Geometrical Objects on Subbundles

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Abstract
The reductions to a vector subbundle of a pull back vector bundle are studied.
They are related to the Finsler splittings (defined earlier by one of the authors) and to geometrical objects, defined here.

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1 Reductions of vector bundles

Let \( \xi = (E, \pi, M) \) be a vector bundle (denoted in the sequel v.b.), with the fibre \( F \cong \mathbb{R}^n \), \( G \subset GL_n(\mathbb{R}) \) a Lie subgroup and \( \xi' = (E', \pi', M) \) be a vector subbundle (denoted in the sequel v.s.b.), with the fibre \( F' \cong \mathbb{R}^k \subset \mathbb{R}^n \). Let us denote as \( L(\xi) = (L(E), p, M) \) the principal bundles (p.b.s) of the frames of the v.b. \( \xi \) and the induced p.b. \( L\xi(\xi) = \pi'^*L(\xi) = (\pi'^*L(E) = LE'(E), p_1, E') \), which is also the p.b.s of the frames of the v.b. \( \xi'(\xi) = \pi'^*\xi = (\pi'^* (E) = E'(E), \pi_1, E') \).

Definition 1.1 If the p.b. \( L(\xi)_{G} \) is a reduction of the group \( GL_n(\mathbb{R}) \) of \( L(\xi) \) to \( G \), then there is a local trivial bundle \( \xi_G \), associated with the p.b. \( L(\xi)_{G} \), defined by the left action of \( G \) on \( F \) (it is used the left action of \( GL_n(\mathbb{R}) \) on \( F \) restricted to \( G \)). We say that the bundle \( \xi_G \) is the \( G \)-reduced bundle of \( \xi \). If \( H \) is a subgroup of \( G \) and there is a reduction of the group \( G \) of \( L(\xi)_G \) to \( H \), in an analogous way we say that \( \xi_H \) is a \( H \)-reduced bundle of \( \xi_G \).

Notice that a reduction of the group \( G \) of \( L(\xi)_G \) to \( H \) is also a reduction of the group \( GL_n(\mathbb{R}) \) of \( L(\xi) \) to \( H \).

Example 1.1 Consider the subgroup of the automorphisms which invariate the vector subspace \( F' \cong \mathbb{R}^k \) of \( \mathbb{R}^n \):

\[
1) G_{G} = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : A \in GL_k(\mathbb{R}), B \in GL_n \ (\mathbb{R}), C \in M_{k,n} \ (\mathbb{R}) \right\} \subset GL_n(\mathbb{R}).
\]

The p.b. \( L(\xi)_{G_{G}} \) always exists and it consists of all the frames of \( L(\xi) \) which extend frames on \( \xi' \); we call in the sequel these frames as frames on \( \xi \), adapted to \( \xi' \). For the same \( G_{G} \) as above, we can consider the p.b. \( L(\xi)'_{G_{G}} \), which also consists of frames on \( L(\xi)' \) which extend frames on \( \xi'(\xi) \), called as frames on \( \xi'(\xi) \), adapted to \( \xi'(\xi) \).

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Example 1.2 Let $G_0$ be as above, $F'' \cong \mathbb{R}^n$ a vector subspace of $F$, so that $F = F' \oplus F''$ and $H_0 \subset G_0$ the subgroup of the elements which invariate the vector subspaces $F'$ and $F''$:

$$H_0 = \left\{ \left( \begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) ; A \in GL_k(\mathbb{R}), B \in GL_n(\mathbb{R}) \right\}. \tag{2}$$

The p.b. $L(\xi)_{H_0}$ always exists and it is a reduction of the group $G_0$ of $L(\xi)_{G_0}$ to $H_0$. It consists of frames of $L(\xi)_{G_0}$ which are adapted also to other subbundle $\xi'' = (E''', \pi'', M)$ of $\xi$. It follows that at every point $x \in M$ we have the direct sum of vector spaces $E_x = E'_x \oplus E''_x$. Such a reduction is also called a Whitney sum of the v.b.s $\xi$ and $\xi''$ and it is denoted as $\xi \oplus \xi''$. This is equivalent with a left splitting $S$ of the inclusion morphism $i : \xi \to \xi$, when $\xi'' = \ker S$. In the case of the p.b. $L\xi(\xi)$, a reduction of the group $G_0$ of $L\xi(\xi)_{G_0}$ to $H_0$ is equivalent to a left splitting $S$ of the inclusion $i' = \pi^* i : \pi^* \xi = \xi''(\xi'') \to \pi^* \xi = \xi''(\xi)$, as we have called Finsler splitting (see. [4]). In this case $\xi''(\xi)$ has an $H_0$-reduction as Whitney sum $\xi'(\xi') \oplus \ker S$.

It is well known that the reduction of the structural group $G$ of a p.b. $P$ to a subgroup $H \subset G$ is equivalent to the existence of a global section in a fibre bundle associated with $P$, which have the fibre $G/H$, defined by the natural action of $G$ on $G/H$ [1, pg.57, Propzition 5.6]. A direct computation leads to the following:

Proposition 1.1 There is a canonical identification

$$G_0/H_0 \cong M_k = \left\{ \left( \begin{array}{cc} 0 & P \\ 0 & I_{n \times k} \end{array} \right) ; P \in M_{k,n}(\mathbb{R}) \right\}, \tag{3}$$

the classes being at left, such that the left action $\circ$ of the group $G_0$ on $M_k$ is the adjunction, and

$$\left( \begin{array}{cc} E & G \\ 0 & F \end{array} \right) \circ \left( \begin{array}{cc} 0 & P \\ 0 & I_{n \times k} \end{array} \right) = \left( \begin{array}{cc} 0 & (E \cdot P + G) \cdot F \\ 0 & I_{n \times k} \end{array} \right). \tag{4}$$

Given the v.s.b. $\xi'$, the $G_0$-reductions of the group $GL_n(\mathbb{R})$ of the v.b.s $L(\xi)$ and the p.b. $L\xi(\xi)$ are uniquely defined. It follows that considering the bundles with the fibres $GL_n(\mathbb{R})/G_0$, associated with the p.b.s of frames $L(\xi)$ and $L\xi(\xi)$, the sections in these bundles, which correspond to the reductions of $GL_n(\mathbb{R})$ to $G_0$, are uniquely determined by $\xi'$. In the case of the Example 1.2, the $H_0$-reductions of the group $G_0$ of the p.b.s $L(\xi)_{G_0}$ and $L\xi(\xi)_{G_0}$ are equivalent with sections in the bundles $F_1$ and $F_2$ which are associated with these p.b.s and have as fibres $G_0/H_0$.

2 Reductions of the group $G^r_{m,n}$

We use in this section some ideas from [2, Cap. IV, Sectiunea 7], but in a more general setting.

Let $\xi = (E, p, M)$ be a v.b., having the fibre $F \cong \mathbb{R}^n$, dim $M = m$ and $G \subset GL_n(\mathbb{R})$ a Lie subgroup. We suppose that the structural group $GL_n(\mathbb{R})$ of $L(\xi)$ is reducible to $G$. Denote as $G^1_{m,n}$ the subgroup of $GL(m + n, \mathbb{R})$ which consists of the
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matrices of the form \( \begin{pmatrix} A^i_j & 0 \\ 0 & B^a_b \end{pmatrix} \), \((A^i_j) \in GL_m(\mathbb{R}), (B^a_b) \in G.\) Given \( r \in \mathbb{N^*} \), the \( r \)-prolongation of the group \( G^r_{m,n} \), denoted as \( G^r_{n,m} \), is the set of the elements which have the form

\[
a = (A^i_j, A^i_{j_1, j_2}, \ldots, A^i_{j_1, j_2, \ldots, j_p}; B^a_b, B^a_{b_1, b_2}, \ldots, B^a_{b_1, b_2, \ldots, b_p}),
\]

where the components are symmetric in the indices \( j_b \in \sum_i m \), and \((A^i_j) \in GL_m(\mathbb{R})\), \((B^a_b) \in G\) and if we fix \( j_1, j_2, \ldots, j_p \), then \( B^a_{b_1, \ldots, b_p} \in g \), where \( g \) is the Lie algebra of the Lie group \( G \). The composition law of two elements of the form (5) can be done looking at the components as multi-linear maps. Thus, let \( a : (A, B) = ((A_1(\cdot), A_2(\cdot), \ldots, A_r(\cdot, \ldots, \cdot)), B_1(\cdot), B_1(\cdot, \ldots, \cdot), B_r(\cdot, \ldots, \cdot)) \) and \( b : (C, D) \) be two elements in \( G^r_{n,m} \). We denote \( a \circ b = (A', B') \), where the expression of this composition law, using coordinates, can be found in [2, pag. 70]. Notice that if \( H \subset G \) is a subgroup, then \( H^r_{m,n} \subset G^r_{n,m} \) is a subgroup.

Consider the p.b. \( OG^r \varepsilon^\xi \) on the base \( E \), with the group \( G^r_{n,m} \), defined by the structural functions

\[
\varphi_{UV}(u) = (\frac{\partial x'}{\partial x}(x), \ldots, \frac{\partial x'}{\partial x_{r-1}}(x), g^a_a(x), \ldots, \frac{\partial^r \varphi}{\partial x_{r-1} \partial x_{r-2}}(x)),
\]

where \( \pi(u) = x \), and \( \{g^a_a(x)\} \) are structural functions of the vertical bundle \( V\xi \), constant on the fibres, defined on an open cover of \( E \) of the form \( \{U = \pi^{-1}(V), V \subset M, V \text{ open}\} \). These structural functions proceed from some structural function on \( M \), so the definition is coherent and is equivalent to that used in [2]. We shall use the condition in this form in order to construct some reductions. In the case \( G = GL_n(\mathbb{R}) \) we get the definitions used in [2].

Let \( \xi' \) be a v.s.b. of the v.b. \( \xi \). We denote \( OG\xi' (\xi')^r = i^*OG\xi'^r \), where \( i : E' \to E \) is the inclusion. From now to the end of the section we study the reductions of the group \( G^r_{m,n} \) of the p.b. \( OG\xi' (\xi')^r \) to the subgroup \( H^r_{m,n} \), where \( G_0 \) and \( H_0 \) are given by the formulas (1) and (2). We consider first the case \( r = 1 \).

**Proposition 2.1** Let \( \xi' \) be a v.s.b. of the v.b. \( \xi \). Then a Finsler splitting of the inclusion \( i : \xi' \to \xi \) induces a \( H_0 \)-reduction of the v.b. \( \xi(\xi') \) to a bundle \( \xi(\xi') \oplus \eta \), where \( \eta = \ker S \) is isomorphic with \( \xi'(\xi'^r) \), \( \xi'^r = \xi/\xi' \), such that the bundle \( \xi(\xi') \oplus \eta \) is isomorphic to the bundle \( \xi(\xi' \oplus \xi'^r) \). Conversely, every \( H_0 \)-reduction of \( \xi(\xi') \) as \( \xi(\xi') \oplus \eta \) defines a Finsler splitting \( S \) of the inclusion \( i \), such that \( \eta = \ker S \).

**Proof.** The second statement follows from Example 1.2. In order to prove the first statement, it suffices to prove that \( \eta \) is isomorphic with \( \xi'(\xi'^r) \). Considering local coordinates, adapted to the v.b.s structures: on \( M, E', E'' \) and \( E \), it can be shown that \( \eta \) is isomorphic with \( \xi'(\xi'^r) \). The same reason shows that \( \xi'(\xi') \oplus \eta \) is isomorphic with \( \xi'(\xi') \oplus \xi'(\xi'^r) \). Q.e.d.

**Theorem 2.1** Let \( \xi' \) be a v.s.b. of the v.b. \( \xi \) and \( r \geq 1 \)

1) Every Finsler splitting of the inclusion \( i : \xi' \to \xi \) defines a canonical reduction of the group \( G^r_{m,n} \) of \( OG\xi' (\xi'^r) \) to \( H^r_{m,n} \), the reduced p.b. being \( OH\xi' (\xi'^r) \).

2) Every reduction of the group \( G^r_{m,n} \) of \( OG\xi' (\xi'^r) \) to \( H^r_{m,n} \) is \( OH\xi' (\xi'^r) \) and it is induced by a Finsler splitting, as above.
Proof. 1) Taking on $E(E)$ a vectorial atlas, which has the structural functions from $H_0$, we obtain structural functions on the p.b. $OG_0 \xi'(\xi)'$, which take values in the subgroup $H_{0,m,n}^r$. 2) Considering a reduction as in hytosis and some structural functions on $OH_0 \xi'(\xi)'$, as in the definition, it follows some structural functions on the p.b.s $OH_0 \xi'(\xi)'$, with $1 \leq t' \leq r$. Taking $r' = 1$ and using the second part and the proof of Proposition 2.1, we obtain a Finsler splitting $S$. Q.e.d.

3 Geometrical objects

Let $\xi$ be a v.b., $G \subset GL_n(\mathbb{R})$ a Lie subgroup and $\xi'$ a v.s.b. of $\xi$.

Definition 3.1 A space of geometrical $G$-objects of order $r$ is a manifold $\Theta$ so that there is a left action of the group $G_{m,n}^r$ on $\Theta$. Consider now the fibre bundle with the fibre $\Theta$, associated with the p.b. $OG\xi'$, which correspond to this action. A section in this bundle is a field of geometrical $G$-objects on the v.b. $\xi$.

In an analogous way, we can consider the fibre bundle with the fibre $\Theta$, associated with the principal bundle $OG\xi'$. A section in this bundle is a field of geometrical $G$-objects on the v.b. $\xi$, restricted to the v.s.b. $\xi'$.

In the case $G = GL_n(\mathbb{R})$ we obtain the definitions used in [2] of a space of geometrical objects of order $r$ and of a field of geometrical objects on a v.b.

Example 3.1 Take $\Theta = \mathbb{R}^k$, $G_0$ given by the formula (1) and the left action of $G_0$ on $\Theta$ given by $\left( \begin{array}{cc} A & C \\ 0 & B \end{array} \right) v = Av$. It is obvious that this action induces a left action of the group $G_{0,m,n}^r$ on $\Theta$. It follows a field of geometrical $G_0$-objects of order 1 on the v.b. $\xi$, which is in fact a section in the v.b. $\xi(\xi')$ and a field of geometrical $G_0$-objects of order 1 on the v.b. $\xi$, restricted to the v.s.b. $\xi'$, which is in fact a section in the v.b. $\xi(\xi')$.

The second part of the example above, can be extended as follows:

Proposition 3.1 Every field of geometrical objects of order $r \geq 1$ on the v.b. $\xi'$ defines canonically a field of geometrical $G_0$-objects of order $r$ on the v.b. $\xi$, restricted to the v.s.b. $\xi'$.

Example 3.2 A d-connection on the v.b. $\xi'$ induces a field of geometrical $G_0$-objects of order 2 on the v.b. $\xi$, restricted to the v.s.b. $\xi'$.

Notice that a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.s.b. $\xi'$ is also such $H_0$-objects. So, the fields of geometrical $G_0$-objects from Proposition 3.1 and from Example 3.2 are also $H_0$-objects. A remarkable example of a field of geometrical $G_0$-objects of order $k$ on the v.b. $\xi$, restricted to the v.s.b. $\xi'$, is given by the following direct consequence of the result [1, pg 57, Proposition 5.6], already stated and used in the first section:

Proposition 3.2 Let $G_0$ and $H_0$ be given by (1) and (2), and $\xi'$ be a v.s.b. of the v.b. $\xi$. Then every reduction of the group $H_{0,m,n}^r$ of the p.b. $OG_0 \xi'(\xi)'$ to $H_{0,m,n}^r$ is uniquely defined by a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.s.b. $\xi'$, of order $r$. 

Example 3.3 Every Finsler splitting $S$ of the inclusion $\iota : \xi \to \xi$ is defined by a field of geometrical $G_0$-objects on the v.b. $\xi$, restricted to the v.s.b. $\xi'$, of order 1. According he above Proposition 3.2, this $G_0$-object is a section $S_0$ in the bundle $\mathcal{F}_0$, which is associated with the p.h. $OG_0(\xi)(\xi)'$ and defined by the left action of $G^{1}_{\text{om},n}$ on $G_0/H_0$ (in the form (3), determined by the left action of $G_0$ on $G_0/H_0$ given by formula 4. This formula can be related to the change rule of the local components of a Finsler splitting, which give also the change rule of the local form of the section $S_0$.

The action (4) can be extended in the general case, but this will be done elsewhere. An example of a field of geometrical $H_0$-objects on the v.b. $\xi$, restricted to the v.s.b. $\xi'$, of order 2, is given by the following action of $H^{2}_{\text{om},n}$ on $F_0 = \mathbb{R}^d$, where $d = m^2 + mk^2 + m(n-k)^2 + m^2n + k^2n + (n-k)^2n$. Writting $F_0$ as $(L_j^i, L_{\beta j}^i, L_{\alpha kj}, C_{\gamma j}, C_{\beta j}, C_{\beta j}, C_{\gamma j}, C_{\beta j}, C_{\beta j})$, we define the action of $H^{2}_{\text{om},n}$ on $F_0$, by an element $(\begin{pmatrix} A^j_{\alpha} & A^j_{\beta j} & (B^j_{\alpha} & 0) \\ 0 & B^j_{w} & (B^j_{\alpha} & 0) \end{pmatrix})$, as

$L^{j}_{\beta j'} = (A^j_{\alpha} L_{\beta j} - A^j_{\beta j} L_{\alpha}) A^i_{j} A^i_{j'}, L^{j}_{\beta j'} = (A^j_{\alpha} L_{\beta j} - A^j_{\beta j} L_{\alpha}) B^j_{\beta j} A^i_{j'},$

$L^{k}_{\alpha i'} = (B^j_{\alpha} L_{\alpha j} - B^j_{\alpha j}) B^j_{\alpha} A^i_{j} A^i_{j'}, C^j_{\gamma j} = A^j_{\alpha} A^j_{\beta j} B^j_{\gamma j} C_{\gamma j},$

$C^j_{\alpha i'} = A^j_{\alpha} A^j_{\beta j} B^j_{\alpha} C_{\gamma j}, C^j_{\beta j} = B^j_{\alpha} B^j_{\beta j} B^j_{\alpha} C_{\gamma j}, C^j_{\gamma j} = B^j_{\alpha} B^j_{\beta j} B^j_{\alpha} C_{\gamma j}, C^j_{\alpha i'} = B^j_{\alpha} B^j_{\alpha} C_{\gamma j}, C^j_{\beta j} = B^j_{\alpha} B^j_{\alpha} C_{\gamma j}, C^j_{\gamma j} = B^j_{\alpha} B^j_{\alpha} C_{\gamma j}.$

It follows that there is a local trivial bundle $\mathcal{F}_1$ with the fibre $F_1$, associated with the p.b. $OH_0(\xi)(\xi)'$.

Let $S$ be the Finsler splitting which correspond to the reduction of the group $G_0$ of $OG_0(\xi)(\xi)'$ to $H_0$, according to Theorem 2.1.

Definition 3.2 A restricted $d$-connection on $\xi$ (related to the v.s.b. $\xi'$ and the Finsler splitting $S$) is a section in the above bundle $\mathcal{F}_1$.

If follows that a restricted $d$-connection is uniquely determined by the local functions on $E'(E)$

$$(7) \quad (L^i_{\beta j}, L_{\alpha kj}, C_{\gamma j}, C_{\beta j}, C_{\beta j}, C_{\gamma j}, C_{\beta j}, C_{\beta j})$$

which have as variables $(x^i, y^a)$. They are given on domains of local maps on $E'$, which belong to a vectorial atlas on $E'$, which proceed from one on $E$. The coordinates on the fibres change following the rules $y^{a'} = g_{a'}^{a} (x^i) y^a$, $y^{a'} = g_{a'}^{a} (x^i) y^a$. The local functions (7) change according the rules

$L^{j}_{\beta j'} = \frac{\partial y^j}{\partial x^i} \bigg( \frac{\partial y^j}{\partial x^i} \frac{\partial y^j}{\partial x^i} L^i_{\beta j} + \frac{\partial y^j}{\partial x^i} \frac{\partial y^j}{\partial x^i} \bigg), L^{j}_{\beta j'} = g_{a'}^{a} \bigg( \frac{\partial y^j}{\partial x^i} \frac{\partial y^j}{\partial x^i} L^i_{\beta j} + \frac{\partial y^j}{\partial x^i} \frac{\partial y^j}{\partial x^i} \bigg),$ $L^{k}_{\alpha i'} = g_{a'}^{a} \bigg( \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^i} L^i_{\alpha j} + \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^i} \bigg), C^j_{\gamma j} = g_{a'}^{a} g_{b'}^{b} \frac{\partial y^j}{\partial x^i} C_{\gamma j}, C^j_{\beta j} = g_{a'}^{a} g_{b'}^{b} \frac{\partial y^j}{\partial x^i} C_{\beta j}, C^j_{\gamma j} = g_{a'}^{a} g_{b'}^{b} \frac{\partial y^j}{\partial x^i} C_{\gamma j}.$

If we compare the above formulas with those of a d-connection on the v.b. $\iota \xi$ [2, ec. (7.5), pag. 72], we obtain:
Theorem 3.1 Let $\xi'$ be a v.b. of the v.b. $\xi$, $N$ a non-linear connection on $\xi$ and $S$ a Finsler splitting of the inclusion $i: \xi' \to \xi$.

1) Every linear $d$-connection on $\xi$ defines canonically a restricted $d$-connection on $\xi$ related to $\xi'$.

2) Every restricted $d$-connection on $\xi$ related to $\xi'$ defines canonically a $d$-connection on $\xi'$ and a linear Finsler $\xi'$-connection on $\xi'' = \xi/\xi'$.

3) A $d$-connection on $\xi'$ and a linear Finsler $\xi'$-connection on $\xi'' = \xi/\xi'$, defines canonically, using the Finsler splitting $S$, a restricted $d$-connection on $\xi$ related to $\xi'$.

The proof of the theorem will be given elsewhere. For the definition of a linear Finsler $\xi'$-connection on a v.b. $\xi''$, over the same base, see [4].

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References


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