Measurability of linear operators in the Skorokhod topology

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Abstract

It is proved that bounded linear operators on Banach spaces of “cadlag” functions are measurable with respect to the Borel \( \sigma \)-algebra associated with the Skorokhod topology.

1 Introduction and notation.

Throughout this paper \( \mathbb{C}^n \) is understood to be equipped with an inner product \( \langle \cdot, \cdot \rangle \), defined by

\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i
\]

for all \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( \mathbb{C}^n \). We shall write \( |x| = \sqrt{\langle x, x \rangle} \) for all \( x \in \mathbb{C}^n \).

A function \( f : [0, 1] \to \mathbb{C}^n \) is said to be a cadlag function (“continu à droite, limite à gauche”) if for all \( t \in [0, 1] \) one has:

\[
\lim_{s \uparrow t} f(s) = f(t+) = f(t) \quad \text{and} \quad \lim_{s \downarrow t} f(s) = f(t-) \quad \text{exists}
\]

As can be proved in an elementary way, for every cadlag function \( f \) and every \( \varepsilon > 0 \) the set

\[
\{ t \in [0, 1] : |f(t) - f(t^-)| \geq \varepsilon \}
\]
is finite. It follows from this that a cadlag function can be uniformly approximated by step functions on $[0, 1]$. Consequently, every cadlag function is a bounded Borel function. The linear space of all cadlag functions assuming values in $C^n$ will be denoted by $\mathcal{D}(C^n)$ or, if there can be no confusion, simply by $\mathcal{D}$.

Now $\mathcal{D}$ is equipped with the supremum norm $\|\cdot\| :$

$$\|f\| = \sup\{|f(t)| : t \in [0, 1]\}$$

In this way $\mathcal{D}$ becomes a non-separable Banach space, we shall denote it by $\mathcal{D}_B$.

In [8] and [9] Skorokhod introduced on $\mathcal{D}$ a weaker topology which turns it into a Polish space. We shall refer to this topology as the Skorokhod topology. The space $\mathcal{D}$, equipped with this topology, will be denoted by $\mathcal{D}_S$.

It can be proved (see Billingsley [1]) that the identity map $I : \mathcal{D}_S \to \mathcal{D}_B$ is continuous in every $f$ which is continuous on $[0, 1]$. In particular $I$ is continuous in the origin.

The map $I$ is of course not continuous everywhere on $\mathcal{D}_S$. It thus appears that the topology on $\mathcal{D}_S$ is not translation invariant; consequently $\mathcal{D}_S$ is not a topological vector space.

Although the Skorokhod topology is not compatible with the linear structure on $\mathcal{D}$, the corresponding Borel $\sigma$-algebra is. In fact we shall see (theorem 3) that it presents the “cylindrical” $\sigma$-algebra on the Banach space $\mathcal{D}_B$.

In the sequel the only thing that we shall need in connection to the Skorokhod topology is that for all $t \in [0, 1]$ the map

$$f \to f(t)$$

is a Borel function on $\mathcal{D}_S$ (see Billingsley [1]).

It follows from this that for all $t \in [0, 1]$ the map

$$f \to f(t-) = \lim_{n \to \infty} f(t - \frac{1}{n}) ,$$

being the pointwise limit of a sequence of Borel functions, is also a Borel function on $\mathcal{D}_S$.

2 The dual space of the Banach space $\mathcal{D}_B$

In this section we are going to study the structure of continuous linear forms on $\mathcal{D}_B(C^n)$, that is, we are going to describe the dual space $\mathcal{D}^*_B$ of $\mathcal{D}_B$ (see also Corson [2]).

For any index set $I$ and any $\varphi : I \to C^n$ we define:

$$\sum_{a \in I} |\varphi(a)| = \sup\{\sum_{a \in F} |\varphi(a)| : F \text{ a finite subset of } I\}$$

If $\sum_{a \in I} |\varphi(a)| < +\infty$, then the limit

$$\lim_{F : |F| \to \infty} \sum_{a \in F} \varphi(a) = \sum_{a \in I} \varphi(a)$$
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exists in $\mathbb{C}^n$, where the filtration on the collection of finite sets $F$ is understood to be defined by inclusion.

The set of all $\varphi : I \to \mathbb{C}^n$ such that $\sum_{a \in I} |\varphi(a)| < +\infty$ will be denoted by $\ell^1(I, \mathbb{C}^n)$.

If $m_1, \ldots, m_n$ are complex Borel measures on $[0, 1]$ then we shall write:

$$m = (m_1, \ldots, m_n)$$

For all $m$ and all $\varphi \in \ell^1([0, 1], \mathbb{C}^n)$ we define a map $[m, \varphi] : \mathcal{D} \to \mathbb{C}$ by:

$$[m, \varphi](f) = \sum_{i=1}^{n} \int f_i \, d\overline{m_i} + \sum_{a \in [0,1]} (f(a) - f(a^-), \varphi(a)),$$

where $f = (f_1, \ldots, f_n) \in \mathcal{D}(\mathbb{C}^n)$.

The following theorem is stated in the notations introduced above:

**Theorem 1.** (i) For all $m = (m_1, \ldots, m_n)$ and $\varphi \in \ell^1([0, 1], \mathbb{C}^n)$ the map $[m, \varphi] : \mathcal{D}_B(\mathbb{C}^n) \to \mathbb{C}$ is a continuous linear form.

(ii) For every continuous linear form $l$ on the Banach space $\mathcal{D}_B(\mathbb{C}^n)$ there exists a unique $m = (m_1, \ldots, m_n)$ and a unique $\varphi \in \ell^1([0, 1], \mathbb{C}^n)$ such that $l = [m, \varphi]$.

**Proof.** The proof of (i) is left to the reader.

We prove statement (ii) in the case where $n = 1$. The general case can easily be deduced from this, for $\mathcal{D}_B(\mathbb{C}^n)$ is in an obvious way the direct sum of copies of $\mathcal{D}_B(\mathbb{C})$.

Let $l$ be an arbitrary continuous linear form on $\mathcal{D}_B = \mathcal{D}_B(\mathbb{C})$. By Riesz’s representation theorem the restriction of $l$ to the subspace $C([0, 1])$ of continuous functions on $[0, 1]$ defines a complex Borel measure on $[0, 1]$. This measure will be denoted by $m$.

The continuous linear form $\tilde{l}$ on $\mathcal{D}_B$ is defined by

$$\tilde{l}(f) = l(f) - \int f \, dm \quad \text{for all } f \in \mathcal{D}.$$

Now one has $\tilde{l}(f) = 0$ for every $f \in C([0, 1])$.

For every finite set $F \subset [0, 1]$ we define the linear subspace $\mathcal{M}_F$ by:

$$\mathcal{M}_F = \{ f \in \mathcal{D} : f(a) - f(a^-) = 0 \text{ if } a \notin F \}$$

In other words, $\mathcal{M}_F$ comprises those $f \in \mathcal{D}$ which have a possible jump in the points of $F$ only.

For every $a \in (0, 1]$ and sufficiently small $\delta > 0$ we define the function $1_a^\delta$ by:

$$1_a^\delta(t) = \begin{cases} 
\frac{1}{\delta}(t-a+\delta) & \text{if } t \in (a-\delta, a) \\
0 & \text{elsewhere on } [0, 1]
\end{cases}$$

If $f \in \mathcal{M}_F$, then for sufficiently small $\delta > 0$ the function

$$f + \sum_{a \in F} \{ f(a) - f(a^-) \} 1_a^\delta$$
is an element of $C([0,1])$. Therefore:

$$\tilde{l} \left( f + \sum_{a \in F} \{ f(a) - f(a-) \} \cdot 1_a^\delta \right) = 0$$

Consequently we have for all $f \in \mathcal{M}_F$

$$\tilde{l}(f) = -\sum_{a \in F} \{ f(a) - f(a-) \} \cdot \tilde{l}(1_a^\delta)$$

Keeping $a$ fixed, the difference of two functions of type $1_a^\delta$ is in $C[0,1]$. We see in this way that the expression $\tilde{l}(1_a^\delta)$ does not depend on $\delta$. For every $a \in [0,1]$, define $\varphi(a) = -\tilde{l}(1_a^\delta)$. We then have:

$$\tilde{l}(f) = \sum_{a \in F} \varphi(a) \{ f(a) - f(a-) \} \quad \text{for all } f \in \mathcal{M}_F$$

Our next goal is to prove that $\varphi \in \ell^1([0,1], \mathbb{C})$. For every finite $F \subset [0,1]$ we define the “complex saw tooth function” $f_F$ in the following way:

1. $f_F(a) = \frac{\varphi(a)}{|\varphi(a)|}$ if $a \in F$ and $\varphi(a) \neq 0$
2. $f_F(a) = 1$ if $a \in F$ and $\varphi(a) = 0$
3. $f_F$ is a linear function on each connected component of $F^c$, such that for all $a \in F$ one has $f_F(a+) = f_F(a)$ and $f_F(a-) = 0$

Now $\|f_F\| \leq 1$ for all $F$. Therefore we have:

$$\sup_{F} \sum_{a \in F} |\varphi(a)| = \sup_{F} |\tilde{l}(f_F)| < +\infty$$

It follows from this that $\varphi \in \ell^1([0,1], \mathbb{C})$, so the map

$$f \rightarrow \sum_{a \in [0,1]} \{ f(a) - f(a-) \} \cdot \varphi(a)$$

is continuous on $\mathcal{D}_B$. For all $f \in \bigcup_F \mathcal{M}_F$ we have

$$\tilde{l}(f) = \sum_{a \in [0,1]} \{ f(a) - f(a-) \} \cdot \varphi(a) \quad (\ast)$$

The linear space $\bigcup \mathcal{M}_F$ being dense in $\mathcal{D}_B$, this implies that $(\ast)$ holds for all $f \in \mathcal{D}_B$. In this way we see, by definition of $\tilde{l}$, that $l = [\overline{m}, \overline{\varphi}]$.

Unicity of $m$ and $\varphi$ can be proved easily; this is left to the reader.
Next, let \( \Omega \) be an arbitrary set, \( \mathcal{F} \) a \( \sigma \)-algebra of subsets of \( \Omega \) and \( M \) a topological space. A map \( X : \Omega \rightarrow M \) is said to be \( \mathcal{F} \)-measurable (or simply measurable if no confusion can arise) if \( X^{-1}(A) \in \mathcal{F} \) for all Borel sets \( A \) in \( M \). If \( M \) is a Banach space then a map \( X : \Omega \rightarrow M \) is said to be scalarly measurable if for every continuous linear form \( l \) on \( M \) the composition \( l \circ X : \Omega \rightarrow \mathbb{C} \) is measurable. A well-known theorem in functional analysis (due to B.J. Pettis [6]) states that in case of a separable Banach space, measurability is equivalent to scalar measurability. If \( M \) is non-separable then this statement is in general not true. In fact, it is easy to construct a counterexample in case \( M = \mathcal{D}_B(\mathbb{C}) \):

**Example.** Let \( \Omega = [0,1] \) and let \( \mathcal{F} \) be the \( \sigma \)-algebra consisting of all Borel sets in \([0,1]\). Define \( X : \Omega \rightarrow \mathcal{D}_B(\mathbb{C}) \) by:

\[
X(s) = 1_{[0,s)} \quad \text{for all } s \in [0,1]
\]

For any continuous linear form \( l = [m, \varphi] \) we have:

\[
l(X(s)) = m\{[0,s]\} + \varphi(s) \quad \text{for all } s \in [0,1]
\]

The condition that \( \sum_a |\varphi(a)| < +\infty \) implies that the set of points \( s \) for which \( \varphi(s) \neq 0 \) is at most countably infinite. Keeping this in mind, measurability of the map \( s \rightarrow l(X(s)) \) can be proved by easy verification. It thus appears that \( X \) is scalarly measurable.

Next we are going to prove that \( X : \Omega \rightarrow \mathcal{D}_B \) is not measurable.

Let \( A \subset [0,1] \) be a set which is not Borel. Define

\[
\mathfrak{A} = \{1_{[0,s)} : s \in A\} \subset \mathcal{D}_B
\]

Denote the convex hull of \( \mathfrak{A} \) by \( \mathfrak{C} \). It is not hard to prove that for all \( t \not\in A \)

\[
\|1_{[0,t]} - f\| \geq \frac{1}{2}
\]

for every \( f \in \mathfrak{C} \), and consequently also for every \( f \) in the closure \( \overline{\mathfrak{C}} \) of \( \mathfrak{C} \) in \( \mathcal{D}_B \).

In this way it turns out that \( X^{-1}(\overline{\mathfrak{C}}) = A \). This shows that \( X \) is neither measurable in the norm, nor in the weak topology associated with the Banach space \( \mathcal{D}_B \). (To the author it is not known whether the Borel \( \sigma \)-algebras corresponding to the norm and the weak topology on \( \mathcal{D}_B \) really differ (see also Edgar [3]). Talagrand proved in [10] and [11] the existence of Banach spaces where both \( \sigma \)-algebras are different).

### 3 Measurability in the Skorokhod topology.

As announced earlier, the linear space \( \mathcal{D} \) equipped with the Skorokhod topology will be denoted by \( \mathcal{D}_S \). A map \( X : \mathcal{D}_S \rightarrow M \), where \( M \) is a topological space, is said to be measurable if it is measurable with respect to the Borel \( \sigma \)-algebra of \( \mathcal{D}_S \).

**Theorem 2.** Let \( l \) be a continuous linear form on the Banach space \( \mathcal{D}_B(\mathbb{C}^n) \). Then \( l : \mathcal{D}_S(\mathbb{C}^n) \rightarrow \mathbb{C} \) is measurable.
Proof. The proof is split up into three steps.
If \( f = (f_1, \ldots, f_n) \in \mathcal{D}(\mathbb{C}^n) \) and \( m = (m_1, \ldots, m_n) \) where \( m_1, \ldots, m_n \) are complex Borel measures on \([0,1]\), then we shall write
\[
\int \langle f, dm \rangle = \sum_{j=1}^{n} \int f_j \, d\pi_j
\]

**step 1:** If \( \delta_a \) is the Dirac measure in the point \( a \) and if \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \), then we denote
\[
m = c\delta_a = (c_1\delta_a, \ldots, c_n\delta_a)
\]
It is known that the map \( f \mapsto f(a) \) is measurable on \( \mathcal{D}_S \) (see Billingsley [1]), so it follows that, in case \( m = c\delta_a \), the map
\[
f \mapsto \int \langle f, dm \rangle = \langle f(a), c \rangle
\]
is also measurable on \( \mathcal{D}_S \).

**step 2:** Next we are going to prove that for arbitrary complex measures \( m_1, \ldots, m_n \) on \([0,1]\) the map
\[
f \mapsto \int \langle f, dm \rangle
\]
is measurable on \( \mathcal{D}_S \).
For every \( k \in \mathbb{N} \) we define the \( 2^k \) intervals \( I_i^k \) by
\[
I_i^k = [(i - 1)/2^k, i/2^k) \quad i = 1, 2, \ldots, 2^k
\]
Moreover, for every \( f \in \mathcal{D} \) a sequence \( f_k \in \mathcal{D} \) is defined by:
\[
f_k = \left( \sum_{i=1}^{2^k} f(i/2^k) \mathbf{1}_{I_i^k} \right) + f(1) \, \mathbf{1}_{\{1\}}
\]
Now if \( k \to \infty \) one has (because \( f(t+) = f(t) \)) that \( f_k(t) \to f(t) \) for every \( t \in [0,1] \).
For all Borel sets \( A \subset [0,1] \) we write
\[
m(A) = (m_1(A), \ldots, m_n(A))
\]
and we define
\[
m_k = \left( \sum_{i=1}^{2^k} m(I_i^k) \delta_{i/2^k} \right) + m(\{1\})\delta_1
\]
Then
\[
\int \langle f, dm_k \rangle = \sum_{i=1}^{2^k} \langle f(i/2^k), m(I_i^k) \rangle + \langle f(1), m\{1\} \rangle = \int \langle f_k, dm \rangle
\]
So by Lebesgue’s bounded convergence theorem, we have for all \( f \in \mathcal{D} \)

\[
\int \langle f, dm \rangle = \lim_{k \to \infty} \int \langle f, dm_k \rangle
\]

By step 1 the maps

\[
f \mapsto \int \langle f, dm_k \rangle
\]

are measurable on \( \mathcal{D}_S \). It follows from this that the map

\[
f \mapsto \int \langle f, dm \rangle ,
\]

being the pointwise limit of a sequence of measurable maps, is measurable on \( \mathcal{D}_S \).

step 3: If \( \varphi \in \ell^1([0, 1], \mathbb{C}^n) \) then the map

\[
f \mapsto \sum_{a \in [0, 1]} \langle f(a) - f(a-), \varphi(a) \rangle
\]

is measurable on \( \mathcal{D}_S \).

To prove this, we observe that the set \( \{ a \mid \varphi(a) \neq 0 \} \) is at most countably infinite. Measurability is now easily verified, for the maps

\[
f \mapsto f(a) \quad \text{and} \quad f \mapsto f(a-)
\]

are measurable on \( \mathcal{D}_S \).

Finally, by step 2, step 3, and theorem 1 we conclude that every continuous linear form on \( \mathcal{D}_B \) is measurable on \( \mathcal{D}_S \). This proves the theorem.

The following theorem gives a characterization of the Borel \( \sigma \)-algebra of \( \mathcal{D}_S \).

**Theorem 3.** The Borel \( \sigma \)-algebra of \( \mathcal{D}_S \) is generated by the maps \( l: \mathcal{D}_S \to \mathbb{C} \), where \( l \in \mathcal{D}_B^* \).

**Proof.** This is a direct consequence of theorem 2 and the fact that the maps of type \( f \mapsto f(a) \) generate the Borel \( \sigma \)-algebra of \( \mathcal{D}_S \) (see Billingsley [1] or apply Fernique’s theorem, see Schwartz [7]).

The theorem above enables us to prove:

**Theorem 4.** If \( T: \mathcal{D}_B(\mathbb{C}^m) \to \mathcal{D}_B(\mathbb{C}^n) \) is a bounded linear operator then \( T: \mathcal{D}_S(\mathbb{C}^m) \to \mathcal{D}_S(\mathbb{C}^n) \) is measurable.

**Proof.** To prove that \( T: \mathcal{D}_S(\mathbb{C}^m) \to \mathcal{D}_S(\mathbb{C}^n) \) is measurable it is, by theorem 3, sufficient to prove that for all \( l \in \mathcal{D}_B^*(\mathbb{C}^n) \) the composition \( l \circ T: \mathcal{D}_S(\mathbb{C}^m) \to \mathbb{C} \) is measurable. This is trivial, because \( l \circ T \in \mathcal{D}_B^*(\mathbb{C}^m) \).

**Closing remarks**

In stochastic analysis one is sometimes encountered with variables assuming values in \( \mathcal{D}_S \). By theorem 3, measurability of such variables is equivalent to scalar measurability with respect to the Banach space \( \mathcal{D}_B \). There is no loss of measurability if bounded linear transformations are applied (see for example J. Kormos e.a. [4] or T. van der Meer [5]).
References


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