots, n - 1. \]

If we substitute instead the continuous variable \( t \in (-\infty, +\infty) \) the discrete variable \( v = 0, 1, 2, \ldots, \) then we obtain

\[ x_v = a_0 + \sum_{j=1}^{n-1} \left( a_j \cos \frac{2\pi jv}{n} + b_j \sin \frac{2\pi jv}{n} \right). \]

In the sequel we call the formula (1.2) a discrete Fourier series of a periodic sequence \( \{x_v\} \). We apply the discrete Fourier series to an n-polygon in the plane. The n-polygon is a polygon with \( n \) sides having the same interior angles equal to \( 2\pi - \frac{2\pi}{n} \), see [2,1].

In the paper n-polygon is "a discrete curve".

With reference to formula (1.1) we recall the following relations. There exists a strict correspondence between a property of curve (1.1) and a property of the function \( f \). For example the following are known:

**Lemma A.** (see [5,1]). A curve (1.1) is closed iff Fourier coefficients \( A_1, B_1 \) of \( f \) vanish, i.e.: \( A_1 = B_1 = 0 \).

**Lemma B.** (see [6]). If a closed curve represented by (1.1) is a curve with constant width, then Fourier coefficients \( A_{2n}, B_{2n} \) of \( f \) vanish, i.e.: \( A_{2n} = B_{2n} = 0 \).

**Lemma C.** (see [2,1]). If a closed curve represented by (1.1) is a curve with constant perimeter of a circumscribed n-polygon, then the Fourier coefficients \( A_{mj}, B_{mj}, \ j = 1, 2, 3, \ldots \) vanish, i.e.: \( A_{mj} = B_{mj} = 0 \).

**Remark A.**
If \( f \) is a constant function (different from zero), then equation (1.1) forms a circle. This means that in this case all Fourier coefficients of \( f \) vanish with the exception of \( A_0 \).

In a discrete domain, n-polygon is represented as the sum

\[ k \mapsto z_k = \sum_{v=0}^{k} x_v e^{i \frac{2\pi v}{n}}, \]

where \( \{x_v\} \) is a sequence and \( k = 0, 1, \ldots, n - 1 \).

There exists a correspondence between a property of an n-polygon and a property of a sequence \( \{x_v\} \). At the discrete domain a counterpart of a curve with constant width is a 2n-polygon with constant diagonal (see p.7).

For 2n-polygon with constant diagonal the following counterpart of the Barbier theorem is satisfied:

\[ L = \frac{\sin \frac{\pi}{n}}{\frac{2\mu}{2n}} d, \]
where $L$ denotes the perimeter of $2n$-polygon and $d$ is the length of a constant diagonal.

A counterpart of a curve with constant perimeter of a circumscribed $m$-polygon is an $mn$-polygon with constant perimeter of a $b$-circumscribed $m$-polygon defined as follows:

Let $P$ be a convex polygon with vertices $w_1, w_2, \ldots, w_n$, $n > 2$. To circumscribe a polygon with $k$ sides ($3 \leq k \leq n$) on polygon $P$, we arbitrary choose vertices

\[ w_{i_1}, w_{i_2}, \ldots, w_{i_k}. \]

Next we draw a straight line $l_{i_1}, l_{i_2}, \ldots, l_{i_k}$ through vertices $w_{i_1}, w_{i_2}, \ldots, w_{i_k}$. We consider only straight lines $l_{i_1}, l_{i_2}, \ldots, l_{i_k}$ passing through the outside angles of polygon $P$. The point of intersection of successive straight lines $l_{i_s}, l_{i_{s+1}}, s = 1, 2, \ldots, k - 1$ is a vertex of the circumscribed polygon. We call the circumscribed polygon $b$-circumscribed on polygon $P$ if and only if all straight lines $l_{i_1}, l_{i_2}, \ldots, l_{i_k}$ are bisectrices of the outside angles of the polygon $P$.

A property of a plane curve (represented by (1.1)) and "a discrete theory plane curve" are connected with the main result of the paper.

Perimeter $2\pi r$ of a circle with radius $r$ can be obtained as the limit of perimeters of well-shaped regular polygons circumscribed on the circle. The above-mentioned idea and the counterpart of the Barbier theorem suggest that perimeter $\pi d$ for an oval with constant width $d$ can be obtained in the similarly way. To reach this aim we prove the following:

**Theorem 1.1** Every $2n$-polygon circumscribed on an oval with constant width $\delta$ is a $2n$-polygon with constant diagonal equal to

\[ \delta \frac{\cos \frac{\pi}{2n}}{n}. \]

**Theorem 1.2** Every $mn$-polygon circumscribed on an oval with constant perimeter $l$ of a circumscribed $m$-polygon is $mn$-polygon with constant perimeter $\frac{l}{\cos \frac{\pi}{mn}}$ of $b$-circumscribed $m$-polygon.

## 2 Properties of a periodic sequence

A periodic sequence has some properties similar to a property of a periodic function. Therefore we recall (see [6,1]) those properties of a periodic function concerning of the discrete domain. Let $f$ be $2\pi$-periodic function having uniformly convergent Fourier series,

\[ f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} [A_n \cos(nt) + B_n \sin(nt)] . \]

**Theorem A.** Expression $f(t) + f(t + \pi)$ is a constant function iff Fourier coefficients $A_{2j}, B_{2j}, j = 1, 2, \ldots$ vanish.

**Theorem B.** Expression
\[ f(t) + f(t + \frac{2\pi}{m}) + f(t + 2\frac{2\pi}{m}) + \ldots + f \left( t + (m - 1)\frac{2\pi}{m} \right) \]

is a constant function iff Fourier coefficients \( A_{mj}, B_{mj}, \quad j = 1, 2, \ldots \) vanish.

Theorems A and B have the following counterparts at the discrete domain:

Let \( x_v \) be a 2\( n \)-periodic sequence, i.e.: \( x_{v+2n} = x_v, \quad v = 0, 1, 2, \ldots \). In this case sequence \( x_v \) has the discrete Fourier sum in the form

\[ x_v = a_0 + \sum_{j=1}^{2n-1} [a_j \cos \frac{\pi j v}{n} + b_j \sin \frac{\pi j v}{n}] \]

We prove two following lemmas:

**Lemma 2.1** If \( \{x_v\} \) is a 2\( n \)-periodic sequence, then the discrete Fourier sum of \( x_v + x_{v+n} \) has the form

\[ x_v + x_{v+n} = 2a_0 + 2 \sum_{l=1}^{n-1} [a_{2l} \cos \frac{2\pi l v}{n} + b_{2l} \sin \frac{2\pi l v}{n}] \]

Moreover

**Lemma 2.2** If

\[ x_v = a_0 + \sum_{j=1}^{2n-1} [a_j \cos \frac{\pi j v}{n} + b_j \sin \frac{\pi j v}{n}] \]

is the discrete Fourier sum of 2\( n \)-periodic sequence \( \{x_v\} \), then the sequence \( \{x_v + x_{v+n}\} \) is a constant function if and only if \( a_{2l} = b_{2l} = 0 \) for \( l = 1, 2, \ldots, n - 1 \).

**Proof.** To prove the lemma we verify that if \( x_v + x_{v+n} = c, \quad v = 0, 1, \ldots \), then \( a_{2l} = b_{2l} = 0, \quad l = 1, 2, \ldots, n - 1 \). Indeed we have

\[ a_{2l} = \frac{1}{2n} \sum_{\mu=0}^{2n-1} x_{\mu} \cos (2\pi \frac{\mu}{n}) = \]

\[ = \frac{1}{2n} (x_0 + x_1 \cos \frac{2\pi l}{n} + x_2 \cos \frac{4\pi l}{n} + \ldots + x_n \cos \frac{2\pi l}{n} + x_{n+1} \cos \frac{4\pi l}{n} + \ldots + x_{2n} \cos \frac{4\pi l}{n} (n+1) + \ldots) = \]

\[ = \frac{1}{2n} ((x_0 + x_n) + (x_1 + x_1+n) \cos \frac{2\pi l}{n} + (x_2 + x_2+n) \cos \frac{4\pi l}{n} + \ldots) = \]

\[ = \frac{c}{2n} (1 + \cos \frac{2\pi l}{n} + \cos \frac{4\pi l}{n} + \ldots) = 0, \]

because

\[ 1 + e^{i\pi} + \ldots + e^{i\frac{(n-1)n}{n}} = 0 \]

hence sum

\[ \sum_{v=0}^{n-1} \cos \frac{v\pi}{n} = 0 \]
vanishes. Similarly we compute that $b_{2l}=0, l=1,2,...,n-1$. So lemmas 1 and 2 are strict counterparts of the relation between Fourier coefficients of a $2\pi$-periodic function $f(t)$ and the function $f(t)+f(t+\pi)\equiv C$. Theorem B has the following discrete counterpart:

**Lemma 2.3** If $\{x_v\}$ is a $m\cdot n$-periodic sequence and the discrete Fourier sum

$$x_v = a_0 + \sum_{j=1}^{m-1} \left[ a_j \cos \frac{2\pi j v}{m \cdot n} + b_j \sin \frac{2\pi j v}{m \cdot n} \right]$$

is given, then the sequence $x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)n}$ has the discrete Fourier sum in the form

$$x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)n} =$$

$$= m a_0 + m \sum_{l=1}^{n-1} \left[ a_{md} \cos \frac{2\pi lv}{n} + b_{md} \sin \frac{2\pi lv}{n} \right].$$

Moreover

**Lemma 2.4** If the discrete Fourier sum of $m\cdot n$-periodic sequence is given, then the sequence $x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)n}$ is constant if and only if

$$a_{ml} = b_{ml} = 0, \quad l = 1, 2, \ldots, n-1.$$

### 3 Convex polygons

Let $x_v$ be an $n$-periodic sequence of real numbers. In this section we consider a polygon line represented by (1.3):

$$k \mapsto z_k = \sum_{v=0}^{k} x_v e^{i\frac{2\pi v k}{n}}.$$

The correspondence (1.3) describes the polygon line whenever $n > 2$. This means that for each fixed sequence $x_v$ points $z_k, \quad k = 0, 1, 2, \ldots$ are vertices of the polygon line in the Euclidean plane, see Fig 1.
Fig.1 Polygon line $z_k$.

Obviously the value $x_k, k = 0, 1, \ldots$ is equal to the distance between points $z_{k-1}$ and $z_k$. Polygon line with vertices $z_k$ becomes a convex polygon if we assume that:

(a) $x_v > 0,$

(b) $a_1 = b_1 = 0.$

Indeed, applying assumptions (a) and (b), we easily compute that

$$z_{k+n} = \sum_{v=0}^{k+n} x_v e^{i2\pi v/n} = z_k + \sum_{v=k+1}^{k+n} x_v e^{i2\pi v/n}.$$ 

Next we analyse the sum

$$S = \sum_{v=k+1}^{k+n} x_v e^{i2\pi v/n} = \sum_{v=k+1}^{k+n} x_v \cos \frac{2\pi v}{n} + i \sum_{v=k+1}^{k+n} x_v \sin \frac{2\pi v}{n}.$$ 

By the periodicity of sequence $x_v$ we have:

$$\sum_{v=k+1}^{k+n} x_v \cos \frac{2\pi v}{n} =$$

$$= x_{k+1} \cos \frac{2\pi(k+1)}{n} + x_{k+2} \cos \frac{2\pi(k+2)}{n} + \ldots +$$

$$+ x_n \cos \frac{2\pi n}{n} + x_{n+1} \cos \frac{2\pi(n+1)}{n} + \ldots + x_{k+n} \cos \frac{2\pi(k+n)}{n} =$$

$$= x_0 + x_1 \cos \frac{2\pi}{n} + \ldots + x_k \cos \frac{2\pi k}{n} + x_{k+1} \cos \frac{2\pi(k+1)}{n} + \ldots + x_{n-1} \cos \frac{2\pi(n-1)}{n} = na_1.$$ 

Similarly we compute that
\[ \sum_{v=k+1}^{k+n} x_v \sin \frac{2\pi v}{n} = nb_1. \]

Finally we obtain
\begin{equation}
(3.5) \quad z_{k+n} = z_k + n(a_1 + ib_1).
\end{equation}

Equality (3.5) implies the following:

**Lemma 3.1** The polygon line with vertices \( z_k \) is closed if and only if the coefficients \( a_1 \) and \( b_1 \) of the discrete Fourier sum of sequence \( x_v \) vanish.

The above-mentioned lemma is the strict counterpart of Lm.A.

A polygon line with vertices \( z_k \) becomes a well-shaped regular polygon with \( n \) sides whenever \( x_v \) is a constant sequence. Therefore, comparing Remark A with formula (1.3), we state that a well-shaped regular polygon is a discrete counterpart of a circle.

## 4 On \( n \)-polygons with constant diagonal

In this section we examine \( 2n \)-polygons represented by formula

\begin{equation}
(4.6) \quad z_k = \sum_{v=0}^{k} x_v e^{i \frac{2\pi v}{n}},
\end{equation}

where \( n \geq 2 \) and the sequence \( x_v \) satisfies the following conditions:

(i) \( x_v > 0 \),

(ii) \( x_{v+2n} = x_v \quad v = 0, 1, \ldots \),

(iii) \( x_v + x_{v+n} = c, \quad v = 0, 1, \ldots \),

(iv) \( a_1 = b_1 = 0 \).

Then the sector between points \( z_k \) and \( z_{k+n} \) is a diagonal of the polygon. Such a diagonal is called \( \frac{1}{2} \)-diagonal of \( 2n \)-polygon because the number all vertices of the polygon between \( z_k \) and \( z_{k+n} \) is equal the number all vertices of the polygon between \( z_{k+n} \) and \( z_k \).

Now we prove the main result of the paper.

**Theorem 4.1** If vertices of a \( 2n \)-polygon are determined by formula (4.6) and the sequence \( x_v \) satisfies conditions (i)-(iv), then all \( \frac{1}{2} \)-diagonals of the polygon have the same length.

**Proof.** We consider a \( 2n \)-polygon represented by equation (4.6). Let \( p_k \) denote a \( \frac{1}{2} \)-diagonal of the polygon. We put

\[ T_k = e^{i \frac{2\pi k}{n}} e^{i \frac{2\pi v}{n}} = e^{i \frac{2\pi k + 1}{2n}} \quad \text{and} \quad N_k = i T_k. \]
The vectors \( T_k, N_k \) are parallel to bisectrices of outside and inside angle at vertices \( z_k, k = 0, 1, \ldots \) of \( 2n \)-polygon, respectively. Vectors \( T_k, N_k \) establish a basis for \( k = 0, 1, \ldots \). Therefore

\[
p_k = D_k T_k - d_k N_k,
\]

where \( d_k = [p_k, T_k] \) and \( D_k = [p_k, N_k] \) are determinants of two pairs of vectors \( p_k, T_k \) and \( p_k, N_k \), respectively. Now we obtain the discrete Fourier sum of \( d_k \) and \( D_k \). First

\[
d_k = [p_k, T_k] = \sum_{v=k+1}^{k+n} x_v e^{i \frac{2\pi}{n} \frac{(2k+1)\pi}{n} v}
= \sum_{v=k+1}^{k+n} x_v \sin \left( \frac{(2k+1)\pi}{n} - \frac{\pi v}{n} \right)
= \sum_{v=k+1}^{k+n} x_v \sin \left( \frac{2k-2v+1}{2n} \right).
\]

On the other hand we have the following formula

\[
x_v = a_0 + \sum_{l=1}^{n-1} [a_{2l+1} \cos \left( \frac{(2l+1)\pi}{n} v \right) + b_{2l+1} \sin \left( \frac{(2l+1)\pi}{n} v \right)],
\]

and we insert it into the formula \( d_k \). Hence we obtain

\[
d_k = a_0 \sum_{v=k+1}^{k+n} \frac{\sin \left( \frac{2k-2v+1}{2n} \right)}{2n} +
= \frac{a_0}{\sin \frac{\pi}{2n}} \sin \frac{\pi}{2n} =
\]

Similarly we compute that \( D_k = 0 \). So we have

\[
p_k = \frac{a_0}{\sin \frac{\pi}{2n}} T_k.
\]

This means that every \( \frac{1}{2} \)-diagonal of the \( 2n \)-polygon has the same length equal to

\[
|p_k| = \frac{a_0}{\sin \frac{\pi}{2n}}.
\]

\[\square\]
Let $L$ be a perimeter of $2n$-polygon with constant diagonal and let $d$ be a length of an $\frac{1}{2}$-diagonal. Then we have the following counterpart of the Barbier’s formula:

\begin{equation}
L = \pi \frac{\sin \frac{\pi}{2n}}{x_{2n-1}} \cdot d
\end{equation}

because

\begin{equation}
\frac{1}{2n}(x_0 + x_1 + \ldots + x_{2n-1}) = \frac{1}{2n}L.
\end{equation}

The relation $D_k = 0$ means that

**Corollary 4.1** Each $\frac{1}{2}$-diagonal of $2n$-polygon determined by a sequence $x_v$ satisfying conditions (i)-(iv) is a bisectrix of an inside angle of the polygon.

To define a $2n$-polygon with constant diagonal by formula (4.6) we need a sequence $x_v$ satisfying conditions (i)-(iv). This means that we solve the linear system $n + 1$ equations with $2n$ unknown quantities. We solve these equations for $n = 3,4$. The sequence $x_0 = d, x_1 = m - d, x_2 = d, x_3 = m - d, x_4 = d, x_5 = m - d$ determines 6-polygon with constant diagonal by formula (4.6) for fixed numbers $d > 0$ and $m - d > 0$. Then

\begin{align*}
z_0 &= d, \\
z_1 &= d + (m - d)e^{i\frac{\pi}{6}}, \\
z_2 &= z_1 + de^{i\frac{\pi}{6}}, \\
z_3 &= z_2 + (m - d)e^{i\frac{\pi}{6}}, \\
z_4 &= z_3 + de^{i\frac{\pi}{6}}, \\
z_5 &= z_4 + (m - d)e^{i\frac{5\pi}{6}}.
\end{align*}

To define a 8-polygon with constant diagonal we apply the following sequence.

\begin{align*}
x_0 &= \frac{a}{2}, \\
x_1 &= \frac{a}{2} + c - \frac{1}{\sqrt{2}}, \\
x_2 &= \frac{a}{2} + c + \frac{1}{\sqrt{2}}, \\
x_3 &= \frac{a}{2} + e - \frac{1}{\sqrt{2}}, \\
x_4 &= \frac{a}{2} + e + \frac{1}{\sqrt{2}}, \\
x_5 &= \frac{a}{2} - c - \frac{1}{\sqrt{2}}, \\
x_6 &= \frac{a}{2} + c - \frac{1}{\sqrt{2}}, \\
x_7 &= \frac{a}{2} - c + \frac{1}{\sqrt{2}},
\end{align*}

where $a, c, e$ are arbitrary numbers changed such that $x_v > 0, v = 0, 1, \ldots, 7$.

Now we present a simple method of defining a $2n$-polygon with constant diagonal. Let $f$ be a $2\pi$-periodic real positive function such that

\begin{equation}
f(t) + f(t + \pi) = C \quad \text{for all } t.
\end{equation}

Then the Fourier series of $f$ has the form (see[8]):

\begin{equation}
f(t) = \frac{1}{2}A_0 + \sum_{j=0}^{\infty} [A_{2j+1} \cos((2j + 1)t) + B_{2j+1} \sin((2j + 1)t)].
\end{equation}

Moreover we assume that the series is uniformly convergent to $f$. Keeping the above-mentioned notions we prove the following lemma:
Lemma 4.1 For each fixed $t$ the sequence
\[ x_v = f(t + v \frac{\pi}{n}), \quad v = 0, 1, \ldots \]
determines the $2n$-polygon with constant diagonal by formula (4.6).

**Proof.** Conditions (i) and (ii) are obvious. We verify the remaining relations.
(iii)
\[ x_v + x_{v+n} = f(t + v \frac{\pi}{n}) + f(t + (v + n) \frac{\pi}{n}) = C, \]
(iv)
\[ 2n a_1 = \sum_{v=0}^{n-1} f(t + v \frac{\pi}{n}) \cos \frac{v\pi}{n} = \]
\[ = \sum_{v=0}^{n-1} \left( \frac{1}{2} A_0 + \sum_{j=1}^{\infty} [A_{2j+1} \cos((2j + 1)(t + v \frac{\pi}{n})) + B_{2j+1} \sin((2j + 1)(t + v \frac{\pi}{n})) \right) \cos \frac{v\pi}{n} = \]
\[ = \frac{1}{2} A_0 \sum_{v=0}^{n-1} \cos \frac{v\pi}{n} + \sum_{j=1}^{\infty} \left( A_{2j+1} \sum_{v=0}^{n-1} \cos((2j + 1)(t + v \frac{\pi}{n})) \cos \frac{v\pi}{n} + B_{2j+1} \sum_{v=0}^{n-1} \sin((2j + 1)(t + v \frac{\pi}{n})) \cos \frac{v\pi}{n} \right) = 0. \]

To verify that the sums:
\[ \sum_{v=0}^{n-1} \cos((2j + 1)(t + v \frac{\pi}{n})) \cos \frac{v\pi}{n}, \]
\[ \sum_{v=0}^{n-1} \sin((2j + 1)(t + v \frac{\pi}{n})) \cos \frac{v\pi}{n}, \]
vanish, we apply simple trigonometric relations and we successively compute that
\[ \sum_{v=0}^{n-1} \cos((2j + 1)(t + v \frac{\pi}{n}) + \frac{v\pi}{n}) = \]
\[ = \frac{\sin((j+1)\frac{\pi}{n}) - 2jt - t}{2 \sin(j+1)\frac{\pi}{n}} - \frac{\sin(j(\frac{1}{n} - 4) + \frac{1}{n} - 4)\pi - 2jt - t}{2 \sin((\frac{1}{n} + \frac{1}{n})\pi)} = 0 \]
\[ \sum_{v=0}^{n-1} \cos((2j + 1)(t + v \frac{\pi}{n}) - \frac{v\pi}{n}) = \]
\[ = \frac{\sin(\frac{\pi}{n} - 2jt - t)}{2 \sin(\frac{\pi}{n})} - \frac{\sin(j(\frac{1}{n} - 4)\pi - 2jt - t)}{2 \sin(\frac{\pi}{n}) \sin(\frac{v\pi}{n} + (2j + 1)(t + \frac{\pi}{n}))} = 0 \]
\[ \sum_{v=0}^{n-1} \sin((2j + 1)(t + v \frac{\pi}{n}) + \frac{v\pi}{n}) = \]

\[ \frac{\cos((\frac{j}{n} + \frac{1}{n})\pi - 2jt - t)}{2\sin((\frac{j}{n} + \frac{1}{n})\pi))} - \frac{\cos((j(\frac{1}{n} - 4) + \frac{1}{n} - 4)\pi - 2jt - t)}{2\sin((\frac{j}{n} + \frac{1}{n})\pi))} = 0. \]

\[ \sum_{v=0}^{n-1} \sin((2j + 1)(t + v \frac{\pi}{n}) - \frac{v\pi}{n}) = \]

\[ \frac{\cos(\frac{j\pi}{n} - 2jt - t)}{2\sin(\frac{j\pi}{n})} - \frac{\cos(j(\frac{1}{n} - 4)\pi - 2jt - t)}{2\sin(\frac{j\pi}{n} j)} = 0. \]

To verify the above-mentioned equalities the computer program "Derive" was used. Therefore \(a_1 = 0\). Similarly we compute that \(b_1 = 0\).

### 4.1 On \(2n\)-polygons circumscribed on an oval with constant width

In this subsection we prove the Th.1.1, i.e.: All \(2n\)-polygons circumscribed on an oval with constant width \(\delta\) are \(2n\)-polygons with constant diagonal equal to

\[ \frac{\delta}{\cos \frac{\pi}{n}}. \]

**Proof.** Let an oval in arc length parametrization be represented by equation

\[ s \mapsto z(s) = x(s) + iy(s). \]

We will denote a curvature, tangent and normal vectors at point \(z(s)\) by \(k(s), T_s, N_s\), respectively. Moreover we define \(K(s) = \int_0^s k(t)dt\). Now we apply function \(\varphi(s) = K^{-1}(K(s) + \frac{\pi}{n})\), where \(K^{-1}\) is an inverse function of \(K\). Denoting \(\varphi^n(s) = \underbrace{\varphi(\varphi \ldots \varphi(s) \ldots)}_{v\text{-times}}\) we easy observe that \(\varphi^n = K^{-1}(K(s) + \pi)\). Obviously

\[ |z(s) - z(\varphi^n(s))| = |z(\varphi(s) - z(\varphi^n(s)))| = \delta, \quad \text{see Fig. 2.} \]
Fig. 2. 2n-polygon circumscribed on an oval.

Next we consider the following expressions
\[ d_v = [x(\varphi^v(s)) - x(\varphi^{v+1}(s)), T_{\varphi^v(s)}], \quad v = 0, 1, \ldots, 2n - 1, \]
\[ D_v = [x(\varphi^v(s)) - x(\varphi^{v+1}(s)), N_{\varphi^v(s)}], \quad v = 0, 1, \ldots, 2n - 1. \]

Applying the same considerations as in [6, p. 373] we solve the following system of equations:
\[ z(\varphi^v(s)) + \xi_v T_{\varphi^v(s)} = z(\varphi^{v+1}(s)) + \eta_v T_{\varphi^{v+1}(s)}, \quad v = 0, 1, \ldots. \]

Hence we obtain the points
\[ A : z(s) + [-D_0 - d_0 \cot \frac{\pi}{n}] T_s, \]
\[ B : z(\varphi^n(s)) + [-D_n - d_n \cot \frac{\pi}{n}] (-T_s). \]

Now we compute the length of the diagonal AB:
\[ |AB| = |z(s) - z(\varphi^n(s)) + [(-D_n - D_0) + (-d_n - d_0) \cot \frac{\pi}{n}] T_s|, \]
but
\[ z(s) - z(\varphi^n(s)) = -\delta N_s \]
\[ D_n + D_0 = -\delta \sin \frac{\pi}{n} \]
\[ d_n + d_0 = \delta (1 - \cos \frac{\pi}{n}). \]

Inserting these relations we express the length |AB| as follows
\[ |AB| = | -\delta N_s - \delta [\sin \frac{\pi}{n} + (1 - \cos \frac{\pi}{n}) \cot \frac{\pi}{n}] T_s| = \]
\[ = \delta| - N_\delta + \tan \frac{\pi}{2n} T_n| = \frac{\delta}{\cos \frac{\pi}{2n}}. \]

This implies that a perimeter of 2n-polygons (circumscribed on the oval) tends to \( \pi \delta \). Indeed we have
\[
\pi \left( \frac{\sin \frac{\pi n}{2n}}{\cos \frac{\pi n}{2n}} \right) \delta \xrightarrow{n \to \infty} \pi \delta.
\]

5 On \( m \)-polygons circumscribed on an \( m \cdot n \)-polygon

The results of this section are discrete counterparts of theorems presented in papers [1,3,4].

In the section we examine \( m \cdot n \)-polygons represented by the formula
\[(5.8) \quad z_k = \sum_{v=0}^{k} x_v e^{i \frac{2\pi v}{m}}, \]
where \( n \geq 2, \ m \geq 3 \) and a sequence \( x_v \) satisfies the following conditions:

1° \( x_v > 0, \)

2° \( x_{v+m} = x_v, \quad v = 0,1, \ldots, \)

3° \( x_v + x_{v+n} + x_{v+2n} + \ldots + x_{v+(m-1)n} = c, \quad v = 0,1, \ldots, \)

4° \( a_1 = b_1 = 0. \)

We consider an \( m \)-polygon b-circumscribed on an \( m \cdot n \)-polygon. For a fixed integer \( k \) we draw bisectors of outside angles in vertices
\[ z_k, z_{k+n}, z_{k+2n}, \ldots, z_{k+(m-1)n}. \]

This \( m \)-polygon b-circumscribed on \( m \cdot n \)-polygon has vertices defined as a point of intersection of two successive bisectors passing through vertices \( z_{k+jn}, z_{k+(j+1)n} \).

Keeping notions as before we show

**Theorem 5.1** All \( m \)-polygons b-circumscribed on an \( m \cdot n \)-polygon have the same perimeter whenever the sequence \( x_v \) satisfies conditions 1° – 4°.

**Proof.** To prove the theorem we denote vectors parallel to bisectors of inside and outside angels at vertex \( z_k \) of the polygon by \( N_k \) and \( T_k \), respectively. Applying Fig.3 we easily observe that
\[ T_k = e^{i \frac{2\pi k}{m}} e^{i \frac{2\pi n}{m}} = e^{i \frac{2\pi k}{m}} \text{ and } N_k = iT_k. \]

To compute the perimeter of b-circumscribed \( m \)-polygon we use the following vectors
\[ T_{k+jn} = e^{i \frac{2\pi k}{m}} T_k, \quad \text{and} \quad N_{k+jn} = iT_{k+jn}, j = 1,2, \ldots m-1, \]
where \( \varepsilon = \cos \frac{2\pi}{m} + i \sin \frac{2\pi}{m} \). Next we solve the following system of equations
\[ z_{k+jn} + \xi_{k+jn} \varepsilon e^{i \frac{2\pi k}{m}} T_k = z_{k+(j+1)n} + \eta_{k+jn} \varepsilon e^{i \frac{2\pi k}{m}} T_k, \quad j = 0,1,2, \ldots m-1. \]

The geometrical meaning of the above-mentioned equations is illustrated in Fig.3
Solving these equations we obtain
\[
\eta_{k+j} = \frac{[z_{k+j+n} - z_{k+j}, \epsilon^j T_k]}{\sin \frac{2\pi m}{m}}, \quad \xi_{k+j} = \frac{[z_{k+j+n} - z_{k+j}, \epsilon^{j+1} T_k]}{\sin \frac{2\pi m}{m}}.
\]
Let \( L_k \) denote a perimeter of b-circumscribed \( m \)-polygon. The Fig.3 suggests that
\[
L_k = \sum_{v}^{m-1} (\xi_{k+v} - \eta_{k+v}).
\]
Inserting all formulas on \( \xi_{\theta}, \eta_{\theta} \) we obtain
\[
L_k = \frac{1}{\sin \frac{2\pi m}{m}} \sum_{v=0}^{m-1} ([z_{k+(v+1)n} - z_{k+v}, \epsilon^{v+1} T_k] - [z_{k+(v+1)n} - z_{k+v}, \epsilon^v T_k]) =
\]
\[
\begin{align*}
&= \frac{1}{\sin \frac{\pi}{m}} \sum_{v=0}^{m-1} \left( [e^{\frac{im}{m}}(z_{k+(v+1)n} - z_{k+vn}), T_k] - [e^{\frac{i(m-v)}{m}}(z_{k+(v+1)n} - z_{k+vn}), T_k] \right) = \\
&= \frac{1}{\sin \frac{\pi}{m}} \left[ (2 - \varepsilon - \frac{1}{\varepsilon}) \sum_{v=0}^{m-1} e^{\frac{i(m-v)}{m}} z_{k+vn}, T_k \right],
\end{align*}
\]

but \(2 - \varepsilon - \frac{1}{\varepsilon} = 2 - 2\text{Re}(\varepsilon) = 4\sin^2 \frac{\pi}{m}\). Hence putting

\[p_k = \sum_{v=0}^{m-1} e^{\frac{i(m-v)}{m}} z_{k+vn}\]

we express \(L_k\) as follows

\[L_k = 2[p_k, T_k] \tan \frac{\pi}{m}\]

Now introducing notions \(d_k = [p_k, T_k]\) and \(D_k = [p_k, N_k]\) we express vector \(p_k\) as follows

\[p_k = D_k T_k - d_k N_k\]

In conclusion we show that discrete Fourier sums of sequences \(d_k\) and \(D_k\) have the form

\[d_k = [p_k, T_k] = \left[ \sum_{j=0}^{m-1} e^{-j} \sum_{v=0}^{n-1} x_v e^{i \frac{2\pi}{mn} (v-1)}, e^{i \frac{2\pi}{mn} j} \right] = \]

\[= \sum_{j=0}^{m-1} \sum_{v=0}^{n-1} x_v \sin \frac{(2k + 1 - 2v + 2jn)\pi}{mn} = \]

\[= \sum_{j=0}^{m-1} \sum_{v=0}^{n-1} \left( a_0 + \sum_{l=1}^{m-1} \sum_{s=1}^{m-1} a_{ml+s} \cos \frac{2\pi(ml + s)v}{mn} + \\
+ b_{ml+s} \sin \frac{2\pi(ml + s)v}{mn} \right) \sin \frac{(2k - 2v + 1 + jn)\pi}{mn} = \]

\[a_0 \sum_{j=0}^{m-1} \sum_{v=0}^{n-1} \sin \frac{(2k - 2v + 2n j + 1)\pi}{mn} + \]

\[+ \sum_{l=1}^{m-1} \sum_{s=1}^{m-1} a_{ml+s} \sum_{j=0}^{m-1} \sum_{v=0}^{n-1} \cos \frac{2\pi(ml + s)v}{mn} \sin \frac{(2k - 2v + 2n j + 1)\pi}{mn} + \]

\[+ \sum_{l=1}^{m-1} \sum_{s=1}^{m-1} b_{ml+s} \sum_{j=0}^{m-1} \sum_{v=0}^{n-1} \sin \frac{2\pi(ml + s)v}{mn} \sin \frac{(2k - 2v + 2n j + 1)\pi}{mn} = \frac{ma_0}{2\sin \frac{\pi}{mn}}.\]

Similarly we compute that \(D_k = 0\). Hence we finally obtain

\[L_k = 2[p_k, T_k] \tan \frac{\pi}{m} = 2 \frac{ma_0}{2\sin \frac{\pi}{mn}} \tan \frac{\pi}{m}.\]

This means that all b-circumscribed m-polygons have the same perimeter independent of index \(k\).
Moreover we observe that every vector \( \mathbf{p}_k = -d_k \mathbf{n}_k \) has the same length equal to \( \frac{m a_0}{2 \sin \frac{x}{mm}} \). We put \( d = |\mathbf{p}_k| \), then we express perimeter \( L \) of \( mn \)-polygon with a constant perimeter of \( b \)-circumscribed \( m \)-polygon by the following relation

\[
L = \frac{2\pi}{m} \left( \frac{\sin \frac{x}{mm}}{x} \right) d,
\]

because \( mnL = x_0 + x_1 + \ldots + x_{mn-1} \). Tending to infinity with \( n \) we obtain

\[
L = \frac{2\pi d}{m}.
\]

Formula (5.9) is a discrete counterpart of Th.1.[3] and becomes formula (1.4) for \( m = 2 \). To define an \( mn \)-polygon with a constant perimeter of a \( b \)-circumscribed \( m \)-polygon we apply a \( 2\pi \)-periodic positive function \( f \) such that

\[
\sum_{v=0}^{m-1} f(t + v \frac{2\pi}{mn}) = C.
\]

We assume that function \( f \) has uniformly convergent Fourier series and this series has a form

\[
f(t) = \frac{1}{2} A_0 + \sum_{i=1}^{\infty} [A_i \cos(lt) + B_i \sin(lt)],
\]

where \( A_{m,j} = B_{m,j} = 0, \quad j = 1, 2, \ldots \) see [4,1]. Putting

\[
x_v = f(t + \frac{2\pi v}{mn}), \quad v = 0, 1, \ldots
\]

we obtain (for a fixed variable \( t \)) a sequence which satisfies conditions \( 1^o-4^o \). Therefore a \( mn \)-polygon represented by equation

\[
z_k = \sum_{v=0}^{k} x_v e^{i \frac{2\pi v}{mn}}
\]

has the constant perimeter of each \( m \)-polygon \( b \)-circumscribed on it.

5.1 On an oval with constant perimeter of a circumscribed \( m \)-polygon and on an \( mn \)-polygon

In subsection 4.1 we proved that every \( 2n \)-polygon circumscribed on an oval with constant width \( d \) is the polygon with a constant diagonal equal to \( \frac{d}{\cos \frac{x}{2n}} \). In this subsection we prove Theorem 1.2, i.e.:

All \( m \)-polygons \( b \)-circumscribed on an \( mn \)-polygon which is circumscribed on an oval with a constant perimeter of a circumscribed \( m \)-polygon have the same perimeter equal to
\[
\frac{l}{m} \cos \frac{\pi}{mn},
\]
where \( l \) denotes the length of \( m \)-polygon circumscribed on this oval.

**Proof.** We keep notion as before. Let \( z(s) \) be an oval with a constant perimeter of a circumscribed \( m \)-polygon. Putting \( \varphi(s) = K^{-1}(K(s) + \frac{2\pi}{mn}) \) we easy observe that \( \varphi^{mn}(s) = s + L \) and that \( mn \)-polygon circumscribed on the oval is tangent (to the oval) at points \( z(\varphi^v(s)) \), \( v = 0, 1, 2, \ldots \). Then vertices of an \( mn \)-polygon circumscribed on the oval are expressed as follows:

\[
z(\varphi^v(s)) + \xi_v T_{\varphi^v(s)}, \quad v = 0, 1, 2, \ldots,
\]

where

\[
\xi = -D_v - d_v \cot \frac{2\pi}{mn}, \quad D_v = [z(\varphi^v(s)) - z(\varphi^{v+1}(s)), N_{\varphi^v(s)}]
\]

and \( d_v = [z(\varphi^v(s)) - z(\varphi^{v+1}(s)), T_{\varphi^v(s)}] \). Now we consider \( m \)-polygon b-circumscribed on \( mn \)-polygon, see Fig.4.

![Fig.4. m-polygon b-circumscribed on mn-polygon which is circumscribed on the oval.](Image)

We denote by \( T_{k+j} \) the tangent vector at point \( z(\varphi^{k+j}\nu(s)) \) for fixed \( k \) and \( j = 0, 1, \ldots, m - 1 \). To compute the perimeter of \( m \)-polygon b-circumscribed on \( mn \)-polygon we solve the following system of equations (Fig.3 and Fig.4):

\[
z(\varphi^{k+j}\nu(s)) + \xi_{k+j} T_{k+j} + \xi_{k+j+1} \mu_{k+j} e^{i \frac{\pi}{mn}} =
\]

\[
z(\varphi^{k+(j+1)}\nu(s)) + \xi_{k+(j+1)} T_{k+(j+1)} + \eta_{k+(j+1)} e^{i \frac{\pi}{mn}},
\]

where \( j = 0, 1, \ldots, m - 1 \).

Moreover the length of sectors \( |z(\varphi^{k+j}\nu(s))A_{k+j}| \) and \( |z(\varphi^{k+(j+1)}\nu(s))A_{k+(j+1)}| \) is denoted by \( \xi_{k+j} \) and \( \xi_{k+(j+1)} \), respectively. The length of sectors \( |A_{k+j}B_{k+j}| \) and \( |B_{k+j}A_{k+(j+1)}| \) is denoted by \( \eta_{k+j} \) and \( \eta_{k+(j+1)} \), respectively. Moreover vectors \( T_{k+j} e^{i \frac{\pi}{mn}} \) and \( T_{k+(j+1)} e^{i \frac{\pi}{mn}} \) are parallel to bisectrices of outside angles of \( mn \)-polygon. Obviously perimeter \( l_k \) of \( m \)-polygon b-circumscribed on \( mn \)-polygon is equal to
\[ I_k = \sum_{j=0}^{m-1} (\xi_{k+jn}^1 - \eta_{k+jn}^1). \]

At first we compute \( \xi_{k+jn}^1 \)
\[
\xi_{k+jn}^1 = \frac{1}{-T_{k+jn} e^{i \frac{\pi}{nm}}, T_{k+(j+1)n} e^{i \frac{\pi}{nm}}} \left( z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n} e^{i \frac{\pi}{nn}} \right) +
\]
\[
+ [\xi_{k+jn} T_{k+jn} - \xi_{k+(j+1)n} T_{k+(j+1)n}, T_{k+(j+1)n} e^{i \frac{\pi}{nn}}] =
\]
\[
= \frac{1}{\sin \frac{\pi}{mm}} \left( z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n} e^{i \frac{\pi}{mm}} \right) +
\]
\[
+ \xi_{k+jn} \sin \frac{(2n + 1)\pi}{mm} - \xi_{k+(j+1)n} \sin \frac{\pi}{mm} \right)
\]

and
\[
\eta_{k+jn}^1 = \frac{1}{\sin \frac{2\pi}{mm}} \left( z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n} e^{i \frac{\pi}{mm}} \right) +
\]
\[
+ \xi_{k+jn} \sin \frac{\pi}{mm} + \xi_{k+(j+1)n} \sin \frac{(2n - 1)\pi}{mm} \right).
\]

Inserting relation \( T_{k+jn} = \varepsilon^j T_k, \quad \varepsilon^m = 1, \varepsilon \neq 1 \) we obtain
\[
W = \sum_{j=0}^{m-1} \left( z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n} e^{i \frac{\pi}{mm}} \right) +
\]
\[
+ \sum_{j=0}^{m-1} [z(\varphi^{k+jn}(s)) - z(\varphi^{k+(j+1)n}(s)), T_{k+(j+1)n} e^{i \frac{\pi}{mm}}] =
\]
\[
= [\varepsilon p_k - p_k - p_k + \varepsilon p_k, T_k e^{i \frac{\pi}{mm}}],
\]

where
\[
p_k = \sum_{j=0}^{m-1} \varepsilon^{m-j} z(\varphi^{k+jn}(s)).
\]

By (3.1)[4,p.374] \( p_k = -dN_k \) (where \( d \) denotes \( n \)-width of the oval) we obtain
\[
W = (\varepsilon + \varepsilon - 2)[p_k, T_k e^{i \frac{\pi}{mm}}] = -4d \sin^2 \frac{\pi}{m} \cos \frac{\pi}{mn}.
\]

It is easy to observe that
\[
\sum_{j=0}^{m-1} \xi_{k+jn} = \sum_{j=0}^{m-1} \xi_{k+(j+1)n}.
\]

The sum
\[
\sum_{j=0}^{m-1} \xi_{k+jn}
\]

is equal to
\[
\sum_{j=0}^{m-1} \xi_{k+jn} = - \sum_{j=0}^{m-1} ([z(\varphi^{k+jn}(s)) - z(\varphi^{k+jn+1}(s)), N_{k+jn}] + \\
+z(\varphi^{k+jn}(s)) - z(\varphi^{k+jn+1}(s)), T_{k+jn}]) \cot \frac{2\pi}{mn} = \\
= - \sum_{j=0}^{m-1} ([e^{m-j}z(\varphi^{k+jn}(s)) - e^{m-j}z(\varphi^{k+jn+1}(s)), N_{k}] + \\
[e^{m-j}z(\varphi^{k+jn}(s)) - e^{m-j}z(\varphi^{k+jn+1}(s)), T_{k}] \cot \frac{2\pi}{mn}) = \\
= -(\{p_k - p_{k+1}, N_k\} + [p_k - p_{k+1}, T_k] \cot \frac{2\pi}{mn} = \\
= d(\sin \frac{2\pi}{mn} - \tan \frac{\pi}{mm} \cos \frac{2\pi}{mn}).
\]

Finally we obtain perimeter \( l_k \) as the following expression

\[
l_k = \frac{-4d \sin^2 \frac{\pi}{m} \cos \frac{\pi}{mm}}{-\sin \frac{2\pi}{m}} + \\
+ \frac{1}{-\sin \frac{2\pi}{m}} \left( \sum_{j=0}^{m-1} \xi_{k+jn} (\sin \frac{(2n+1)\pi}{mn} - \sin \frac{\pi}{mn}) \right) + \\
+ \sum_{j=0}^{m-1} \xi_{k+(j+1)n} (-\sin \frac{\pi}{mm} - \sin \frac{(2n-1)}{mn}) = \\
= 2d \tan \frac{\pi}{m} (\cos \frac{\pi}{mm} + \sin \frac{\pi}{mm} (\sin \frac{2\pi}{mm} - \tan \frac{\pi}{mm} \cos \frac{2\pi}{mn})) = \\
= 2d \tan \frac{\pi}{m} \frac{1}{\cos \frac{\pi}{mn}}.
\]

But by [4, p. 373] \( l = 2d \tan \frac{\pi}{m} \) is equal to the perimeter of \( m \)-polygon circumscribed on an oval. Therefore

\[
l_k = \frac{l}{\cos \frac{\pi}{mm}}.
\]

This means that all \( m \)-polygons b-circumscribed on \( mn \)-polygon have the same perimeter. Moreover if \( n \) tends to infinity then perimeter \( l_k \) tends to the perimeter of \( m \)-polygon circumscribed on an oval. \( \square \)
References


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