On Finite Type Closed Curves on the Pseudo-Hyperbolic Space $H^3(-c^2)$

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Abstract

We obtain some nonexistence theorems of certain finite type closed curves on the pseudo-hyperbolic space $H^3(-c^2)$ in the Minkowski spacetime $E_1^4$.

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1 Introduction

First, we will survey briefly the fundamental concepts and properties in the pseudo-Riemannian geometry. We refer mainly to O’Neill([9]) and Chen([3],[4]). For the general concepts in the Riemannian geometry, refer to the book of Kobayashi and Nomizu([8]).

Let $M$ be a $C^\infty$-class differentiable manifold of dimension $n$ and $g$ a $C^\infty$-class differentiable symmetric nondegenerate tensor field of type $(0,2)$ on $M$. The pseudo-Riemannian metric $g_p$ at every point $p$ of $M$ defines the scalar product on the tangent space $T_p(M)$ of $M$ at $p$. The index of $g_p$ is not necessarily constant in general. If the index of $g_p$ is constant $t(0 \leq t \leq n)$ on $M$, then we call $g$ a pseudo-Riemannian metric of signature $(t,n-t)$. And a $C^\infty$-class differentiable manifold $(M,g)$ furnished with a pseudo-Riemannian metric $g$ is called a pseudo-Riemannian manifold. A pseudo-Riemannian manifold of signature $(0,n)$ means a Riemannian manifold. Let $v$ be a tangent vector to a pseudo-Riemannian manifold $M$ with a pseudo-Riemannian metric $g$. Then $v$ is said to be

- spacelike if $g(v,v) > 0$ or $v = 0$;
- lightlike if $g(v,v) = 0$ and $v \neq 0$;
- timelike if $g(v,v) < 0$.

The simplest example of pseudo-Riemannian manifold is a pseudo-Euclidean space.

Let $(x^1, x^2, \ldots, x^m)$ be a point in the set $R^m$ of all ordered $m$-tuples of real numbers. For each $t(0 \leq t \leq m)$, we define a scalar product $g_0$ on $T_p(R^m)$ at the point $p$ of $R^m$ by
\[ g_0(v_p, w_p) = -\sum_{i=1}^{t} v^i w^i + \sum_{i=t+1}^{m} v^i w^i, \]

where \( v_p = \sum_{i=1}^{m} v^i \partial / \partial x^i \) and \( w_p = \sum_{i=1}^{m} w^i \partial / \partial x^i \). \( E^m_t \) denotes a \( R^m \) with a canonical pseudo-Riemannian metric \( g_0 \). In this case, \( g_0 \) is called a \textit{pseudo-Euclidean metric} of signature \((t, m-t)\) and \( E^m_t \) is called a \textit{pseudo-Euclidean space} of signature \((t, m-t)\).

In particular, \( E^1_0 \) is called a \textit{Minkowski spacetime}.

From now on, we will use \( <,> \) instead of a pseudo-Euclidean metric \( g_0 \). And we denote by \( H^m_t(-c^2) = \{ p \in E^m_{t+1} | < p, p > = -c^2 \} \). In this case, it is called the \textit{pseudo-hyperbolic space} of radius \( c > 0 \) and center \( 0 \) in \( E^m_{t+1} \). For a vector \( a_0 = (a_1, a_2, \ldots, a_t, \ldots, a_m) \) in \( E^m_t \),

\[ \tilde{a}_0 = (-a_1, -a_2, \ldots, -a_t, a_{t+1}, a_{t+2}, \ldots, a_m) \]

is called the \textit{conjugate vector} of \( a_0 \). In [7] and [10], the authors proved the following

**Theorem A.** \textit{Only 1-type closed curve \( \gamma(s) \) on \( H^m_t(-c^2) \) is an intersection of \( H^m_t(-c^2) \) and a 2-plane \( P \) lying in \( \Pi_{a_0} \), where \( P \) is determined by two timelike vectors and \( \Pi_{a_0} \) denotes a hyperplane through \( a_0 \) which is orthogonal to the conjugate vector \( \tilde{a}_0 \) in the sense of Euclidean scalar product.}

Ishikawa([7]), and Shin and Pyo([11]) also proved some nonexistence theorems concerning finite type closed curves on pseudo-hyperbolic spaces \( H^2(-c^2) \) and \( H^4(-c^2) \).

For instance,

**Theorem B.** \textit{There exists neither 2-type closed curves nor 3-type closed curves on \( H^2(-c^2) \).}

**Remark.** Finite type curves in a Euclidean space were investigated in [1], [2], [5], [6] \textit{etc.}

The purpose of this article is to prove some theorems on nonexistence of certain finite type closed curves on the pseudo-hyperbolic space \( H^3(-c^2) \) in the Minkowski spacetime \( E^4_1 \).

2 Preliminaries

Every closed curve \( \gamma : [0, 2\pi r] \rightarrow E^m_t \) of the length \( 2\pi r \) in \( E^m_t \) may be regarded as an isometric immersion of a circle of radius \( r \) into \( E^m_t \). We use the arc length \( s \) as a parameter of \( \gamma \). Then the Laplacian \( \Delta \) on the circle is given by \( \Delta = -d^2 / ds^2 \) and the eigenvalues are \( \{ (l/r)^2, l = 1, 2, \ldots \} \). The corresponding eigenspace \( V_l \) is constructed by using \( \cos(ls/r) \) and \( \sin(ls/r) \). Hence, every closed curve \( \gamma : [0, 2\pi r] \rightarrow E^m_t \) has the spectral decomposition

\[ \gamma(s) = a_0 + \sum_{l=1}^{\infty} \{ a_l \cos(ls/r) + b_l \sin(ls/r) \}, \]

where \( a_l, b_l \) are some vectors in \( E^m_t \) (see [2],[5]). In particular, if \( \gamma \) is a \( k \)-type closed curve of the length \( 2\pi \) on \( H^m_t(-c^2) \), then \( \gamma \) can be expressed as
(2.1) \[ \gamma(s) = a_0 + \sum_{i=1}^{k} \left\{ a_i \cos(p_i s) + b_i \sin(p_i s) \right\}, \]

where \( a_i \) or \( b_i \) is nonzero vector in \( E^m_1 \) for each \( i = 1, 2, \cdots, k \), \( p_i \) are the positive integers with \( p_1 < p_2 < \cdots < p_k \) and \( s \) is the arc length parameter of \( \gamma \). Because of \( \gamma(s) \) being on \( H^m_1(-c^2) \) and \( a_0 \) the center of mass of \( \gamma \), \( a_0 \) is a timelike vector in \( E^{m+1}_1 \) (see [7]). Furthermore, from \( \langle \gamma(s), \gamma(s) \rangle = -c^2 \), we have the following

(2.2) \[ 2 < a_0, a_0 > + 2c^2 + \sum_{i=1}^{k} D_{ii} = 0, \]

(2.3) \[ \sum_{p_i=l} M_i + \sum_{2p_i=l} A_{ii} + 2 \sum_{p_i+p_j=m} A_{ij} + 2 \sum_{p_i-p_j=m} D_{ij} = 0, \]

(2.4) \[ \sum_{p_i=l} \tilde{M}_i + \sum_{2p_i=l} \tilde{A}_{ii} + 2 \sum_{p_i+p_j=m} \tilde{A}_{ij} - 2 \sum_{p_i-p_j=m} \tilde{D}_{ij} = 0, \]

for each \( l \in \{ p_i, 2p_i, p_i + p_j, p_i - p_j : 1 \leq j < i \leq k \} \), where

\[
\begin{align*}
M_i &= 4 < a_0, a_i >, & \tilde{M}_i &= 4 < a_0, b_i >, \\
A_{ij} &= < a_i, a_j >, & \tilde{A}_{ij} &= < a_i, b_j > + < b_i, a_j >, \\
D_{ij} &= < a_i, a_j > + < b_i, b_j >, & \tilde{D}_{ij} &= < a_i, b_j > - < b_i, a_j >.
\end{align*}
\]

From now on, we call the real numbers \( M_i \) and \( \tilde{M}_i \) (resp. \( A_{ii} \) and \( \tilde{A}_{ii} \), \( A_{ij} \) and \( \tilde{A}_{ij} \), or \( D_{ij} \) and \( \tilde{D}_{ij} \)) to be corresponding to the integer \( p_i \) (resp. \( 2p_i, p_i + p_j, \) or \( p_i - p_j \)). Since \( s \) is the arc length parameter of \( \gamma(s) \), we have

(2.5) \[ 2 = \sum_{i=1}^{k} p_i^2 D_{ii}, \]

(2.6) \[ \sum_{2p_i=l} p_i^2 A_{ii} + 2 \sum_{p_i+p_j=m} p_i p_j A_{ij} - 2 \sum_{p_i-p_j=m} p_i p_j D_{ij} = 0, \]

(2.7) \[ \sum_{2p_i=l} p_i^2 \tilde{A}_{ii} + 2 \sum_{p_i+p_j=m} p_i p_j \tilde{A}_{ij} + 2 \sum_{p_i-p_j=m} p_i p_j \tilde{D}_{ij} = 0. \]

Moreover, if \( \langle \gamma^{(r)}(s), \gamma^{(r)}(s) \rangle \) is constant \( (r = 1, 2, \cdots) \), then we have

(2.8) \[ \sum_{2p_i=l} p_i^{2r} A_{ii} + 2 \sum_{p_i+p_j=m} (p_i p_j)^r A_{ij} + (-1)^r 2 \sum_{p_i-p_j=m} (p_i p_j)^r D_{ij} = 0, \]
\[
(2.9) \quad \sum_{p_i > l} p_i^{2r} \bar{A}_{ii} + 2 \sum_{p_i + p_j - 1 > j} (p_i p_j)^r \bar{A}_{ij} - (-1)^r 2 \sum_{p_i - p_j - 1 > j} (p_i p_j)^r \bar{D}_{ij} = 0.
\]

Next, let \( \gamma \) be a \( k \)-type closed curve on \( H_t^m(-c^2) \) given in (2.1). Divide the set \( A = \{ \sqrt{1}, \sqrt{1} + \sqrt{A}, \sqrt{1} - \sqrt{A} : -\infty \leq \sqrt{1} \leq \infty \} \) as the union of the subsets as follows:

\[
(2.10) \quad A = A_\infty \cup A_\epsilon \cup \cdots \cup A_N,
\]

where all elements in each subset \( A_i (\sqrt{1} = \infty, \epsilon, \cdots, N) \) are equal to each other and if \( n_1 \neq n_2 \), then every element in \( A_i \) is not equal to any element in \( A_j \).

3 Main Results

Let \( \gamma \) be a closed \( k \)-type curve on \( H_t^m(-c^2) \) in \( E_t^{m+1} \). Then \( \gamma \) is expressed as \( \gamma(s) = a_0 + \sum_{i=1}^{k} \{ a_i \cos(p_i s / r) + b_i \sin(p_i s / r) \} \), where \( a_i \), \( b_i \) is nonzero vector in \( E_t^{m+1} \) for each \( i = 1, 2, \cdots, k \) and \( p_i \) are the positive integers satisfying \( p_1 < p_2 < \cdots < p_k \). Here \( s \) is the arc length parameter of \( \gamma \) and the length of \( \gamma \) is \( 2\pi r \). Therefore every \( k \)-type closed curve \( \gamma(s) \) of the length \( 2\pi r \) may be described as

\[
(3.1) \quad \gamma(s) = a_0 + \sum_{i=1}^{k} \{ a_i \cos(p_i s) + b_i \sin(p_i s) \},
\]

where \( a_i \neq 0 \) or \( b_i \neq 0 \) for each \( i \). We prove our results for \( r = 1 \), because the proof for case \( r \neq 1 \) is the same as one for case of \( r = 1 \).

**Lemma 3.1([7]).** (1) \( \gamma^{(r)}(s) \), \( \gamma^{(r)}(s) > 0 \) is constant \( (r = 1, 2, \cdots, l) \) and the number of members in \( A_i \) is less than \( 1 \) equal to \( l + 1 \), then \( M_i \) and \( \bar{M}_i \) (resp. \( A_{ii} \), \( A_{ij} \) and \( \bar{A}_{ij} \), or \( D_{ij} \) and \( \bar{D}_{ij} \)) of corresponding to the integer \( p_i \) (resp. \( 2p_i, p_i + p_j, \) or \( p_i - p_j \)) in \( A_i \) vanish.

(2) In particular, for every \( k \)-type closed curve \( \gamma(s) \) on \( H_t^m(-c^2) \) in \( E_t^{m+1} \), we have

\[
A_{kk} = \bar{A}_{kk} = 0, \\
A_{(k-1)(k-1)} = \bar{A}_{(k-1)(k-1)} = 0, \\
A_{(k-1)(k-1)} = \bar{A}_{(k-1)(k-1)} = 0.
\]

Now, let \( \gamma(s) \) be a \( k \)-type closed curve on \( H^2(-c^2) \) in a Minkowski spacetime \( E_1^4 \) as (3.1). Then we can obtain the following lemmas.

**Lemma 3.2.** If \( \gamma(s) \) satisfies the following conditions

\[
(3.2) \quad M_k = \bar{M}_k = M_{k-1} = 0 \text{ and } D_{k(k-1)} = \bar{D}_{k(k-1)} = 0,
\]

then either (1) \( \{ a_0, a_{k+1}, a_k, b_k \} \) forms a basis for \( E_1^4 \), or (2) \( b_{k-1} \) is a lightlike vector and \( \{ a_0, b_{k-1}, a_k, b_k \} \) is a basis of \( E_1^4 \).

**Proof.** Since \( a_0 \) is a timelike vector in \( E_1^4 \), from the first equation of (3.2), we know that \( a_{k+1}, a_k \) and \( b_k \) are spacelike vectors. Hence \( a_k \) and \( b_k \) are nonzero vectors because
\[ \langle a_k, a_k \rangle < \langle b_k, b_k \rangle \] and \( \gamma(s) \) is of \( k \)-type. From Lemma 3.1(2) and the second equation of (3.2), we have

\[ \begin{align*}
\langle a_{k-1}, a_{k-1} \rangle & = \langle b_{k-1}, b_{k-1} \rangle, \\
\langle a_k, b_k \rangle & = \langle a_{k-1}, b_{k-1} \rangle = 0, \\
\langle a_k, a_{k-1} \rangle & = \langle b_k, b_{k-1} \rangle = 0,
\end{align*} \]

and

\[ \langle a_k, b_{k-1} \rangle = \langle b_k, a_{k-1} \rangle = 0. \]

If \( a_{k-1} \neq 0 \), then \( a_0, a_{k-1}, a_k, b_k \) are linearly independent vectors in \( E_4^1 \) and hence \( \{a_0, a_{k-1}, a_k, b_k\} \) is a basis of \( E_4^1 \).

Suppose \( a_{k-1} = 0 \). Since \( \langle a_{k-1}, a_{k-1} \rangle = \langle b_{k-1}, b_{k-1} \rangle = 0 \) and \( \gamma(s) \) is of \( k \)-type, \( b_{k-1} \) is a lightlike vector. If we put \( Aa_0 + Bb_{k-1} + Ca_k + Db_k = 0 \), then we can obtain \( A = B = C = D = 0 \) because \( \langle a_0, b_{k-1} \rangle \neq 0 \)(see [11]). Therefore we complete the proof.

**Remark.** If the \( k \)-type closed curve \( \gamma \) on \( F^3(\mathbb{C}^2) \) satisfies \( M_{k-1} = 0 \) and \( b_{k-1} \) is a lightlike vector in \( E_4 \), then \( a_{k-1} = 0 \) because \( \langle a_{k-1}, a_{k-1} \rangle = \langle b_{k-1}, b_{k-1} \rangle = 0 \).

**Lemma 3.3.** Suppose that \( \{a_0, a_{k-1}, a_k, b_k\} \) is a basis of \( E_4^1 \) satisfying (3.2). Then \( b_{k-1} \) is a parallel nonzero vector to \( a_0 \).

**Proof.** Put \( b_{k-1} = Aa_0 + Ba_{k-1} + Ca_k + Db_k \). Combining Lemma 3.1(2) and (3.2), we have \( B = C = D = 0 \) because \( a_{k-1}, a_k \) and \( b_k \) are nonzero spacelike vectors. Since \( \langle a_0, b_{k-1} \rangle \neq 0 \), \( b_{k-1} = Aa_0 \neq 0 \) for a constant \( A \). Next, we can obtain the following

**Lemma 3.4.** Suppose that \( \{a_0, a_{k-1}, a_k, b_k\} \) is a basis of \( E_4^1 \) satisfying (3.2). If a pair \( \{a_i, b_i\} (i = 1, 2, \cdots, k-2) \) satisfies

\[ A_{ki} = \tilde{A}_{ki} = 0 \]

and

\[ \langle a_{k-1}, a_i \rangle = \langle a_{k-1}, b_i \rangle = 0, \]

then \( A_{ii} = \tilde{A}_{ii} = 0 \) if and only if \( M_i = \tilde{M}_i = 0 \).

**Proof.** Put \( a_i = Aa_0 + Ba_{k-1} + Ca_k + Db_k \) and \( b_i = Ea_0 + Fa_{k-1} + Ga_k + Hb_k \). Combining Lemma 3.1(2), (3.2) and our assumptions, we have \( B = C = D = 0 \) and \( D = -G \). Hence \( a_i = Aa_0 + Ca_k + Db_k \) and \( b_i = Ea_0 - Da_k + Cb_k \) for some constants \( A, C, D \) and \( E \).

Suppose \( A_{ii} = \tilde{A}_{ii} = 0 \). Then we get \( AE = 0 \) and \( A^2 - E^2 = 0 \) because \( \langle a_k, a_k \rangle = \langle b_k, b_k \rangle \) and \( \langle a_k, b_k \rangle = 0 \), and hence \( A = E = 0 \). Therefore \( M_i = 4 < a_0, a_i \rangle = 0 \) and \( M_i = 4 < a_0, b_i \rangle = 0 \).

Conversely, if \( M_i = \tilde{M}_i = 0 \), then we have \( a_i = Ca_k + Db_k \) and \( b_i = -Da_k + Cb_k \) for some constants \( C \) and \( D \). Hence \( \langle a_i, a_i \rangle = \langle b_i, b_i \rangle \) and \( \langle a_i, b_i \rangle = 0 \).

**Lemma 3.5.** Suppose that \( \{a_0, a_{k-1}, a_k, b_k\} \) is a basis of \( E_4^1 \) satisfying (3.2). If a pair \( \{a_i, b_i\} (i = 1, 2, \cdots, k-2) \) is satisfying

\[ A_{ki} = \tilde{A}_{ki} = 0, \quad D_{ki} = \tilde{D}_{ki} = 0 \]

and

\[ \langle a_{k-1}, a_i \rangle = \langle a_{k-1}, b_i \rangle = 0, \]

then \( a_i \) and \( b_i \) are parallel to \( a_0 \).
Proof. If we put \( a_i = Aa_0 + Bb_{k-1} + Ca_k + Db_k \) and \( b_i = Ea_0 + Fa_{k-1} + Ga_k + Hb_k \), then we have, from Lemma 3.1(2), (3.2) and our assumptions, \( B = C = D = 0 \), \( F = G = H = 0 \). It follows that \( a_i = Aa_0 \) and \( b_i = Ea_0 \) for some constants \( A \) and \( E \). Finally, we have the following lemma.

**Lemma 3.6.** Suppose that \( \{a_0, b_{k-1}, a_k, b_k\} \) is a basis of \( E^4 \) satisfying (3.2). If a pair \( \{a_i, b_i\} \) \((i = 1, 2, \cdots, k-2)\) is satisfying

\[
A_{ki} = \tilde{A}_{ki} = 0
\]

and

\[
< b_{k-1}, a_i > = < b_{k-1}, b_i > = 0,
\]

then \( A_{ii} = \tilde{A}_{ii} = 0 \).

**Proof.** If we put \( a_i = Aa_0 + Bb_{k-1} + Ca_k + Db_k \) and \( b_i = Ea_0 + Fa_{k-1} + Ga_k + Hb_k \), Combining Lemma 3.1(2), (3.2) and our assumptions, we have \( C = H \) and \( D = -G \). Since \( b_{k-1} \) is lightlike and \( < a_0, b_{k-1} > \neq 0 \), \( A = E = 0 \). Hence \( a_i = Bb_{k-1} + Ca_k + Db_k \) and \( b_i = Fb_{k-1} - Da_k + Cb_k \) for some constants \( B, C, D \) and \( F \). Therefore \( < a_i, a_i > = < b_i, b_i > \) and \( < a_i, b_i > = 0 \).

From now on, we prove the following nonexistence theorems for a \( k \)-type \((k \geq 2)\) closed curve \( \gamma(s) \) on \( H^3(-c^2) \).

**Theorem 3.1.** There exists no 2-type closed curve \( \gamma(s) \) on \( H^3(-c^2) \).

**Proof.** We assume the existence of the 2-type closed curve

\[
\gamma(s) = a_0 + a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) + b_2 \sin(p_2 s)
\]

on \( H^3(-c^2) \). From Lemma 3.1, we see

\[
M_1 = \tilde{M}_1 = 0, \quad M_2 = \tilde{M}_2 = 0.
\]

Hence \( a_1, b_1, a_2 \) and \( b_2 \) are spacelike vectors in \( E^4 \). Furthermore \( a_1, b_1, a_2 \) and \( b_2 \) are nonzero vectors because \( A_{11} = A_{22} = 0 \) and \( \gamma(s) \) is of 2-type. We also have

\[
\tilde{A}_{11} = \tilde{A}_{22} = 0, \quad A_{21} = \tilde{A}_{21} = 0, \quad D_{21} = \tilde{D}_{21} = 0.
\]

Therefore \( a_0, a_1, b_1, a_2, b_2 \) are linearly independent vectors in \( E^4 \). It contradicts.

**Theorem 3.2.** There exists no 3-type closed curve \( \gamma(s) \) on \( H^3(-c^2) \) satisfying \( M_2 = 0 \) and \( D_{32} = \tilde{D}_{32} = 0 \).

**Proof.** We assume the existence of the 3-type closed curve

\[
\gamma(s) = a_0 + a_1 \cos(p_1 s) + b_1 \sin(p_1 s) + a_2 \cos(p_2 s) + b_2 \sin(p_2 s) + a_3 \cos(p_3 s) + b_3 \sin(p_3 s)
\]

on \( H^3(-c^2) \) satisfying the assumptions \( M_2 = 0 \) and \( D_{32} = \tilde{D}_{32} = 0 \).

First, if we assume that \( a_2 \neq 0 \), then \( \{a_0, a_2, a_3, b_3\} \) is a basis of \( E^4 \) satisfying (3.2) by Lemmas 3.1 and 3.2.

**Case 1.** In case of \( \{p_1, p_2, p_3\} = \{p_1, 2p_1, 3p_1\} \), it follows that \( A = \{ \sqrt{\infty}, \sqrt{\varepsilon} - \sqrt{\infty}, \sqrt{\varepsilon} - \sqrt{\varepsilon} \} \cup \{ \sqrt{\varepsilon}, \sqrt{\varepsilon}, \sqrt{\varepsilon} - \sqrt{\infty} \} \cup \{ \sqrt{\infty} + \sqrt{\varepsilon}, \sqrt{\varepsilon} \} \cup \{ \sqrt{\varepsilon} + \sqrt{\varepsilon}, \sqrt{\varepsilon} \} \cup \{ \sqrt{\varepsilon} + \sqrt{\varepsilon}, \sqrt{\varepsilon} \} \cup \{ \sqrt{\varepsilon} \} \). Applying (2.3), (2.4), (2.6) and (2.7) for the subclasses \( \{p_1, p_2 - \)
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\( p_1, p_3 - p_2 \) and \( \{2p_1, p_2, p_3 - p_1\} \) of \( \mathcal{A} \), and combining Lemmas 3.1 and 3.4, we obtain

\[
A_{21} = \tilde{A}_{21} = 0, \quad D_{21} = \tilde{D}_{21} = 0,
\]

\[
A_{31} = A_{31} = 0, \quad M_1 = M_1 = 0,
\]

\[
A_{11} = A_{11} = 0, \quad D_{31} = \tilde{D}_{31} = 0
\]

by our assumptions. Furthermore, we have \( \tilde{M}_2 = 0 \). Hence \( b_2 \) is a spacelike vector in \( E^1_t \). It is a contradiction to Lemma 3.3.

**Case 2.** In case of \( \{p_1, p_2, p_3\} = \{p_1, 2p_1, 4p_1\} \), it follows that \( \mathcal{A} = \{\sqrt{\infty}, \sqrt{\epsilon}, -\sqrt{\infty}, \sqrt{\epsilon}, \sqrt{\alpha} - \sqrt{\epsilon}, \sqrt{\alpha} + \sqrt{\epsilon}, \sqrt{\beta} - \sqrt{\epsilon}, \sqrt{\beta} + \sqrt{\epsilon}, \sqrt{\infty} + \sqrt{\epsilon}, \sqrt{\infty} + \sqrt{\beta}, \sqrt{\infty} + \sqrt{\beta}, \sqrt{\epsilon} + \sqrt{\beta}, \sqrt{\epsilon} + \sqrt{\beta} \} \). Applying (2.3), (2.4), (2.6) and (2.7) for the subclass \( \{2p_1, p_2, p_3 - p_2\} \) of \( \mathcal{A} \), we get \( \tilde{M}_2 = 0 \) by the assumption \( D_{32} = \tilde{D}_{32} = 0 \). Hence Lemma 3.3 leads a contradiction.

**Case 3.** In case of \( \{p_1, p_2, p_3\} = \{p_1, 3p_1, 5p_1\} \), \( \mathcal{A} = \{\sqrt{\infty} \cup \sqrt{\epsilon}, \sqrt{\alpha} - \sqrt{\epsilon}, \sqrt{\alpha} + \sqrt{\epsilon}, \sqrt{\beta} - \sqrt{\epsilon}, \sqrt{\beta} + \sqrt{\epsilon}, \sqrt{\infty} + \sqrt{\epsilon}, \sqrt{\infty} + \sqrt{\beta}, \sqrt{\epsilon} + \sqrt{\beta}, \sqrt{\epsilon} + \sqrt{\beta} \} \). From Lemma 3.1(1), we obtain \( \tilde{M}_2 = 0 \). This is a contradiction.

**Case 4.** Let \( \{p_1, p_2, p_3\} \neq \{p_1, 2p_1, 4p_1\}, \{p_1, 3p_1, 5p_1\} \) or \( \{p_1, 2p_1, 3p_1, 4p_1\} \). In this case, each subset \( \mathcal{A}_n \) of \( \mathcal{A} \) consists of at most two elements. Hence, we have \( \tilde{M}_2 = 0 \) by Lemma 3.1(1). It contradicts.

Summarizing all cases, we complete the proof of this theorem in the case of \( a_2 \neq 0 \).

Now, let \( a_2 = 0 \). Then, by Lemmas 3.1(1) and 3.2, \( \{a_0, b_2, a_3, b_3\} \) forms a basis for \( E^1_t \) satisfying (3.2). In Case 1, applying (2.3), (2.4), (2.6) and (2.7) for the subclass \( \{p_1, p_2, p_3 - p_2\} \) of \( \mathcal{A} \), and combining the condition \( D_{32} = \tilde{D}_{32} = 0 \) and Lemma 3.1(1), we have

\[
A_{31} = \tilde{A}_{31} = 0, \quad A_{21} = \tilde{A}_{21} = 0, \quad D_{31} = \tilde{D}_{31} = 0
\]

Hence \( A_{11} = \tilde{A}_{11} = 0 \) by Lemma 3.6. Applying (2.3), (2.4), (2.6), (2.7) and the above equation for the subclass \( \{2p_1, p_2, p_3 - p_2\} \) of \( \mathcal{A} \), we get \( \tilde{M}_2 = 0 \). Since \( b_2 \) is a lightlike vector by Lemma 3.2, it contradicts.

The other cases are also impossible.

Therefore we complete the proof of this theorem.

For a 3-type closed curve \( \gamma(s) = a_0 + \sum_{i=1}^{3} \{a_i \cos(p_i s) + b_i \sin(p_i s)\} \) on \( H^3(-c^2) \), if \( a_2 = 0 \), then we have \( M_2 = 0 \) and \( \langle b_2, a_3 \rangle \leq \langle b_2, b_3 \rangle \geq 0 \) by Lemma 3.1(2). Hence \( D_{32} = \tilde{D}_{32} = 0 \). Therefore, from Theorem 3.2, we have the following corollary.

**Corollary 3.1.** There exists no 3-type closed curve

\[
\gamma(s) = a_0 + \sum_{i=1}^{3} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}
\]

on \( H^3(-c^2) \) satisfying \( a_2 = 0 \).

**Corollary 3.2.** There exists no 3-type closed curve with constant curvature on \( H^3(-c^2) \).

**Proof.** Let
be a 3-type closed curve with constant curvature on $H^3(-c^2)$. Then each subclass of $\mathcal{A}$ consists of at most three elements. From Lemma 3.1(1), we get

$$M_2 = \tilde{M}_2 = 0, \quad M_3 = \tilde{M}_3 = 0.$$ 

Hence $a_2, b_2, a_3$ and $b_3$ are spacelike vectors. Furthermore, they are nonzero vector because $A_{22} = A_{33} = 0$ and $\gamma(s)$ is of 3-type. Therefore $a_0, a_2, b_2, a_3, b_3$ are linearly independent vectors in $E_1^4$ by Lemma 3.1. This implies a contradiction.

Next, we get the following

**Theorem 3.3.** There exists no 4-type closed curve with constant curvature on $H^3(-c^2)$ satisfying $D_{43} = \tilde{D}_{43} = 0$.

**Proof.** Assume the existence of the 4-type closed curve

$$\gamma(s) = a_0 + \sum_{i=1}^{4} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}$$

satisfying our assumptions. If $\{p_1, p_2, p_3, p_4\} = \{p_1, 2p_1, 3p_1, 4p_1\}$, then $\mathcal{A} = \{\sqrt{e}, -\sqrt{e}, \sqrt{3}, -\sqrt{3}, \sqrt{\Delta}, -\sqrt{\Delta}\} \cup \{\sqrt{e}, \sqrt{e}, \sqrt{3}, -\sqrt{3}, \sqrt{\Delta}, -\sqrt{\Delta}\} \cup \{\sqrt{3}, \sqrt{3}, \sqrt{\Delta}, -\sqrt{\Delta}\} \cup \{\sqrt{\Delta}, \sqrt{\Delta}, \sqrt{3}, -\sqrt{3}\} \cup \{\sqrt{3}, \sqrt{\Delta}, \sqrt{\Delta}, \sqrt{3}\} \cup \{\sqrt{\Delta}, \sqrt{\Delta}, \sqrt{3}, \sqrt{3}\} \cup \{\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}\}$.

Let $A$ be the subclass consisting of all elements in $\mathcal{A}$ to be equal to $p_i$. Then the number of elements in $A_3$ (and $A_\Delta$) is less than or equal to three in this case. Hence, from Lemma 3.1, we have

$$M_3 = \tilde{M}_3 = 0, \quad M_4 = \tilde{M}_4 = 0, \quad A_{33} = A_{44} = 0, \quad A_{34} = A_{43} = 0.$$ 

Since $D_{43} = \tilde{D}_{43} = 0$, we get $a_0, a_3, b_3, a_4, b_4$ are linearly independent vectors in $E_1^4$ by the same way as the proof Corollary 3.2. It contradicts.

In case of $\{p_1, p_2, p_3, p_4\} \neq \{p_1, 2p_1, 3p_1, 4p_1\}$, we can also imply a contradiction by the same way.

From Theorem 3.3, we can obtain the following corollary.

**Corollary 3.3.** There exists no 4-type closed curve

$$\gamma(s) = a_0 + \sum_{i=1}^{4} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}$$
on $H^3(-c^2)$ satisfying $a_3 = 0$.

**Theorem 3.4.** There exists no 5-type closed curve $\gamma(s)$ on $H^3(-c^2)$ with $D_{54} = \tilde{D}_{54} = 0$ satisfying $\gamma^{(l)}(s) < \gamma^{(l)}(s)$ is constant ($l = 2, 3$).

**Proof.** Assume the existence of the 5-type closed curve

$$\gamma(s) = a_0 + \sum_{i=1}^{5} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}$$


satisfying our conditions. Let \( \{p_1, p_2, p_3, p_4, p_5\} = \{p_1, 2p_1, 3p_1, 4p_1, 5p_1\} \), it follows that \( \mathcal{A} = \{ \sqrt{\infty}, \sqrt{e} - \sqrt{\infty}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e} \} \cup \{ e, \infty, \sqrt{\alpha} \} \). Applying Lemma 3.1(1) for the subclasses \( \{p_1, 2p_2, p_3, p_4, p_5 - p_1\} \) and \( \{p_5, p_1 + p_4, p_2 + p_3\} \) of \( \mathcal{A} \), we obtain
\[
M_4 = \tilde{M}_4 = 0, \quad M_5 = \tilde{M}_5 = 0.
\]

Hence \( a_4, b_4, a_5 \) and \( b_5 \) are spacelike vectors in \( E_4^1 \). Furthermore, from Lemma 3.1(2), we have
\[
A_{44} = A_{55} = 0, \quad A_{54} = A_{54} = 0.
\]

Therefore \( a_0, a_4, b_4, a_5, b_5 \) are linearly independent vectors in \( E_4^1 \) because \( D_{54} = \tilde{D}_{54} = 0 \) and \( \gamma(s) \) is of 5-type. It contradicts.

By the same way, in case of \( \{p_1, p_2, p_3, p_4, p_5\} \neq \{p_1, 2p_1, 3p_1, 4p_1, 5p_1\} \), we can also imply a contradiction.

Finally, we get the following theorem.

**Theorem 3.5.** There exists no 6-type closed curve \( \gamma(s) \) on \( H^3(-c^2) \) with \( D_{65} = \tilde{D}_{65} = 0 \) satisfying \( \gamma^{(l)}(s), \gamma^{(l)}(s) > 0 \) is constant \( (l = 2, 3) \).

**Proof.** Assume the existence of the 6-type closed curve
\[
\gamma(s) = \alpha_0 + \sum_{i=1}^{6} \{a_i \cos(p_i s) + b_i \sin(p_i s)\}
\]
satisfying our conditions.

**Case 1.** Let \( \{p_1, p_2, p_3, p_4, p_5, p_6\} = \{p_1, 2p_1, 3p_1, 4p_1, 5p_1, 6p_1\} \), it follows that \( \mathcal{A} = \{ \sqrt{\infty}, \sqrt{e} - \sqrt{\infty}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e}, \sqrt{\alpha} - \sqrt{e} \} \cup \{ e, \infty, \sqrt{\alpha} \} \). Applying Lemma 3.1(1) for the subclasses \( \{p_5, p_1 + p_4, p_2 + p_3, p_6 - p_1\} \) and \( \{p_6, 2p_3, p_1 + p_3, p_2 + p_4\} \) of \( \mathcal{A} \), we obtain
\[
M_5 = \tilde{M}_5 = 0, \quad M_6 = \tilde{M}_6 = 0.
\]

And, from Lemma 3.1(2), we have
\[
A_{55} = A_{66} = 0, \quad A_{65} = A_{65} = 0.
\]

Hence \( a_5, b_5, a_6, b_6 \) are linearly independent vectors in \( E_4^1 \) by our assumptions. It contradicts.

In case of \( \{p_1, p_2, p_3, p_4, p_5, p_6\} \neq \{p_1, 2p_1, 3p_1, 4p_1, 5p_1, 6p_1\} \), we can also imply a contradiction.
References


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