Generalized Invexities and Global Minimum Properties
Ștefan Mititelu

Abstract

New definitions for all types of invexity and generalized invexity of the arbitrary real functions are given. Direct implications between the invexity and generalized invexity types are established. Moreover, it is shown that for a (strict) invex, (strict) pseudoinvex or (strict) quasiinvex function every (strict) local minimum point is one of (strict) global.

Mathematics Subject Classification: 49J52, 26B25
Key words: Generalized convexity. Invexity.

0. Introduction

The concept of convexity plays a very important role in the optimization theory. Various convex models and methods of the convex programming are used also in the study of the Riemannian manifolds [10]. The convex, pseudoconvex and quasiconvex functions [1] knew various generalizations. We quote in the following some of these.

Let $A$ be a nonempty open set of $\mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}$ be a differentiable function on $A$. Hanson [5] generalized the differentiable convex, pseudoconvex and quasiconvex functions respectively, by the following definitions.

**Definition I.** The differentiable function $f$ is *invex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u \in A : f(x) - f(u) \geq \eta(x, u) \nabla f(u).$$

(‘ is the sign of transposition and $\nabla f$ is the gradient of $f$).

**Definition II.** The differentiable function $f$ is *pseudoinvex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u \in A : \eta'(x, u) \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

**Definition III.** The differentiable function $f$ is *quasivex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u \in A : \eta(x, u) \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

0. Introduction

The concept of convexity plays a very important role in the optimization theory. Various convex models and methods of the convex programming are used also in the study of the Riemannian manifolds [10]. The convex, pseudoconvex and quasiconvex functions [1] knew various generalizations. We quote in the following some of these.

Let $A$ be a nonempty open set of $\mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}$ be a differentiable function on $A$. Hanson [5] generalized the differentiable convex, pseudoconvex and quasiconvex functions respectively, by the following definitions.

**Definition I.** The differentiable function $f$ is *invex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u : f(x) - f(u) \geq \eta(x, u) \nabla f(u).$$

(‘ is the sign of transposition and $\nabla f$ is the gradient of $f$).

**Definition II.** The differentiable function $f$ is *pseudoinvex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u : \eta(x, u) \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$

**Definition III.** The differentiable function $f$ is *quasivex* on $A$ if there is a vector function $\eta : A \times A \rightarrow \mathbb{R}^n$ such that

$$\forall x, u : \eta(x, u) \nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u).$$
\( \forall x, u \in A : f(x) \leq f(u) \Rightarrow \eta'(x, u) \nabla f(u) \geq 0. \)

Later the types of invexity, pseudoinvexity and quasiinvexity have been introduced, specially in the differentiable case, by Jayakumar [6] and Preda [8]. Craven [3] introduced the invexity notion for the Lipschitz functions. In the nonsmooth case Giorgi and Mititelu [4] and Mititelu and Stancu-Minasian [7] defined the nonsmooth invex, pseudoinvex and quasiinvex functions using the upper Dini and Clarke directional derivatives, respectively. So, let \( f \) be the same function \( f \) and vectors \( u \in A \) and \( v \in \mathbb{R}^n \). The symbol

\[
f^+_\rho(u; v) = \lim_{\lambda \to 0} \sup \frac{f(u + \lambda v) - f(u)}{\lambda} \quad (u + \lambda v \in A)
\]

is called the upper Dini directional derivative of \( f \) at the point \( u \) in the direction \( v \). Let \( \rho \) be a real number. Then the definitions of all types of the invexity, pseudoinvexity and quasiinvexity in respect to \( f \), given by Giorgi and Mititelu, are the following:

**Definition 1' (Invexities).** The function \( f \) is said to be \( \rho \)-invex on \( A \) (shortly \( \rho I_+ \)), if there exist vector functions \( \eta, \theta : A \times A \to \mathbb{R}^n \) such that

\[
(\rho I_+) \quad \forall x, u \in A : f(x) - f(u) \geq f^+_\rho(u; \eta(x, u)) + \rho \|\theta(x, u)\|^2.
\]

If

1. \( \rho > 0 \) the function \( f \) is called strongly invex;
2. \( \rho = 0 \) the function \( f \) is called invex;
3. \( \rho < 0 \) the function \( f \) is called weakly invex;
4. \( \forall x \in A, x \neq u : f(x) - f(u) > f^+_\rho(u; \eta(x, u)) \) the function \( f \) is called strictly invex.

**Definition 2' (Pseudoinvexities).** The function \( f \) is said to be \( \rho \)-pseudoinvex on \( A(\rho I_+) \) if there exist vector functions \( \eta, \theta : A \times A \to \mathbb{R}^n \) such that

\[
(\rho PI_+) \quad \forall x, u \in A : f^+_\rho(u; \eta(x, u)) + \rho \|\theta(x, u)\|^2 \geq 0 \Rightarrow f(x) \geq f(u).
\]

If

1. \( \rho > 0 \) the function \( f \) is called strongly pseudoinvex;
2. \( \rho = 0 \) the function \( f \) is called pseudoinvex;
3. \( \rho < 0 \) the function \( f \) is called weakly pseudoinvex;
4. \( \forall x, u \in A, x \neq u : f^+_\rho(u; \eta(x, u)) \geq 0 \Rightarrow f(x) > f(u) \) the function \( f \) is called strictly pseudoinvex.

**Definition 3' (Quasiinvexities).** The function \( f \) is said to be \( \rho \)-quasiinvex on \( A \) if vector functions \( \eta, \theta : A \times A \to \mathbb{R}^n \) exist such that

\[
(\rho QI_+) \quad \forall x, u \in A : f(x) \leq f(u) \Rightarrow f^+_\rho(u; \eta(x, u)) + \rho \|\theta(x, u)\|^2 \leq 0.
\]

If

1. \( \rho > 0 \) the function \( f \) is called strongly quasiinvex;
2. \( \rho = 0 \) the function \( f \) is called quasiinvex;
3. \( \rho < 0 \) the function \( f \) is called weakly quasiinvex;
4. \( \forall x, u \in A, x \neq u : \forall \lambda \in (0, 1) : f(x) \leq f(u) \Rightarrow f(u + \lambda \eta(x, u)) < f(u) \) the function \( f \) is called strictly quasiinvex.
Generalized Invexities and Global Minimum Properties

(3c) \( \forall x, u \in A, x \neq u, \forall \lambda \in (0,1) : f(x) < f(u) \Rightarrow f(u + \lambda \eta(x,u)) < f(u) \) the function \( f \) is called semistrictly quasiinvex.

Remark 1. Some of Definitions 1'-3' have sense when function \( f'(u) \) is finite.

The purpose of this paper is to give definitions for the nonsmooth invex, pseudoinvex and quasiinvex functions without use directional derivatives. The new definitions are defined at a point. The direct implications between all types of invexity, pseudoinvexity and quasiinvexity are given. Moreover, some properties of global minimum of the nonsmooth invex, pseudoinvex and quasiinvex functions are established.

1 Invexity and generalized invex types

We propose for all types of invexity and incavity definitions as follows.

Definitions 1. (Invexities at a point). The functions \( f \) is said to be \( \rho \)-invex at a point \( u \in A(\rho I) \) if there exist vector functions \( \eta, \theta : A \times A \to \mathbb{R}^n \) such that

\[
(\rho I) \quad \forall x \in A, \forall \lambda \in [0,1] : f(u + \lambda \eta(x,u)) \leq f(u) + \lambda [f(x) - f(u)] - \rho \lambda \|\theta(x,u)\|^2.
\]

If

(1a) \( \rho > 0 \) the function \( f \) is called strongly invex at \( u(SgI) \);
(1b) \( \rho = 0 \) the function \( f \) is called invex at \( u(I) \);
(1c) \( \rho < 0 \) the function \( f \) is called weakly invex at \( u(WI) \);
(1d) \( \forall x \in A, x \neq u, \forall \lambda \in (0,1) : f(u + \lambda \eta(x,u)) < f(u) + \lambda [f(x) - f(u)] \), the function \( f \) is called strictly invex at \( u(SI) \).

B (Incavities at a point). The function \( f \) is said to be \( \rho \)-incave at \( u \) if the function \(-f\) is \( \rho \)-invex at \( u \). If \( \rho > 0 \), \( \rho = 0 \) or \( \rho < 0 \) \( f \) is called strongly incave, incave or weakly incave at \( u \). The function \( f \) is said to be strictly incave at \( u \) if the function \(-f\) is strictly invex at \( u \).

In this section we suppose that \( f'_{+}(u) \) is finite (see Remark 1).

Theorem 1.1. Definitions 1' with \( u \) fixed and 1A are equivalent.

Proof. If \( u \) is fixed in Definition 1' then from relation (\( \rho I \)) we obtain:

\[
(\rho I)_{+} \quad x \in A : f(x) - f(u) \geq f'_{+}(u; \eta(x,u)) + \rho \|\theta(x,u)\|^2.
\]

Function \( f'_{+}(u, \cdot) \) is positively homogeneous and then, by multiplication of \( (\rho I)_{+} \) with an arbitrary scalar \( t \geq 0 \), we obtain

\[
x \in A : tf(x) - tf(u) \geq f'_{+}(u; t\eta(x,u)) + t\rho \|\theta(x,u)\|^2 \quad \text{for all} \quad t \geq 0,
\]

\[
x \in A : tf(x) - tf(u) \geq \limsup_{\mu \to 0} \frac{f(u + \mu t \eta(x,u)) - f(u)}{\mu} + t\rho \|\theta(x,u)\|^2
\]

for all \( t \geq 0 \) and \( \mu > 0 \),

\[
x \in A : tf(x) - tf(u) \geq \frac{f(u + \mu t \eta(x,u)) - f(u)}{\mu} + t\rho \|\theta(x,u)\|^2 \quad \text{and even}
\]

(1) \( x \in A, t \geq 0 : \tilde{\mu} tf(x) - \tilde{\mu} tf(u) \geq f(u + \tilde{\mu} t \eta(x,u)) - f(u) + \tilde{\mu} t\rho \|\theta(x,u)\|^2.
\]

Noting \( \lambda = \tilde{\mu} t \) in (1) it results \( (\rho I) \) from Definition 1A.
Conversely, for an arbitrary \( \lambda > 0 \) in the relation \((\rho I)\) by Definition 1A we obtain

\[
\frac{f(u + \mu \eta(x,u)) - f(u)}{\lambda} \leq f(x) - f(u) - \rho ||\theta(x,u)||^2
\]

and taking upper limit by \( \lambda \downarrow 0 \) in (2) we obtain

\[
f^*_\lambda(u; \eta(x,u)) \leq f(x) - f(u) - \rho ||\theta(x,u)||^2,
\]

which is \((\rho I_\lambda)\) by Definition 1'.

**Theorem 1.2.** Between the invexity types the following direct implications hold:

(a) Strongly invex (SgI) and \( x \neq u \Rightarrow \theta(x,u) \neq 0 \) \( \Rightarrow \) Strictly invex (SI);

(b) Strictly invex (SI) \( \Rightarrow \) Invex (I) \( \Rightarrow \) Weakly invex (WI).

**Proof.** See above Theorem 1.1 and Théorème 2.1 from [4].

For the pseudoinvexity and pseudoincavity types at a point we propose definitions as follows.

**Definition 2. A (Pseudoinvexitites at a point).** The function \( f \) is said to be \( \rho \)-pseudoinvex at \( u \in A \) \((\rho PI)\) if there exist vector functions \( \eta, \theta : A \times A \rightarrow \mathbb{R}^m \) such that

\[
(\rho PI) \quad \forall x \in A; \forall \lambda \in [0,1]: f(u + \lambda \eta(x,u)) + \rho \lambda ||\theta(x,u)||^2 \geq f(u) \Rightarrow f(x) \geq f(u).
\]

If

(2a) \( \rho > 0 \) the function \( f \) is called strongly pseudoinvex at \( u \) (Sg PI);

(2b) \( \rho = 0 \) the function \( f \) is called pseudoinvex at \( u \) (PI);

(2c) \( \rho < 0 \) the function \( f \) is called weakly pseudoinvex at \( u \) (WPI);

(2d) \( \forall x \in A, x \neq u, \forall \lambda \in (0,1) : f(u + \lambda \eta(x,u)) \geq f(u) \Rightarrow f(x) > f(u) \),

the function \( f \) is called strictly pseudoinvex at \( u \) (SPI).

**B (Pseudoincavities at a point).** The function \( f \) is said to be \( \rho \)-pseudoincrea at \( u \) if the function \(-f\) is \( \rho \)-pseudoinvex at \( u \). If \( \rho > 0 \), \( \rho = 0 \) or \( \rho < 0 \) then \( f \) is called strongly pseudoincrea, pseudoincrea or weakly pseudoincrea at \( u \). The function \( f \) is said to be strictly pseudoincrea at \( u \) if the function \(-f\) is strictly pseudoinvex at \( u \).

In what follows is necessary the next lemma.

**Lemma.** For all \( x \in A \) and \( \rho \in \mathbb{R} \) the following equivalence holds

\[
(3) \quad f^*_\lambda(u; \eta(x,u)) + \rho ||\theta(x,u)||^2 \geq 0 \iff \quad \forall \lambda \in [0,1]: f(u + \lambda \eta(x,u)) + \rho \lambda ||\theta(x,u)||^2 \geq f(u).
\]

\[
(4) \quad f(u + \lambda \eta(x,u)) + \rho \lambda ||\theta(x,u)||^2 \geq f(u), \quad \forall \lambda \in (0,1).\]

**Proof.** Function \( f^*_\lambda(u; \cdot) \) is positively homogeneous and then for all \( t \geq 0 \) from (3) it results

\[
f^*_\lambda(u; t\eta(x,u)) + t\rho ||\theta(x,u)||^2 \geq f(u)
\]

or equivalently

\[
\limsup_{\mu \downarrow 0} \frac{f(u + \mu \eta(x,u)) - f(u)}{\mu} + t\rho ||\theta(x,u)||^2 \geq 0, \quad \forall t \geq 0.
\]

Then there is a \( \tilde{\mu} > 0 \) such that
\[
\frac{f(u + \tilde{\mu} \eta(x, u)) - f(u)}{\tilde{\mu}} + t\rho\|\theta(x, u)\|^2 \geq 0, \quad \forall t \geq 0
\]
or equivalently,
\[
f(u + \tilde{\mu} \eta(x, u)) - f(u) + \tilde{\mu}t\|\theta(x, u)\|^2 \geq 0, \quad \forall t \geq 0
\]
We denote \( \lambda = \tilde{\mu}t \) in this inequality and one obtains (4).
Conversely, by (4) for \( \lambda > 0 \), we obtain
\[
\frac{f(u + \lambda \eta(x, u)) - f(u)}{\lambda} + \rho\|\theta(x, u)\|^2 \geq 0
\]
and taking upper limit by \( \lambda \downarrow 0 \) in this inequality one obtains inequality (3).

**Theorem 1.3.** Definitions 2' with \( u \) fixed and 2A are equivalent.

**Proof.** One uses the above Lemma.

**Theorem 1.4.** Between the types of pseudoinvexity the following direct implications hold:

(a) Strongly pseudoinvex (Sp PI) and injective \( \Rightarrow \) Strictly pseudoinvex (SPI);

(b) Strictly pseudoinvex (SPI) \( \Rightarrow \) Pseudoinvex (P) \( \Rightarrow \) Weakly pseudoinvex (WPI).

**Proof.** See Théorème 2.3 from [4] and the above Lemma.

**Theorem 1.5.** If the function \( f \) is \( \rho \)-invex at \( u \in A \) then \( f \) is \( \rho \)-pseudoinvex at \( u \).
Moreover, if \( f \) is strictly invex at \( u \) then \( f \) is strictly pseudoinvex at \( u \).

**Proof.** If \( f \) is \( \rho \)-invex at \( u \) then we have
\[
(5) \quad x \in A, \quad \lambda \in [0, 1] : f(u + \lambda \eta(x, u)) + \rho \lambda\|\theta(x, u)\|^2 - f(u) \leq \lambda[f(x) - f(u)]
\]
and if \( f(u + \lambda \eta(x, u)) + \rho \lambda\|\theta(x, u)\|^2 \geq f(u) \) then from (5) it results \( f(x) \geq f(u) \). For the quasiinvexity and quasiancavity types at a point we propose definitions as follows.

**Definition 3. A (Quasiinvexities at a point).** The function \( f \) is said to be \( \rho \)-quasiinvex at \( u \in A \) (\( \rho \)QI) if there are vector functions \( \eta, \theta : A \times A \rightarrow \mathbb{R}^n \) such that
\[
(\rho \text{QI}) \quad \forall x \in A, \quad \forall \lambda \in [0, 1] : f(x) \leq f(u) \Rightarrow f(u + \lambda \eta(x, u)) + \rho \lambda\|\theta(x, u)\|^2 \leq f(u).
\]

If

(3a) \( \rho > 0 \) the function \( f \) is called strongly quasiinvex at \( u \) (Sp QI);

(3b) \( \rho = 0 \) the function \( f \) is called quasiinvex at \( u \) (QI);

(3c) \( \rho < 0 \) the function \( f \) is called weakly quasiinvex at \( u \) (WQI);

(3d) \( \forall x \in A, \ x \neq u, \ \forall \lambda \in (0, 1) : f(x) \leq f(u) \Rightarrow f(u + \lambda \eta(x, u)) < f(u) \) the function \( f \) is called strictly quasiinvex at \( u \) (SQI);

(3e) \( \forall x \in A, \ x \neq u, \ \forall \lambda \in (0, 1) : f(x) < f(u) \Rightarrow f(u + \lambda \eta(x, u)) < f(u) \) the function \( f \) is called semistrictly quasiinvex at \( u \) (SSQI).

**B (Quasiancaviies at a point).** The function \( f \) is said to be \( \rho \)-quasiancave at \( u \) if the function \( -f \) is \( \rho \)-quasiinvex at \( u \). If \( \rho > 0 \), \( \rho = 0 \) or \( \rho < 0 \) then \( f \) is called strongly quasiancave, quasiancave or weakly quasiancave at \( u \). The function \( f \) is said to be strictly quasiancave at \( u \) if the function \( -f \) is strictly quasiinvex at \( u \) and \( f \) is said to be semistrictly quasiancave at \( u \) if \( -f \) is semistrictly quasiinvex at \( u \).

**Theorem 1.6.** Definitions 3' with \( u \) fixed and 3A are equivalent.

**Proof.** One uses the above Lemma.
Theorem 1.7. Between the types of quasiinvexity the next direct implications hold at \( u \):

(a) Strongly quasiinvex (Sg QI) and \( x \neq u \Rightarrow \theta(x, u) \neq 0 \) \( \Rightarrow \) Strictly quasiinvex (SQI);

(b) Strictly quasiinvex (SQI) \( \Rightarrow \) Semistrictly quasiinvex (SSQI);

(c) Semistrictly quasiinvex (SSQI) and lower semicontinuous on \( A \) and \( \eta(\cdot ; u) \)
bounded on \( A \) \( \Rightarrow \) Quasiinvex (QI);

(d) Quasiinvex (QI) \( \Rightarrow \) Weakly quasiinvex (WQI).

Proof. For (a) and (d) one use the Lemma and Théorème 2.7 by [4].

(b) Is obvious.

(c) We must show that

\[
\text{(QI) } \forall x \in A, \forall \lambda \in [0, 1] : f(x) \leq f(u) \Rightarrow f(u + \lambda \eta(x, u)) \leq f(u).
\]

Because \( f \) is (SSQI) at \( u \) and all \( \lambda > 0 \), \( f(u + \lambda \eta(x, u)) < f(u) \) implies \( f(u + \lambda \eta(x, u)) \leq f(u) \), it results that \( f \) is (QI) at \( u \).

We now have to prove that

\[
\forall x \in A, \forall \lambda \in [0, 1] : f(x) = f(u) \Rightarrow f(u + \lambda \eta(x, u)) \leq f(u).
\]

Assume by reductio ad absurdum that (6) is not true. Then

\[
\exists t \in A, \exists \lambda \in (0, 1] : f(t) = f(u) \quad \text{and} \quad f(u + \lambda \eta(t, u)) > f(u).
\]

We denote \( \tilde{x} = u + \tilde{\lambda} \eta(t, u) \) and also

\[
f(\tilde{x}) - f(u) = a(a > 0).
\]

Because \( f \) is lower semicontinuous at \( x \) it results that for any \( \varepsilon > 0 \) there is a \( \delta_\varepsilon > 0 \)
such that for any \( x \in A \) for which \( ||x - \tilde{x}|| < \delta_\varepsilon \) one has \( f(x) > f(\tilde{x}) - \varepsilon \). In particular for \( x = u \) one gets that \( ||u - \tilde{x}|| < \delta_\varepsilon \) implies

\[
f(u) > f(\tilde{x}) - \varepsilon.
\]

Choosing \( \varepsilon = a \) from (7) and (8) one obtain \( a > a \), which is contradictory.

In this proof we suppose that \( ||u - \tilde{x}|| < \delta_\varepsilon \), which is equivalent to \( ||\eta(t, u)|| < \delta_\varepsilon / \tilde{\lambda} \).

From this it follows that the function \( \eta(\cdot, u) \) must be bounded on \( A \).

Theorem 1.8. Between the types of pseudoinvexity and quasiinvexity the next implications hold at \( u \):

(a) Strongly pseudoinvex (Sg PI) \( \Rightarrow \) Strongly quasiinvex (SgQI);

(b) Strictly pseudoinvex (SPI) \( \Rightarrow \) Semistrictly quasiinvex (SSQI);

(c) Weakly pseudoinvex (WPI) \( \Rightarrow \) Weakly quasiinvex (WQI);

(d) Pseudoinvex (QI) \( \Rightarrow \) Semistrictly quasiinvex (SSQI).

Proof. One use the Lemma and Théorème 2.8 by [4].

Direct implications which exist between various types of invexity, pseudoinvexity and quasiinvexity at a point, according to Theorems 1.2, 1.4, 1.5, 1.7 and 1.8 are given in the following block diagram.
weakly invex $\rightarrow$ weakly pseudoinvex $\rightarrow$ weakly quasiinvex

\[ \uparrow \quad \uparrow \quad \uparrow \]

\[ \text{quasiinvex} \]

\[ \uparrow \]

invex $\rightarrow$ pseudoinvex $\rightarrow$ semistrictly quasiinvex

\[ \uparrow \quad \uparrow \quad \uparrow \]

strictly invex $\rightarrow$ strictly pseudoinvex $\rightarrow$ strictly quasiinvex

\[ \uparrow \quad \uparrow \quad \uparrow \]

strongly invex $\rightarrow$ strongly pseudoinvex $\rightarrow$ strongly quasiinvex

Remark 2. Functions 0-invex (case 1A(b)) have been introduced by Ben Israel and Mond in 1986 and its have been denoted "pre-invex functions" by Jeyakumar (see Weir and Mond [11]).

2 Convexity, pseudoconvexity and quasiconvexity types

In the particular case when $A$ is a nonempty convex set, $\eta(x, u) = x - u$ and $\theta(x, u) = x - u$ we recover the types of convexity, pseudoconvexity and quasiconvexity at a point, as follows:

Definition 2.1. A(Convexities at a point). The function $f$ is said to be $\rho$-convex at $u \in A$ $(\rho C)$ if

\[ (\rho C) \quad \forall x \in A, \forall \lambda \in [0, 1]: f(\lambda x + (1 - \lambda) u) \leq \lambda f(x) + (1 - \lambda) f(u) - \rho \lambda \| x - u \|^2. \]

If

$(1'a)$ $\rho > 0$ the function $f$ is called strongly convex at $u$ $(SGC)$;
$(1'b)$ $\rho = 0$ the function $f$ is called convex at $u$ (C);
$(1'c)$ $\rho < 0$ the function $f$ is called weakly convex at $u$ (WC);
$(1'd)$ $\forall x \in A, x \neq u, \forall \lambda \in (0, 1): f(\lambda x + (1 - \lambda) u) < \lambda f(x) + (1 - \lambda) f(u)$ then $f$ is called strictly convex at $u$ (SC).

B (Concavities at a point). The function $f$ is said to be $\rho$-concave at $u$ if the function $-f$ is $\rho$-convex at $u$. If $\rho > 0$, $\rho = 0$ or $\rho < 0$ then $f$ is called strongly concave, concave or weakly concave at $u$. The function $f$ is said to be strictly concave at $u$ if the function $-f$ is strictly convex at $u$.

Definition 2.2. A(Pseudoconvexities at a point). The function $f$ is said to be $\rho$-pseudoconvex at $u \in A$ $(\rho PC)$ if
\[(\rho PC) \quad \forall x \in A, \forall \lambda \in [0, 1] : f(\lambda x + (1 - \lambda)u) + \rho \lambda \|x - u\|^2 \geq f(u) \Rightarrow f(x) \geq f(u).\]

If
\[(2'a) \rho > 0 \text{ the function } f \text{ is called strongly pseudoconvex at } u \text{ (SgPC);}
(2'b) \rho = 0 \text{ the function } f \text{ is called pseudoconvex at } u \text{ (PC);}
(2'c) \rho < 0 \text{ the function } f \text{ is called weakly pseudoconvex at } u \text{ (WPC);}
\]
\[\quad (2'd) \forall x \in A, x \neq u, \forall \lambda \in (0, 1) : f(\lambda x + (1 - \lambda)u) \geq f(u) \Rightarrow f(x) > f(u) \text{ then } f \text{ is called strongly pseudoconvex at } u \text{ (SCP).}\]

**B (Pseudoconcavities at a point).** The function \(f\) is said to be \(\rho\)-pseudoconcave at \(u\) if the function \(-f\) is \(\rho\)-pseudoconvex at \(u\). If \(\rho > 0\), \(\rho = 0\) or \(\rho < 0\) then \(f\) is called strongly pseudoconcave, pseudoconcave or weakly pseudoconcave at \(u\). The function \(f\) is said to be strictly pseudoconcave at \(u\) if the function \(-f\) is strictly pseudoconvex at \(u\).

**Definition 2.3. A (Quasiconcavities at a point).** The function \(f\) is said to be \(\rho\)-quasiconvex at \(u \in A\) \((\rho QC)\) if
\[(\rho QC) \quad \forall x \in A, \forall \lambda \in [0, 1] : f(x) \leq f(u) \Rightarrow f(\lambda x + (1 - \lambda)u + \rho \lambda \|x - u\|^2 \leq f(u).\]

If
\[(3'a) \rho > 0 \text{ the function } f \text{ is called strongly quasiconvex at } u \text{ (SGQC);}
(3'b) \rho = 0 \text{ the function } f \text{ is called quasiconvex at } u \text{ (QC);}
(3'c) \rho < 0 \text{ the function } f \text{ is called weakly quasiconvex at } u \text{ (WQC);}
\]
\[\quad (3'd) \forall x \in A, x \neq u, \forall \lambda \in (0, 1) : f(x) \leq f(u) \Rightarrow f(\lambda x + (1 - \lambda)u) < f(u) \text{ then } f \text{ is called strongly quasiconvex at } u \text{ (SQC).}
\]
\[\quad (3'e) \forall x \in A, x \neq u, \forall \lambda \in (0, 1) : f(x) < f(u) \Rightarrow f(\lambda x + (1 - \lambda)u) < f(u) \text{ then } f \text{ is called semistrictly quasiconvex at } u \text{ (SSQC).}\]

**B (Quasiconcavities at a point).** The function \(f\) is said to be \(\rho\)-quasiconcave at \(u\) if the function \(-f\) is \(\rho\)-quasiconvex at \(u\). If \(\rho > 0\), \(\rho = 0\) or \(\rho < 0\) then \(f\) is called strongly quasiconcave, quasiconcave or weakly quasiconcave at \(u\). The function \(f\) is said to be strictly quasiconcave at \(u\) if the function \(-f\) is strictly quasiconvex at \(u\) and \(f\) is called semistrictly quasiconcave at \(u\) if the function \(-f\) is semistrictly quasiconvex at \(u\).

## 3 Global minimum properties for (generalized) invex functions

In this section we show that for all (strictly) invex, (strictly) pseudoinvex and (strictly) quasianvex functions every (strict) local minimum point is a (strict) global minimum point. The global minimum property for these functions makes them indispensable in the optimization theory. For a semistrictly quasianvex function every local minimum point is one of absolutely minimum point.

**Theorem 3.1.** If \(a\) is a (strict) local minimum point of the function \(f\) in \(A\) and \(f\) is (strictly) invex at \(a\), then \(a\) is a (strict) global minimum point of \(f\) on \(A\).

**Proof.** We suppose that \(a\) is a local minimum point and that \(f\) is invex at \(a\) in respect to \(\eta : A \times A \rightarrow \mathbb{R}^n\). Then there is a neighbourhood \(N\) of a such that \(f(x) \geq f(a), \forall x \in N \cap A\). Let now an arbitrary point \(t \in A\). Then exists a \(\lambda > 0\), enough small, such that \(a + \lambda \eta(t, a) \in N \cap A\). We have \(f(a) \leq f(a + \lambda \eta(t, a))\) and by the
inexivity of $f$ at $a$ it results $f(a) \leq f(a) + \lambda[f(t) - f(a)]$. From this inequality one obtains $f(t) \geq f(a), \forall t \in A$. The proof of the strict variant of the theorem is obtained on the same text.

**Theorem 3.2.** If $a \in A$ is a (strict) local minimum point of the function $f$ in $A$ and $f$ is (strictly) pseudo-inexact at $a$, then $a$ is a (strict) global minimum point of $f$ on $A$.

**Proof.** We suppose that $f$ is pseudoinexact at $a$ in respect to $\eta: A \times A \rightarrow R^n$ and that $a$ is a local minimum point of $f$ in $A$. Then there is a neighbourhood $N$ of $a$ such that

$$f(x) \geq f(a), \quad \forall x \in N \cap A.$$  

Let now $t$ be an arbitrary point in $A$. Then there is a $\lambda > 0$, enough small, such that

$$a + \lambda \eta(t, a) \in N \cap A.$$

According to (9) we have $f(a + \lambda \eta(t, a)) \geq f(a)$. But $f$ is pseudoinexact at $a$ and then $f(a + \lambda \eta(t, a)) \geq f(a)$ implies $f(t) \geq f(a), \forall t \in A$. The strict variant of the theorem is obtained in the same manner.

**Theorem 3.3.** If $a \in A$ is a strict local minimum point of the function $f$ in $A$ and $f$ is strictly quasi-inexact at $a$, then $a$ is a strict global minimum point of $f$ on $A$.

**Proof.** We suppose that $f$ is (SQI) at $a \in A$ in respect to $\eta: A \times A \rightarrow R^n$. Then, equivalently, we have:

$$\forall x \in A, x \neq a, \forall \lambda \in (0, 1): f(a + \lambda \eta(x, a)) \geq f(a) \Rightarrow f(x) > f(a).$$

If $a$ is a strict local minimum point in $A$ then there is a neighbourhood $N$ of $a$ such that

$$f(x) > f(a), \quad \forall x \in N \cap A \setminus a.$$  

At consequence there is an $\varepsilon > 0$ such that $a + \varepsilon \eta(t, a) \in N \cap A \setminus a$, where $t$ is arbitrary in $A$. Then, according to (11), we have

$$f(a + \varepsilon \eta(t, a)) > f(a), \quad \forall \lambda \in (0, \varepsilon], \quad \forall t \in A \setminus a.$$  

Taking now into account relations (10) and (12) we obtain $f(t) > f(a), \quad \forall t \in A \setminus a$ and so, $a$ is the strict global minimum point of $f$ on $A$.

**Theorem 3.4.** Let $f$ be a semistriectly quasi-inexact function at the point $a \in A$. If $a$ is a local minimum point of $f$ in $A$, then $a$ is an absolute minimum point of $f$ on $A$.

**Proof.** If $f$ is (SSQI) at $a$ in respect to $\eta: A \times A \rightarrow R^n$ then, equivalently, one has

$$\forall x \in A, x \neq a, \forall \lambda \in (0, 1): f(a + \lambda \eta(x, a)) \geq f(a) \Rightarrow f(x) \geq f(a).$$

The point $a$ is one of local minimum of $f$ and then there exists a neighbourhood $N$ of $a$, such that $f(x) \geq f(a), \quad \forall x \in N \cap A$. Particulary, there is an $\varepsilon > 0$ such that

$$f(a + \lambda \eta(x, a)) \geq f(a), \quad \forall x \in A, \quad \forall \lambda \in (0, \varepsilon].$$

Combining relations (13) and (14) it results $f(x) \geq f(a), \forall x \in A$.

**Definition 3.1.** Let $V$ and $S$ be two nonempty subset of $A$. $V$ is said to be a "neighbourhood" of $S$ if $S \subset V$ and $(\text{Fr } S) \cap (\text{Fr } V) = \emptyset$ or otherwise $\emptyset \neq (\text{Fr } S) \cap (\text{Fr } V) \subset \text{Fr } A$. 
Definition 3.2. Let \( f : A \to R \) be where the set \( A \) has a nonempty relative interior.

a) The nonempty set \( S \subset A \) is said to be a local minimum subdomain of the function \( f \) in \( A \) if \( f \) is constant on \( S \) and there is a "neighbourhood" \( V \subset A \) of \( S \) such that \( f(y) > f(x), \forall x \in S, \forall y \in V \).

b) If \( M \) is the single local minimum subdomain of the function \( f \) in \( A \) then \( M \) is called the global minimum subdomain of \( f \) on \( A \).

Theorem 3.5. Let \( f \) be a quasiinveex function on \( A \). Then any local minimum subdomain of \( f \) in \( A \) is a global minimum subdomain of \( f \) on \( A \).

Proof. Let \( M \subset A \) be a local minimum subdomain of \( f \) in \( A \). We suppose, ad absurdum, that the quasiinveex function \( f \) admits in \( A \) another local minimum subdomain \( S \), different from \( M \). Evidently \( M \cap S = \emptyset \). Let \( u \in M \) and \( a \in S \) be arbitrary chosen and we suppose

\[
f(a) \leq f(u).
\]

Let \( V \) be a "neighbourhood" of \( M \). We ascertain that

\[
B = \{ u + \lambda \eta(x, u) \in A | \forall x \in A, \forall \lambda \in (0, \bar{\lambda}) \} \cap V \neq \emptyset
\]

where

\[
\bar{\lambda} = \min\{1, \sup\{ \lambda > 0 | u + \lambda \eta(x, u) \in A \} \}.
\]

Let \( y \) be an arbitrary point in \( B \setminus M \). Then there are \( x' \in A \) and \( \lambda' \in (0, \bar{\lambda}) \) such that \( y = u + \lambda' \eta(x', u) \). We can choose \( y \) in \( B \setminus M \) such that \( f(x') < f(u) \) because, according to (15), \( u \) is not an absolutely minimum point. Then, the quasiinveexity of \( f \) at \( u \) imply

\[
f(u + \lambda' \eta(x', u)) \leq f(u).
\]

But \( y = u + \lambda' \eta(x', u) \) in \( V \) implies

\[
f(y) = f(u + \lambda' \eta(x', u)) > f(u).
\]

Relations (16) and (17) are contradictory. Then it follows that the supposition above made, that \( M \) is not the single local minimum subdomain of \( f \) in \( A \), is false.

Remark 3. The preceding theory can be translated in the Riemannian language using the ideas of Udrişte [10].

Acknowledgements. A version of this paper was presented at the First Conference of Balkan Society of Geometers, Politehnica University of Bucharest, September 23-27, 1996.

References


Technical University of Civil Engineering
B-dul Lacul Tei 124, Sector 2, Bucharest
72302, Romania