Classification of Locally Symmetric Contact Metric Manifolds

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Abstract

We complete the classification of 5-dimensional locally symmetric contact metric manifolds stated by D. Blair and J.M. Sierra. Furthermore, in general dimension we prove the existence of a foliation with totally geodesic leaves locally isometric to a Riemannian product $E^{m+1} \times S^n(4)$.

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Introduction

In [6], Z. Olszak proved that for dimensions $2n + 1 \geq 5$ there are not contact metric manifolds of constant curvature unless the constant is 1 and in this case the structure is Sasakian. On the other hand, in [7], S. Tanno proved that a locally symmetric $K$-contact manifold is of constant curvature. Motivated by these results, D. Blair and J.M. Sierra proposed the question of classifying locally symmetric contact manifolds, and in [5] they studied the 5-dimensional case, proving the following theorem.

Theorem. Let $M$ be a complete 5-dimensional locally symmetric contact metric manifold. Then the simply-connected covering space is either $S^5(1)$ or $E^3 \times S^2(4)$ or $H^2(k_1) \times H^2(k_2) \times R$, where $H^2(k_i)$, $i = 1, 2$ is the hyperbolic plane with constant negative curvature $k_i$.

However, whereas $S^5(1)$ and $E^3 \times S^2(4)$ admit such a structure ([2], [3]), the problem of the existence in the third case remained still open. We recall also that the 3-dimensional case has been studied in [4] by Blair and Sharma who proved that a 3-dimensional locally symmetric contact metric manifold is of constant curvature $+1$ or 0.

In this paper we prove that the third possibility in the theorem of Blair and Sierra has to be removed. Moreover, in the general case, we prove that a locally symmetric contact metric manifold $M^{2n+1}$, $2n + 1 > 5$, admits a foliation whose leaves are totally geodesic and locally isometric to the Riemannian product $E^{m+1} \times S^n(4)$, for a suitable $m$. 

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1 Preliminaries

We recall some results on contact metric manifolds and for more details we refer to [1],[3],[5].

A contact metric manifold $M^{2n+1}$ is a $C^\infty$-manifold with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that there exists a unique vector field $\xi$ on $M^{2n+1}$ satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$. A manifold $M^{2n+1}$ is said to be a contact metric manifold if it admits a contact metric structure $(\varphi, \xi, \eta, g)$, where $\varphi$ is a tensor field of type $(1, 1)$ and $g$ is an associated metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \quad d\eta(X, Y) = g(X, \varphi(Y)).$$

Denoting by $L$ the Lie-derivation operator, the tensor field $h = \frac{1}{2}L_\xi \varphi$ is a symmetric operator which anticommutes with $\varphi$. Obviously, $h(\xi) = 0$ and if $\lambda$ is an eigenvalue of $h$ with eigenvector $X$, then $-\lambda$ is an eigenvalue with eigenvector $\varphi(X)$. Moreover, we have $h = 0$ if and only if $\xi$ is a Killing vector field and in this case $M^{2n+1}$ is called a $K$-manifold.

We have the following formulas, for any vector field $X$ on $M^{2n+1}$:

1. $$\nabla_X \xi = -\varphi(X) - \varphi h(X)$$

2. $$\frac{1}{2}(R_{\xi X} \xi - \varphi R_{\xi \varphi(X) \xi}) = h^2(X) + \varphi^2(X)$$

3. $$(\nabla_\xi h)(X) = \varphi(X) - h^2 \varphi(X) - \varphi R_{\xi X} \xi,$$

where $\nabla$ is the Levi-Civita connection and $R$ its curvature tensor field, [5]. Furthermore, in [2], the following theorem is proved.

**Theorem B.** Let $M^{2n+1}$ be a contact metric manifold and suppose that $R(X, Y) \xi = 0$ for all vector fields $X$ and $Y$. Then $M^{2n+1}$ is locally the product of a flat $(n + 1)$-dimensional manifold and an $n$-dimensional manifold of positive constant curvature 4.

Finally, supposing that $M^{2n+1}$ is a locally symmetric contact metric manifold we have $\nabla_\xi h = 0$, [3]. Consequently, (3) gives

4. $$R_{\xi X} \xi = -X + \eta(X) \xi + h^2(X)$$

and the following formulas hold for all orthogonal to $\xi$ unit eigenvectors $X, Y$ of $h$ with eigenvalues $\lambda, \mu$ respectively, ([5] lemma 3.3):

5. $$(\lambda^2 - \mu^2)g(\nabla_\varphi X, Y) = (1 - \lambda)[(1 - \lambda)g(\nabla X \varphi X, Y) - 2\lambda g(\nabla_X \varphi X + (1 + \mu)g(\nabla_X \varphi X, \varphi Y)]$$

and

6. $$(\varphi Y)(\lambda^2) = 2(X \mu)(1 - \mu)g(\varphi Y, X) + 2(1 - \mu^2)g(\nabla_X \varphi Y, X) + 2(1 - \lambda - \mu + \lambda \mu)g(\nabla_X \varphi Y, \varphi X) + 4\lambda(1 - \mu)g(\nabla_X \varphi Y, \varphi X)$$
2 The five-dimensional case

Let $M^5$ be a locally symmetric contact metric manifold. If the tensor field $h$ vanishes, then $M^5$ is a $K$-manifold of constant curvature $+1$ and it is realized by $S^5(1)$ with the standard Sasakian structure, [6], [7].

Now, suppose that $h \neq 0$. As discussed in section 4 of [5], for any $p \in M^5$ there exists a unit vector $X \in T_p(M^5)$ such that $g(X, \xi) = 0$ and $R_{X\xi} \xi = 0$. Using (4), we have

\begin{equation}
   h^2(X) - X = 0
\end{equation}

and since $h(\xi) = 0$, the spectrum of the operator $h$ is given by $\{0, \lambda, -\lambda, \mu, -\mu\}$. We suppose $\lambda \geq 0$, $\mu \geq 0$ and we denote by $\{\xi, e_1, e_2, e_3, e_4\}$ the set of the corresponding eigenvectors. Writing $X = \sum_{i=1}^4 X^i e_i$, and applying (7) we obtain that at least one of $\lambda$ or $\mu$ must be 1, say $\mu$. Moreover, $\phi(e_1) = e_2$, $\phi(e_3) = e_4$ and the eigenvalues are constant along their eigenvectors.

Blair and Sierra distinguished three cases:

i) $\lambda = 1$ ; 

ii) $\lambda = 0$ ; 

iii) $\lambda \neq 0, 1$.

In their proof the first case implies that $M^5$ is locally isometric to the Riemannian product $E^4 \times S^2(4)$ via theorem B, the second one leads to an empty class and the third one implies the local isometry of $M^5$ with $H^2(k_1) \times H^2(k_2) \times R$.

Now, we shall prove that the third possibility has to be excluded, obtaining the following classification theorem.

**Theorem 1** Let $M$ be a complete 5-dimensional locally symmetric contact metric manifold. Then the simply-connected covering space is either $S^5(1)$ or $E^5 \times S^2(4)$.

**Proof.** Let us suppose $\lambda \neq 0, 1$. In this hypothesis, Blair and Sierra proved the following results:

a) The distribution $ [+1] \bigoplus [-1] \bigoplus [\xi]$ is integrable with flat totally geodesic leaves. Here, $[+1]$ and $[-1]$ denote respectively the eigenspaces related to the eigenvalues $+1$ and $-1$ and $[\xi]$ is the distribution spanned by $\xi$.

b) The Levi-Civita connection satisfies the following relations:
\( \nabla \xi = 0 \quad \nabla e_4 = 0 \)

\( \nabla e_1 = -\beta'_1 e_3 \quad \nabla e_2 = -\gamma'_1 e_3 - \gamma_1 e_4 + (1 + \lambda) \xi \)

\( \nabla e_3 = \beta'_1 e_1 + \gamma'_1 e_2 \quad \nabla e_4 = \gamma_1 e_2 \)

\( \nabla e_1 \xi = (1 - \lambda) e_2 \quad \nabla e_2 \xi = -\beta'_2 e_3 - \beta_2 e_4 - (1 - \lambda) \xi \)

\( \nabla e_3 e_2 = -\gamma'_2 e_3 \quad \nabla e_3 e_3 = \beta'_2 e_1 + \gamma'_2 e_2 \)

\( \nabla e_2 e_4 = \beta_2 e_1 \quad \nabla e_3 \xi = (1 - \lambda) e_1 \)

\( \nabla e_3 e_1 = \alpha_3 e_2 \quad \nabla e_3 e_2 = -\alpha_3 e_1 \)

\( \nabla e_3 e_3 = 0 \quad \nabla e_3 e_4 = 2 \xi \)

where \( \beta_2 = \frac{1 - \lambda}{1 + \lambda} \gamma_1 \), \( \lambda \alpha_3 = -\gamma'_1 \), \( \xi(\alpha_3) = 0 \).

c) \( R_{e_1 e_2 \xi} = -((1 + \lambda) \gamma'_2 + (1 - \lambda) \beta'_1) e_3 \).

d) The eigenvalue \( \lambda \) must be a non constant function, and \( \xi(\lambda) = 0 \), \( e_4(\lambda) = 0 \).

First at all, we deduce some other formulas. Taking \( Y = e_4 \) and \( X = e_i, i = 1, 2 \) in (6) we get

\( -e_3 (\lambda^2) = 4(1 - \lambda) g(\nabla e_1, e_4, e_2) + 8 \lambda g(\nabla e_4, e_1, e_2) \)

Then, using b), we obtain \( -e_3 (\lambda^2) = 4(1 - \lambda)\gamma_1 \) and

\( e_3(\lambda) = -2 \frac{1 - \lambda}{\lambda} \gamma_1. \)

Now, condition d) implies \( \gamma_1 \neq 0 \) and applying the first Bianchi identity to \( e_1, e_3, \xi \) and using \( R_{e_3 \xi} = 0 \) we obtain:

\( -2 \gamma_1 + e_3 (\lambda) + (1 + \lambda)(\beta'_1 - \gamma'_2) = 0 \)

Again, using \( R_{e_3 \xi} = 0 \) and c) we find:

\( \gamma'_2 = -\frac{1 - \lambda}{1 + \lambda} \beta'_1 \)

and substituting (8) , (10) in (9) , we get

\( \beta'_1 = \frac{1}{\lambda} \gamma_1. \)

Finally, by direct computation, we have

\( g(R_{e_1 e_2 e_1, e_2}) = - (\gamma'_1)^2 - \frac{(1 - \lambda)^2}{\lambda^2} (\gamma_1)^2 + 1 - \lambda^2. \)
Now, we suppose that \( M^5 = H^2(k_1) \times H^2(k_2) \times R \) and recall that a) holds. Obviously, \( \xi \) has non zero component tangent to \( H^2(k_1) \times H^2(k_2) \), otherwise we have \( R_{XY} \xi = 0 \) for all \( X, Y \) and \( \lambda = 1 \). Moreover, since the foliation spanned by \( \{v_3, v_4, \xi\} \) induces foliations by geodesics on each \( H^2(k_i) \), we can consider \( (f_1, f_2) \) orthonormal vectors tangent to \( H^2(k_1) \), and \( (f_3, f_4) \) orthonormal vectors tangent to \( H^2(k_2) \) such that \( \{f_2, f_4, f_5\} \) span the distribution \( \{+1\} \oplus \{-1\} \oplus \{\xi\} \). It follows that \( e_1 \) and \( e_2 \) belong to the \( \text{span}\{f_1, f_3\} \) and, since the sectional curvature \( K(\{f_1, f_3\}) = 0 \), we have \( K(\{e_1, e_2\}) = 0 \) and (11) implies

\[
1 - \lambda^2 = (\gamma_1)^2 + \frac{(1 - \lambda)^2}{\lambda^2}(\gamma_1)^2 > 0.
\]

On the other hand, writing \( \xi = af_2 + bf_4 + cf_5 \) and using (4) we obtain \( R_{f_1} \xi = (1 - \lambda^2)f_1 \), whereas using the sectional curvature, we get \( R_{f_1} \xi = a^2k_1 \) so that

\[
1 - \lambda^2 = a^2k_1.
\]

We conclude that \( 1 - \lambda^2 < 0 \), contradicting (12).

3 Some results in the higher dimensional case

Let \( M^{2n+1} \) be a locally symmetric contact metric manifold and suppose that \( h \neq 0 \). Arguing as at the beginning of section 2, we consider the set

\[
\{0, +1, -1, \lambda_1, -\lambda_1, \ldots, \lambda_r, -\lambda_r\}
\]

of the distinct eigenvalues of \( h \) such that \( \text{dim}\{0\} = p + 1, \text{dim}\{+1\} = m, \text{dim}\{\lambda_1\} = m_1, \ldots, \text{dim}\{\lambda_r\} = m_r \) and \( 2n + 1 = p + 1 + 2m + 2m_1 + \ldots + 2m_r \).

Here \( [\lambda] \) denotes the eigenspace corresponding to the eigenvalue \( \lambda \).

**Theorem 2** Let \( M^{2n+1}, 2n + 1 > 5, \) be a locally symmetric contact metric manifold and suppose that the spectrum of \( h \) is given by the set \( \{0, +1, -1\} \) with \( +1 \) and \( -1 \) as eigenvalues of multiplicity \( n \) and 0 as simple eigenvalue. Then \( M^{2n+1} \) is locally isometric to the Riemannian product \( E^{2n+1} \times S^n(4) \).

**Proof.** By means of (4), we get \( R_{X} \xi = 0 \) for any eigenvector \( X \in \{\pm 1\} \). Consequently, the sectional curvatures of the tangent 2-planes containing \( \xi \) vanish.

If \( M^{2n+1} \) is irreducible, it is Einstein with \( \text{Ric}(\xi, \xi) = 2n - tr(h^2) = 0 \) and consequently it is Ricci-flat and then flat, contradicting the result of Olszak in [6]. Hence, \( M^{2n+1} \) is reducible and the vanishing of the \( \xi \)-curvatures implies that \( \xi \) has to be tangent to a flat factor. It follows that \( R_{XY} \xi = 0 \) for all tangent vectors \( X, Y \) and theorem B applies.

Now, we suppose \( m < n \), we put \( [0] = [\xi] \oplus V_0 \) (orthogonal sum), and \( H = [\xi] \oplus [\pm 1] \). To prove that the distribution \( H \) is integrable we need some lemma.

**Lemma 1.** For any \( X \in H \) we have \( [\xi, X] \in H \).

**Proof.** Clearly, for \( X \in H \) we have:

\[
X \in [+1] \Rightarrow (\nabla_X \xi = -2\varphi X \in [-1], \nabla_\xi X \in [+1])
\]

\[
X \in [-1] \Rightarrow (\nabla_X \xi = 0, \nabla_\xi X \in [-1])
\]
Finally, $\nabla_{\xi} \xi = 0$ and $[X, \xi] \in \{\pm 1\} \subset H$ follows.

**Lemma 2.** For any $X, Y$ belonging to $[+1]$ we have $\nabla_{\varphi Y} X \in [\pm 1] \subset H$.

**Proof.** We use the following formula stated as formula (5) in [3]

$$
R_{YX} \xi + R_{X} Y - R_{hX} Y \xi - R_{YX} h Y = g(X, Y) \xi - 2\eta(Y) X + \eta(X) Y
$$

(13)

$$
- g(X, h Y) \xi + 2\eta(Y) h Y
- \eta(X) h Y + (\nabla_{\varphi Y} h^2)(X).
$$

obtaining $(\nabla_{\varphi Y} h^2)(X) = 0$, i.e.,

(14) $\nabla_{\varphi Y} X - h^2 (\nabla_{\varphi Y} X) = 0$

and this implies $\nabla_{\varphi Y} X \in [\pm 1]$. Namely, we decompose $\nabla_{\varphi Y} X$ with respect to the direct sum of the eigenspaces:

(15) $\nabla_{\varphi Y} X = A_0 + A_{+1} + A_{-1} + A_{\lambda_1} + A_{\lambda_2} + \ldots + A_{\lambda_r} + A_{\lambda_{r+1}}$

Then, we have

$$
h^2 (\nabla_{\varphi Y} X) = A_0 + A_{+1} + \lambda_1^2 A_{\lambda_1} + \lambda_2^2 A_{\lambda_2} + \ldots + \lambda_r^2 A_{\lambda_r} + \lambda_{r+1}^2 A_{\lambda_{r+1}}.
$$

Using (14) and (15), we get $A_0 = 0, A_{\lambda_1} = 0, A_{\lambda_2} = 0, \ldots, A_{\lambda_r} = 0, A_{\lambda_{r+1}} = 0$, $\lambda_1, \ldots, \lambda_r$ being different from $+1, -1$. Finally, from (15) we obtain $\nabla_{\varphi Y} X = A_{+1} + A_{-1} \in [\pm 1] \subset H$.

**Corollary 1.** For any $X \in [-1]$ and $Y \in [+1]$ we have $\nabla_X Y \in [\pm 1]$.

**Proof.** Apply Lemma 2 to $\varphi X$ and $Y$.

**Lemma 3.** For any $Y \in [+1]$ and $X \in [-1]$, we have $\nabla_{\varphi Y} X \in [\pm 1]$.

**Proof.** From (13), since $g(X, Y) = 0$, we obtain $(\nabla_{\varphi Y} h^2)(X) = 0$ and we continue as in the proof of Lemma 2.

**Corollary 2.** We have: a) $(X \in [-1], Y \in [-1]) \Rightarrow \nabla_X Y \in [\pm 1]$

b) $X, Y \in [-1] \Rightarrow [X, Y] \in [\pm 1]$

**Lemma 4.** For any $X \in [-1]$ and $Y \in [-1]$, we have $\nabla_{\varphi Y} X \in H$.

**Proof.** Using (13) we have:

$$
2R_{XY} \xi + 2R_{\xi X} Y = 2g(X, Y) \xi + (\nabla_{\varphi Y} h^2)(X).
$$

Lemma 1 and Corollary 2 easily imply that $R_{YX} \xi \in [\pm 1]$ and $R_{\xi X} Y \in [\pm 1]$.

It follows

(16) $B = 2g(X, Y) \xi + \nabla_{\varphi Y} X - h^2 (\nabla_{\varphi Y} X) \in [\pm 1]$

On the other hand, decomposing $\nabla_{\varphi Y} X$ as in (15) and computing $h^2 (\nabla_{\varphi Y} X)$, we get
\begin{equation}
B = 2g(X, Y)\xi + A_0 + (1 - \lambda_1^2)A_{\lambda_1} + (1 - \lambda_r^2)A_{\lambda_r} + (1 - \lambda_{r+1}^2)A_{\lambda_{r+1}} + \ldots
\end{equation}

Comparing (16) and (17) we conclude

\[ A_0 = -2g(X, Y)\xi, A_{\lambda_1} = 0, A_{\lambda_1} = 0, \ldots, A_{\lambda_r} = 0, A_{\lambda_r} = 0 \]

so that

\[ \nabla_{\psi} X = -2g(X, Y)\xi + A_{\lambda_1} + A_{\lambda_1} \in H. \]

**Corollary 3.** \( (X \in [+1], Y \in [-1]) \Rightarrow (\nabla_X Y \in H, [X, Y] \in H). \)

**Lemma 5.** For any \( Y \in [-1] \) and \( X \in [+1] \) we have \( \nabla_{\psi} Y X \in [\pm 1] \).

**Proof.** Using (13), since \( g(X, Y) = 0 \), we get

\[ 2R_{XY} \xi + 2R_{\xi X} Y = (\nabla_{\psi} Y h^2)(X) \]

Now, Lemma 1 and the previous corollaries easily imply that \( R_{XY} \xi \in [\pm 1] \) and \( R_{\xi X} Y \in H \), so that

\begin{equation}
\nabla_{\psi} Y X = h^2(\nabla_{\psi} Y X) \in H.
\end{equation}

Again, decomposing \( \nabla_{\psi} Y X \) with respect to the direct sum of eigenspaces, (18) implies \( A_0 = a_{\xi}, A_{\lambda_1} = 0, \ldots, A_{\lambda_r} = 0 \), so that we have

\[ \nabla_{\psi} Y X = a_{\xi} + A_{\lambda_1} + A_{\lambda_1} \]

Now, since \( \psi Y \in [+1] \), we get \( g(\nabla_{\psi} Y X, \xi) = -g(X, \nabla_{\psi} Y \xi) = -2g(x, \psi^2 Y) = 2g(X, Y) = 0 \) and \( \nabla_{\psi} Y X \in [\pm 1] \).

**Corollary 4.** \( (X \in [+1], Y \in [+1]) \Rightarrow (\nabla_X Y \in [\pm 1], [X, Y] \in [\pm 1]. \)

**Proposition 4.1.** The distribution \( H = [\xi] \oplus [\pm 1] \) is integrable with totally geodesic leaves.

**Proof.** The previous lemma and corollaries imply that \( [X, Y] \in H \) for any \( X \in H \) and \( Y \in H \). Thus the distribution \( H \) is involutive and integrable.

Let \( N \) be a maximal integral submanifold. Since \( \nabla_X Y \) is tangent to \( N \) for any vector fields \( X, Y \) tangent to \( N \), the second fundamental form vanishes and \( N \) is totally geodesic.

**Proposition 4.2.** The integral manifolds of the distribution \( H \) are locally isometric to the Riemannian product \( F^{m+1} \times S^n \).

**Proof.** Let \( N \) be an integral manifold of \( H \), a local frame for \( TN \) is given by \( \xi \) and the eigenvectors \( \{ e_i, \psi e_i \}, \quad i \in \{ 1, \ldots, m \} \) corresponding to the eigenvalues \(+1, -1\), and \( N \) has a canonically induced contact metric structure \( (\xi, \psi, g) \) where \( \psi \) is the restriction of \( \varphi \) to \( N \). Moreover, \( N \) turns out to be locally symmetric since it is totally geodesic in the locally symmetric manifold \( M^{2m+1} \). It is easy to verify that \( h' = \frac{1}{2} L_\xi \varphi' \) is the restriction of \( h \) to \( N \). Now, \( h' \) has eigenvalues \(+1, -1\) with multiplicity \( m \) and \( 0 \).
as a simple eigenvalue. Theorem 2 insures that $N$ is locally isometric to $E^{m+1} \times S^m(4)$. Hence, we can conclude with the following theorem:

**Theorem 3** Let $M^{2n+1}$ be a locally symmetric contact metric manifold. Then $M^{2n+1}$ admits a foliation whose leaves are totally geodesic and locally isometric to the Riemannian product $E^{m+1} \times S^m(4)$. The integer $m$ is the multiplicity of the eigenvalue $+1$ of the operator $\frac{1}{2}L_{\xi}\varphi$.

**References**


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