Extrema Constrained by $C^k$ Curves

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Abstract

In this paper we generalize the results obtained in Ref. [17].

§1 raises the following problem: what connection there exists between the local extrema of the function $f : D \subset \mathbb{R}^p \rightarrow R$ and the local extrema of the functions $f \circ \alpha$, $\alpha \in \Gamma$, where $\Gamma$ is a given family of parametrized curves ?

§2 proves the existence of a $C^k$ curve containing a given sequence of points.

§3 solves the problem which was presented in §1 in the case of $C^k$ curves.

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1 Introduction

Let us consider the extremum problem

$$\min f(x), \text{ subject to } x \in M,$$

where $M$ is a subset of $\mathbb{R}^p$ with a given structure. If $M$ is an open set of $\mathbb{R}^p$ which coincides to the domain of $f$, then the extremum problem is called unconstrained; in any other case, the extremum problem is called constrained. Such problems, in which $M$ is a $C^k$, $k \geq 2$, finite-dimensional differentiable were developed recently in Refs. [1]-[4], [6]-[9].

The extremum conditions (necessary and sufficient) depend on the fashion of defining the subset $M$. If $M$ is a differentiable manifold, then they depend also on the geometrical structure of $M$. Frequently, $M$ is considered as the union of a family of its subsets (a plane as the union of straight lines, an integral manifold of a Pfaff system as the union of some integral curves, and so on, Refs. [10]-[16]). Then the following two problems arise:

1) Let $D$ be an open subset in $\mathbb{R}^p$ and $\{A_i\}_{i \in I}$ be a family of subsets of $D$ having a common point $x_* \in A_i, \forall i \in I$. Suppose $x_*$ is a local minimum point for each restriction $f|_{A_i}$ of the function $f : D \rightarrow R$ to the subset $A_i, i \in I$. Is $x_*$ a local minimum point of $f$ ?

2) Let $f : D \subset \mathbb{R}^p \rightarrow R$ and $\alpha_i : I_i \subset R \rightarrow D, i \in J$ a family of parametrized curves. What connection we have between of functions $f \circ \alpha_i$, the extrema of the restrictions $f|_{\alpha_i(I_i)}$, and the extrema of the function $f$ ?


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The problem 1) and the problem 2), for the $C^1$ or $C^2$ curves, were solved in Ref. [17]. In this paper we shall solve the problem 2) for the general case of $C^k$ curves and even for the analytic curves. For that reason we shall recall some notions about the curves.

**Definition 1.1.** Let $I \subset R$ be an interval. A function $\alpha : I \to R^p$ of class $C^k$, $k \geq 1$, is called parametrized curve of class $C^k$ and is denoted by $\alpha$. We shall say that:

1) $\alpha$ passes (just once) through the point $x_* \in R^p$ if there exists (only one) $t_0 \in Int I$ such that $\alpha(t_0) = x_*;

2) $\alpha$ is a simple parametrized curve if $\alpha$ is injective;

3) $\alpha$ is regular at the point $x_* = \alpha(t_0)$ if $\alpha'(t_0) \neq 0$;

4) $\alpha$ has a tangent at the point $x_* = \alpha(t_0)$ if there exists $m \in \mathbb{1}, k$, such that $\alpha^{(m)}(t_0) \neq 0$.

**Definition 1.2.** Two parametrized curves $\alpha : I \to R^p, \beta : J \to R^p$ of class $C^k$ are called equivalent if there exists a diffeomorphism $h : I \to J$ of class $C^k$ such that $\alpha = \beta \circ h$. We shall write $\alpha \sim \beta$.

**Definition 1.3.** The set $\tilde{\alpha}$ of $C^k$ parametrized curves equivalent to $\alpha : I \to R^p$ is called curve of class $C^k$. The curve $\tilde{\alpha}$ has qualities 1) to 4) in the Definition 1.1, if a representative $\alpha$ has these properties.

From now we shall refer to a function $f : D \to R$, where $D$ is an open subset in $R^p$.

**Definition 1.4.** Let $f : D \to R$, let $x_* \in D$, and $\alpha : I \to D$ be a parametrized curve passing through $x_*$. We shall say that:

1) $x_*$ is a minimum point for $f$ constrained by $\alpha$ if for any $t_0 \in I$, with $\alpha(t_0) = x_*$, the point $t_0$ is a local minimum point for $f \circ \alpha$, i.e., there exists a neighborhood $I_{t_0} \subset I$ of $t_0$ such that

$$f(x_*) = f(\alpha(t_0)) \leq f(\alpha(t)), \forall t \in I_{t_0}.$$

2) $x_*$ is a minimum point for $f$ constrained by $\tilde{\alpha}$ if there exists a neighborhood $V$ of $x_*$, such that

$$f(x_*) \leq f(x), \quad \forall x \in V \cap \alpha(I).$$

**Remark.** If $x_*$ is a minimum point of $f$ constrained by the curve $\tilde{\alpha}$, then $x_*$ is a minimum point of $f$ constrained by the parametrized curve $\alpha$. The converse is not true even so $\alpha$ is a simple parametrized curve. However, in case that $\alpha : I \to D$ is a simple and regular parametrized curve and $I$ is a compact set, both notions coincide.

**Definition 1.5.** Let $\Gamma_{x_*}$ be a family of parametrized curves (curves) passing through the point $x_*$. We shall say that $x_*$ is a minimum point of $f$ constrained by the family $\Gamma_{x_*}$ if $x_*$ is a minimum point of $f$ constrained by every curve of the family $\Gamma_{x_*}$.

## 2 $C^k$ curves by given sequences of points

The aim of this paragraph is to show that certain conditions assume the existence of $C^k$ curves which contain a given sequence of points. To this we recall shortly the prolongation theorem of Whitney (Ref. [5]).

Let $K \subset R^p$ be a compact set, $k = (k_1, \ldots, k_p)$ be a multiindex and $|k| = k_1 + \cdots + k_p$. A family of continuous functions $F = (f_k)_{|k| \leq m}, f^k : K \to R$, is called jet of order $m$. Denote $F(x) = f^0(x), x \in K$ and $D^k F = (f^{k+1})_{|k| \leq m-|k|, |k| \leq m}$. Naturally, for any function $g \in C^m (K)$ one can define the jet.
\[ J^m(g) = \left( \frac{\partial^k g}{\partial x^k} \right)_{|k| \leq m}. \]

For any \( x \in \mathbb{R}^p \) and a fixed \( a \in K \), we introduce the Taylor polynomial function

\[ T^n_a F(x) = \sum_{|k| \leq m} \frac{(x - a)^k}{k!} f^k(a). \]

Denote

\[ \hat{T}^n_a F = J^m(T^n_a F), \quad R^n_a F = F - \hat{T}^n_a F. \]

**Prolongation Theorem 2.1.** Let \( F = (f^k)_{|k| \leq m} \). There exists a function \( f \in C^m(\mathbb{R}^p) \) with \( g^m(f) = F \) if and only if

\[ (R^n_a F)^k(y) = O(||y||^{0-\|m||}), \]

when \( |x - y| \to 0 \), for any \( x, y \in K \) and any \( |k| \leq m \).

In the sequel we shall apply the preceding theorem in the case \( p = 1 \) and \( K = \{0\} \cup \{t_n|n \in N\} \); where \( t_n \in R \) and \( t_n \to 0 \).

**Corollary 2.1.** Let us consider \( k \in N^* \). Given the real sequences \( t_n \to 0, x_n^{(0)} \to 0, x_n^{(i)} \to a^{(i)}, i = \overline{1,k} \), there exists \( f \in C^k(R) \) with \( f^i(t_n) = x_n^{(i)}, \forall i \in \overline{0,k} \), if and only if

\[ \frac{x_m^{(p)} - \sum_{i=p}^{k-1} \frac{(t_m - t_n)^{i-p}}{(i-p)!} x_n^{(i)}}{(t_m - t_n)^{k-p}} \to \frac{a^{(k)}}{(k-p)!} \]

for \( m, n \to \infty \) and for any \( p \in \overline{0,k-1} \).

**Lemma 2.1. (Ref. [17])** Let \( (x_n), (y_n) \) be two sequences of real numbers such that

1) \( x_n \neq 0, x_n \neq x_{n+1}, \forall n \in N; \)
2) there exists \( \lim_{n \to \infty} \frac{y_n}{x_n} = r; \)
3) there exists \( \lambda > 0 \) with \( \frac{x_n}{x_{n+1}} - 1 \geq \lambda, \forall n \in N. \)

Then the sequence \( \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \) is convergent towards \( r \).

**Lemma 2.2.** If \( (x_n) \) is a sequence of strictly positive real numbers and \( x_{n+1} \leq \frac{1}{2^k} x_n, \forall n \in N, \) where \( k \in N^* \), then there exists \( \mu > 0 \) such that

\[ \frac{x_n - x_m}{(x_n^{1/k} - x_m^{1/k})^k} \leq \mu, \forall m, n \in N, m \neq n. \]

**Proof.** For \( m = n + p, p \geq 1 \), it follows \( x_n^{1/k} \leq \frac{1}{2^p} x_m^{1/k} \). Denote \( t_n = x_n^{1/k} \). We have

\[ \frac{x_n - x_m}{(x_n^{1/k} - x_m^{1/k})^k} = \frac{t_n^k - t_m^k}{(t_n - t_m)^k} = \frac{p_{n-1} + \ldots + p_{m-1}}{(t_n - t_m)^{k-1}} \leq \]
\[
\frac{t_n^{k-1}}{t_n^{k-1} \left( \frac{1}{2^p} \right)^{k-1}} = 1 - \left( \frac{1}{2^p} \right)^k < \frac{1}{k}.
\]

**Lemma 2.3.** Let \((x_n), (y_n)\) two sequences of real numbers such that \((x_n)\) is strictly monotone,

\[
y_n \to 0, \quad \frac{y_n}{x_n} \to 0, \quad |x_{n+1}| \leq \frac{1}{2^k|x_n|}, \quad \forall n \in \mathbb{N}
\]

where \(k \in \mathbb{N}^*\). Then \(\frac{y_m - y_n}{x_m - x_n} \to 0\) for \(m, n \to \infty\).

**Proof:** Suppose, for example, \(x_n > 0, \forall n \in \mathbb{N}\). Let \(b_n = \frac{y_{n+1} - y_n}{x_{n+1} - x_n}\). From Lemma 2.1 it follows \(b_n \to 0\). Let \(m > n\) and \(\mu_{mn} = \max\{|b_n|, \ldots, |b_{m-1}|\}\). Then, for any \(i \in n, m - 1\) we have

\[
\frac{|y_{i+1} - y_i|}{x_{i+1} - x_i} \leq \mu_{mn}, \quad \text{namely}
\]

\[
-\mu_{mn} \leq \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \leq \mu_{mn}
\]

and therefore

\[
-\mu_{mn} \sum_{i=n}^{m-1} (x_i - x_{i+1}) \leq \sum_{i=n}^{m-1} (y_{i+1} - y_i) \leq \mu_{mn} \sum_{i=n}^{m-1} (x_i - x_{i+1}).
\]

It results

\[
\left| \frac{y_m - y_n}{x_m - x_n} \right| \leq \mu_{mn}.
\]

Now the conclusion is obvious.

**Lemma 2.4.** Let \((x_n), (y_n)\) be two sequences of real numbers such that \((x_n)\) is strictly monotone, \(y_n \to 0, \quad \frac{y_n}{x_n} \to 0, \quad |x_{n+1}| \leq \frac{1}{2^k|x_n|}, \quad \forall n \in \mathbb{N}, \quad \text{where} \quad k \in \mathbb{N}^*\). Then, there exist two functions \(f, g \in C^k(\mathbb{R})\) and a sequence \((t_n)\) of real numbers such that \(t_n \to 0, \quad f(t_n) = x_n, \quad g(t_n) = y_n, \quad f^{(i)}(0) = 0, \quad \forall i \in \{0, k-1\}, \quad g^{(i)}(0) = 0, \quad \forall i \in \{0, k\} \) and \(f^{(k)}(0) \neq 0\).

Moreover, the function \(f\) and the sequence \((t_n)\) do not depend on the sequence \((y_n)\).

**Proof:** Let \(t_n = x_n^{1/k}\). Then, the function \(f(x) = x^{1/k}\) and the sequence \((t_n)\) satisfy the required properties. In order to get the function \(g\) we should apply the Corollary 2.1 to \(t_n = x_n^{1/k}, \quad x_n^{(0)} = y_n, \quad x_n^{(i)} = 0\) and \(g^{(i)} = 0, \quad \forall i \in \{1, k\}\). Obviously, the condition (\(\star\)) from the statement (Corollary 2.1) is fulfilled, for any \(p \geq 1\). For \(p = 0\) this condition becomes \(\frac{y_m - y_n}{(t_m - t_n)^k} \to 0\), for \(m, n \to \infty\). Taking into account that

\[
\frac{y_m - y_n}{(t_m - t_n)^k} = \frac{y_m - y_n}{x_m - x_n} \cdot \frac{x_m - x_n}{(t_m - t_n)^k}
\]

and using the Lemmas 2.2 and 2.3, it results \(\frac{y_m - y_n}{(t_m - t_n)^k} \to 0\), for \(m, n \to \infty\).
Thus we obtain a function $g \in C^k(R)$ which satisfies the required properties.

**Theorem 2.2.** Let $(x_n)$ be a sequence of distinct points of $R^p$, convergent to the point $a \in R^p$. Then, for any $k \in N^*$, there exist a subsequence $(x_{nm})$ and a parametrized curve $\alpha : R \to R^p$ of class $C^k$, which contains the set of points $\{x_n, a\}$ such that $\alpha$ has a tangent at the point $a$. Moreover, if $a = \alpha(t_0)$, then $\alpha^{(i)}(t_0) = 0$, $\forall i \in 1, k - 1$, $\alpha^{(k)}(t_0) \neq 0$ and there exists a sequence $(t_m)$ with $t_m \to t_0$ and $\alpha(t_m) = x_{nm}$.

**Proof.** By a translation, we can suppose $a = (0, \ldots, 0) \in R^p$. Since $u_n = \frac{x_n}{||x_n||}$ is bounded, considering it likely a subsequence, we can assume $u_n \to u \in R^p$. By a rotation, we can suppose $u = (1, 0, \ldots, 0)$. Consequently, if $x_n = (x_{n1}, \ldots, x_{np})$, it follows

$$\frac{x_{n1}}{|x_n|} \sqrt{1 + \left( \frac{x_{n2}}{x_{n1}} \right)^2 + \ldots + \left( \frac{x_{np}}{x_{n1}} \right)^2} \to 1.$$  

Hence $x_{ni} > 0$ for sufficiently large $n$ and $\frac{x_{ni}}{x_{n1}} \to 0$, $\forall i \in \overline{1, p}$. Obviously, there exists a subsequence $(x_{nm})$ such that $x_{nm1} > 0$ and $x_{nm1} \leq \frac{1}{2} x_{nm1}$, $\forall m \in N$.

Applying Lemma 2.4 to the pair of sequences $(x_{nm1})$ and $(x_{nm1})$, $i \in \overline{1, k}$, we obtain the functions $\varphi_i : R \to R$, $i \in \overline{1, p}$, of class $C^k$ and a sequence $(t_m)$ of real numbers such that $t_m \to 0$, $\varphi_i(t_m) = t_{nm1}$, $i \in \overline{1, p}$, $\varphi_i^{(j)}(0) = 0$, $i \in \overline{1, p}$, $j \in 0, k - 1$, $\varphi_i^{(k)}(0) = 0$, $i \in \overline{2, p}$ and $\varphi_i^{(k)}(0) \neq 0$. Then, the parametrized curve $\alpha(t) = (\varphi_1(t), \ldots, \varphi_p(t))$, $t \in R$, has the required properties.

### 3 Minimum constrained by $C^k$ curves

Let $D$ be an open subset in $R^p$ and $x_* \in D$. For any $k \in N^*$ we denote by $\Gamma^k_{x_*}$ the family of all $C^k$ parametrized curves passing just once through the point $x_*$, each having a tangent at $x_*$. Let

$$A^k_{x_*} = \left\{ \alpha \in \Gamma^k_{x_*} | \alpha(t_0) = x_*, \quad \alpha^{(i)}(t_0) = 0, \quad \forall i \in \overline{1, k - 1}, \quad \alpha^{(k)}(t_0) \neq 0 \right\}$$

and

$$B^k_{x_*} = \left\{ \alpha \in \Gamma^k_{x_*} | \alpha(t_0) = x_*, \quad \exists m \in \overline{1, k - 1} \text{ such that } \alpha^{(m)}(t_0) \neq 0 \right\}, \quad \text{for } k \geq 1.$$

**Theorem 3.1.** Let $f : D \to R$. Then, $x_*$ is a local minimum point of $f$, if and only if, there exists $k \in N^*$ such that $x_0$ is a minimum point constrained by the family $A^k_{x_*}$.

**Proof.** We can suppose $f(x_*) = 0$. If $x_*$ would not be a local minimum point of $f$, then there exists a sequence of distinct points $(x_n)$ of $R^p$, with $x_n \to x_*$, and $f(x_n) < 0$, $\forall n \in N$. Taking into account the Theorem 2.2 we find a curve $\alpha \in A^k_{x_*}$ such that $x_*$ is not a minimum point of $f$ constrained by $\alpha$. Contradiction.

It is interesting to remark that Theorem 3.1 does not impose any condition upon the function $f$. Then, more surprising is the fact that Theorem 3.1 fails for the family $B^k_{x_*}$ or for the family of all analytic curves passing through the point $x_*$, even if $f$ is of class $C^\infty$. 
**Examples.** 1) Let $f : R^3 \to R$,

$$f(x, y) = (y^k - x^{k+1})(y^k - 2^k x^{k+1}),$$

where $k \in N$, $k \geq 1$. It is obvious that $f$ is of class $C^\infty$ and that the critical point $x_* = (0, 0)$ is not a local minimum point of $f$. Let us show that $x_* = (0, 0)$ is a minimum point of $f$ constrained by the family $B^k_{x_*}$.

Let $D^- = \{ (x, y) \in R^2 \mid f(x, y) < 0 \}$. For any $(x, y) \in D^-$ it results

$$(*) \quad \frac{|y|}{|x|} < 2|x|^{1/k}$$

and

$$(***) \quad \frac{|y|}{|x|^{(m+1)/m}} > |x|^{-1/m(m+1)},$$

with $|x| < 1$ and $m \in \frac{1}{k-1}$.

Let $\alpha \in B^k_{x_*}$. We can suppose $x_* = \alpha(0)$. If, by reductio ad absurdum, the point $x_*$ would not be a minimum point of $f$ constrained by the parametrized curve $\alpha$, then would exist a sequence $(t_n)$ with $t_n \to 0$ and $\alpha(t_n) = (x_n, y_n) \in D^-$, $\forall n \in N$. Consequently, the numbers $x_n$ and $y_n$ satisfy the above conditions $(*)$ and $(***)$. Obviously, if $\alpha(t) = (x(t), y(t))$, then $x(t) = t^m(a + tf(t))$ and $y(t) = t^m(b + tg(t))$ where $a^2 + b^2 > 0$, $m \in \frac{1}{k-1}$ and $f, g$ are continuous functions. We assume that $a \neq 0$. Hence,

$$\left| \frac{y_n}{x_n} \right| = \left| \frac{b + t_n g(t_n)}{a + t_n f(t_n)} \right|.$$  Using the relation $(*)$, we get that $\left| \frac{y_n}{x_n} \right| \to 0$ and therefore $b = 0$. Hence,

$$\left| \frac{y_n}{x_n} \right| = \frac{|t_n f(t_n)|}{|a + t_n f(t_n)|^{1/(m+1)}} \to |g(0)|.$$

On the other hand, using the relation $(***)$, it results that $\left| \frac{y_n}{x_n} \right|^{1/(m+1)} \to \infty$. Contradiction. Now, we assume that $a = 0$. Then,

$$\left| \frac{x_n}{y_n} \right| = \frac{|f(t_n)|}{|b + t_n g(t_n)|} \to 0,$$

which is a contradiction to the relation $(*)$.

2) Let $g : R \to R$, $g(x) = e^{-1/x^2}$, for any $x > 0$ and $g(x) = 0$ for any $x \leq 0$. Let $f : R^2 \to R$, $f(x, y) = y(y - g(x))$ which is of class $C^\infty$. Also, it is obvious that the critical point $x_* = (0, 0)$. Let us show that $x_* = (0, 0)$ is a minimum point of $f$ constrained by the family $\Gamma^\infty_{x_*}$, where $\Gamma^\infty_{x_*}$ is the family of all analytic parametrized curves passing through the point $x_*$.

Let $D^- = \{ (x, y) \mid f(x, y) < 0 \}$. It follows that for any $(x, y) \in D^-$ we have $x > 0$ and

$$(*) \quad 0 < ye^{1/x^2} < 1$$

Let $\alpha \in \Gamma^\infty_{x_*}$. We can suppose $x_* = \alpha(0)$. If, by reductio ad absurdum, the point $x_*$ would not be a minimum point of $f$ constrained by $\alpha$, then would exist a sequence
(t_n) with t_n \to 0 and \alpha(t_n) = (x_n, y_n) \in D^-, \forall n \in N. Hence, the numbers x_n and y_n satisfy the above condition (*) and x_n \to 0, y_n \to 0. Obviously, if \alpha(t) = (x(t), y(t)), then x(t) = t^p(a + \ldots) and y(t) = t^q(b + \ldots), with ab \neq 0. It follows that

\[ y_n e^{1/x_n^2} = [(x_n^2)^{1/2} e^{1/x_n^2}] y_n (x_n^2)^{-q/2p} \to \infty \cdot \frac{\|y\|}{|\alpha|^q} = \infty, \]

which is in contradiction to the relation (*).

In the following, we shall denote by \overline{\Gamma}^k_{x_\star}, the family of all C^k curves passing through the point \( x_\star \), regular at \( x_\star \).

**Theorem 3.2.** Let \( f : D \subset R^p \to R \). If there exists \( k \in N^+ \) such that for any \( \hat{\alpha} \in \overline{\Gamma}^k_{x_\star} \), the point \( x_\star \) is an extrema point of \( f \) constrained by \( \hat{\alpha} \), then \( x_\star \) is a local extrema point of \( f \).

**Proof.** Let us suppose that \( x_\star \) is not a local extrema point for \( f \) and \( f(x_\star) = 0 \). Then, there exist two sequences \((x_n, y_n)\) of distinct points of \( D \) with \( x_n \to x_\star, y_n \to x_\star, f(x_n) < 0 \) and \( f(y_n) > 0 \), \( \forall n \in N \). By the Theorem 2.2 there exist two subsequences \((x_{n_m}, y_{n_m})\) and \((y_{n_r}, x_{n_r})\), two \( C^k \) parametrized curves \( \alpha \) and \( \beta \), and two sequences of real numbers \((t_{m}, t_{r})\) with \( t_m \to 0, t_r \to 0, t_m > 0, t_r > 0 \) such that \( \alpha(t_{m}) = x_{n_{m}}, \beta(t_{r}) = y_{n_{r}}, \forall m, r \in N \). Then, it is easy to show that there exists a parametrized curve \( \gamma : R \to R^p \) of class \( C^k \) such that \( \gamma(t) = \alpha(t), \forall t \leq 1, \gamma(t) = \beta(1/t), \forall t \geq 3, \gamma(2) = x_\star \), and \( \gamma'(2) \neq 0 \). It follows that \( \gamma \in \overline{\Gamma}^k_{x_\star} \) and \( \gamma \) contains the points \( x_{n_{m}}, y_{n_{r}}, \forall m, r \in N \). Hence, the point \( x_\star \) is not a local extrema point of \( f \).

**References**


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