On a generalized nonlinear equation of Schrödinger type.

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Abstract

Here is established the global existence of smooth solutions to a generalized nonlinear equation of Schrödinger type in the usual Sobolev spaces H^s and certain weighted Sobolev spaces by using Leray-Schauder fixed point theorem and delicate a priori estimates.

1 Introduction

In recent years, the initial (initial-boundary) value problem for the nonlinear Schrödinger (NLS) equation

$$i\partial_t u + u_{2x} + 2|u|^2 u = 0 \tag{1.1}$$

and its generalized forms have been widely studied in a lot of papers such as [3-12, 19, 20]. The high order nonlinear Schrödinger equation is

$$i\partial_t u + u_{4x} + 8|u|^2 u_{2x} + 2u^2 u^*_{2x} + 4|u_x|^2 u + 6u_x^2 u^* + 6|u|^4 u = 0 \tag{1.2}$$

which is the second equation in the Lax hierarchy of NLS equation [1,3,16,21,22].

NLS equations have been of great interest due to their occurrences as mathematical models in several scopes of physics and their implication in the development of solitons and inverse scattering transform theory [1, 2, 4, 14, 16, 17, 21, 22]. Particularly, it is found that the nonlinear Schrödinger equation (1.1) and the following

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generalized Schrödinger type equation

$$i\partial_t u + u_{2x} + 2|u|^2 u + \beta \left(u_{4x} + 8|u|^2 u_{2x} + 2u^2 u^*_{2x} + 4|u_x|^2 u + 6u_x^2 u^* + 6|u|^4 u \right) = 0 \tag{1.3}$$

are closely related to the isotropic Heisenberg spin ferromagnetic chain equation [14, 17 and references therein], which arises in high- energy physics, nuclear physics, condensed matter physics and statistical mechanics, ect.. It has been pointed out in [14, 17] that the Heisenberg spin ferromagnetic chain equation and its natural extension of $4th$ order correction are gauge equivalent to equations (1.1) and (1.3), respectively. Here I would like to apologize for some omission of other papers related to Schrödinger equation since there is a large amount of work on this subject.

The purpose of this paper is to study the convergence of solutions(as α or $\beta \to 0$)to the following initial value problem

$$iu_t + \alpha(u_{2x} + 2|u|^2 u) + \beta(u_{4x} + 8|u|^2 u_{2x} + 2u^2 u^*_{2x} + 4|u_x|^2 u + 6u_x^2 u^* + 6|u|^4 u) = 0 \tag{1.4}$$

$$u(x,0) = \phi(x) \tag{1.5}$$

where subscripts stand for partial differentiation, $i = \sqrt{-1}$, $|u|$ is the norm of complex-valued function u, and u^* is the complex conjugate of u. Here the real couple $(\alpha, \beta) \neq (0,0)$. The problem is discussed in the usual Sobolev spaces H^s and weighted Sobolev spaces J_r^s (see definition below). Furthermore, the main results lie mathematically in that the convergence of solutions for the natural extension of the 4th order correction of the Heisenberg spin ferromagnetic equation to its own ones is, to some extent, valid. Such a kind of analysis is, of course, very important in physics. In a future paper, based on the convergence results of this paper, various physical phenomena and relations between equation (1.4) and the Heisenberg spin ferromagnetic chain equation will be disclosed from a pure physical point of view. The methods here follow the same lines as those of the author's [3].

The paper is organized as follows: Section 2 deals with preliminaries. Section 3 deals with the a priori estimates. Sections 4 and 5 are devoted to solving problem (1.4), (1.5) in H^s and J_r^s, respectively.

2 Preliminaries and Results

This part is to give the notation that will be used throughout this paper and to announce the results of this paper.

As usual, $L^p(R)$, $1 \leq p \leq \infty$, and $H^s(R)$, $s \in R$ will be the usual Lebesgue and Sobolev spaces with norms $| \bullet |_p$ and $\| \bullet \|_s$, respectively. If I is an interval and X a Banach space with norm $\| \bullet \|_X$, $L^p(I; X) \equiv \{u : I \longrightarrow X \;\; such \;\; that \;\; \|u\|_X \in L^p(I)\}$. $W_p^r(0,T; H^k(R))$ denotes the space of function $f(x,t)$ that has derivatives $\partial_t^s \partial_x^h f(t,x) \in L^p(0,T; L^2(R))$ with $0 \leq s \leq r,\;\; 0 \leq h \leq k$. We denote by $C,\;\; C(.,.,.)$ generic constants, not necessarily the same at each occurrence, which depend in an increasing way on the indicated quantities.

Let $Z = \{0, 1, 2, \cdots\}$, $\omega(x) = (1+x^2)^{1/2}$ and $S(R)$ be the Schwartz space of all rapidly decreasing infinitely differentiable functions on R. Then for any $s,\;\; r \in$

R, H_r^s is the completion of $S(R)$ under the norm, $||u||_{r,s} = |\omega^r(1-\partial_x^2)^{\frac{s}{2}}u|_2$. Let $J_r^s = H_r^0 \bigcap H_0^s$ with the norm $|||u|||_{r,s} = (||u||_{r,0}^2 + ||u||_{0,s}^2)^{\frac{1}{2}}$. For H_r^s and J_r^s we have [3, 19]

Lemma 2.1.

a) $H_{r'}^{s'} \subseteq H_r^s, \quad J_{r'}^{s'} \subseteq J_r^s \quad s \leq s', \quad r \leq r'$

b) $[H_{r_1}^{s_1}; H_{r_2}^{s_2}]_\theta = H_{(1-\theta)r_1+\theta r_2}^{(1-\theta)s_1+\theta s_2}, \quad 0 < \theta < 1, \quad s_j, \quad r_j \in R, \quad j = 1, 2$
where $[;]_\theta$ *denotes the complex interpolation.*

c) $J_r^s \subseteq [H_r^0, H_0^s]_\theta = H_{(1-\theta)r}^{\theta s}, \quad 0 < \theta < 1, \quad s, r \in R.$

d) $\bigcap_{r,s\in Z} J_r^s = \bigcap_{r,s\in Z} = S(R)$

e) $(J_r^s)' = H_{-r}^0 + H_0^{-s}$

f) Let $r, \quad s > 0$. *If* $u \in J_r^{rs}$, *then* $u \in H_{r-1}^s$ *and* $||u||_{r-1,s} \leq C(r,s)|||u|||r, rs$

g) Let $r \in R$ *and* $s \geq 0$. *Then for* $h \in Z$ *we have*

$$|||\partial_x^h u|||_{r,s} \leq C(r,s)|||u|||r', s'$$

where $r' = (s+h)r/s$ *and* $s' = s + h$.

h) If $s > \frac{1}{2}$, *then for all* $u, \quad v \in J_r^s$ *we have*

$$|||uv|||_{r,s} \leq C(r,s)|||u|||_{r,s}|||v|||_{r,s}$$

Remark: By the lemma above and Sobolev embedding theorem we see that if $u \in J_r^{r(s+1)}$ with $s, \quad r \in Z$ and $r \neq 0$, then

$$Sup_{x\in R}|\omega(x)^{r-1}\partial_x^s u(x)| \leq C(r,s)||u||_{r-1,s+1} \leq C(r,s)|||u|||_{r,(s+1)r}$$

that is $|\partial_x^s u(x)| = O(|x|^{-(r-1)})$ as $|x| \longrightarrow \infty$.

The following statement follows from a straightforward calculation and its proof is omitted here.

Lemma 2.2. *Let* $\mu(x) \in C_0^\infty(R)$ *such that* $0 \leq \mu \leq 1, \quad \mu = 1$ *if* $|x| \leq 1$ *and* $\mu = 0$ *if* $|x| \geq 2$. *Let* $\mu_\epsilon(x) = \mu(\epsilon x)$ *for* $0 < \epsilon < 1$. *Then as* $\epsilon \longrightarrow 0$

$$\mu_\epsilon(x) \longrightarrow 1 \quad uniformly \quad on \quad any \quad bounded \quad set \quad of \quad R,$$

$$\partial_x^j \mu_\epsilon(x) \longrightarrow 0 \quad uniformly \quad on \quad R \quad for \quad j \neq 0$$

Moreover for any $j \in Z$ *we have*

$$|\partial_x^h(x)| \leq C(j)\epsilon^h(\omega(x))^{-(j-h)}, \quad 0 \leq h \leq j$$

where $C(j) > 0$ *is independent of* ϵ .

Lemma 2.3. *Let* $q, \quad r$ *be any real numbers satisfying* $1 \leq q, \quad r \leq \infty$ *and* $j, \quad m \in Z$ *such that* $j \leq m$. *Then for* u *with* $\partial_x^m u \in L^r, \quad u \in L^q$ *we have*

$$|\partial_x^j u|_p \leq C(j,m,q,r,a)|\partial_x^m u|_r^a |u|_q^{1-a}$$

where $1/p = j + a(1/r - m) + (1-a)1/q$ *for all* a *in the interval* $j/m \leq a \leq 1$.

Lemma 2.4. *(Gronwall's inequality). Suppose that* $g(t), \quad h(t)$ *satisfy the inequality*

$$g(t) \leq M_1 + M_2 \int_0^t g(s)h(s)ds, \quad for \quad any \quad 0 \leq t \leq T,$$

where $M_1, \quad M_2$ *are two nonnegative constants. Moreover,* $h(t)$ *satisfies* $\int_0^T h(t)dt < \infty$*. Then we have*

$$g(t) \leq M_1 exp(M_2 \int_0^T h(t)dt), \quad t \in [0,T]$$

Our results are as follows:

Theorem 2.5. *Let* T *be any given positive constant. For any initial data* $\phi \in H^s, \quad s \in Z$ *and* $s \geq 4$*, then problem (1.4), (1.5) has a solution* u *such that*

$$u \in \bigcap_{k+4h \leq s} W_\infty^h(0,T;H^k(R))$$

where $k, \quad h \in Z$*.*

Theorem 2.6. *Let* T *be any given positive constant. For any initial data* $\phi \in J_r^s(R)$ *with* $s, \quad r \in Z$ *and* $s \geq max(3r, 4)$*,then problem (1.4), (1.5) has a solution* u *such that*

$$u \in \bigcap_{s'+4h \leq s; r's+4hr \leq rs} W_\infty^h(0,T;J_{r'}^{s'})$$

where $s', \quad h \in Z$ *and* $r' \geq 0$ *real.*

Corollary 2.7. *For any* $T > 0$ *fixed, if* $\phi \in S(R)$*, then the solution* u *to problem (1.4), (1.5) belongs to* $C^\infty([0,T];S(R))$*.*

Corollary 2.8. *Let* $u_{\alpha,\beta}$ *be the solution obtained in Theorems 2.5 and 2.6. Then as* $\beta \longrightarrow 0(\alpha \longrightarrow 0)$*,* $u_{\alpha,\beta}$ *converges in*

$$\bigcap_{k+4h \leq s} W_\infty^h(0,T;H^k)(\bigcap_{s'+4h \leq s; r's+4hr \leq rs} W_\infty^h(0,T;J_{r'}^{s'}))$$

to a solution to the following equations

$$iu_t + \alpha(u_{2x} + 2|u|^2 u) = 0$$

and

$$iu_t + \beta\left(u_{4x} + 8|u|2u_{2x} + 2u^2 u_{2x}^* + 4|u_x|^2 u + 6u_x^2 u^* + 6|u|^4 u\right) = 0$$

respecctively.

Corollary 2.8 is the main result of this paper, which will be served as a basis in disclosing some important physical phenomena of the Heisenberg spin ferromagnetic equation in a future paper.

3 Global a priori estimates

In the study of global existence for the dispersive equations, the global a priori estimates play an important role. It is a common technique to use the conservation laws to establish the global a priori estimates of solutions to the lower order dispersive nonlinear equations such as the usual KdV equation, the standard Schrödinger equation (1.1). Generally speaking, the first known explicit conservation laws associated to them are sufficient. But when dealing with the higher order dispersive equations infinite many conservation laws are needed(see [3]). Although it is known that the standard Schrödinger equation has infinite many conservation laws, one finds it difficult to calculate them all explicitly. In this part we succeed in deriving the global a priori estimates of the solution to (1.4), (1.5) by constructing substitutes for the higher order conservation laws corresponding to equation (1.4).

Lemma 3.1. *Let u be the solution of problem (1.4), (1.5) with the given initial data $\phi \in H^s$, $s \in Z$, and $s \geq 4$, then we have the following identities*

$$E_0(t) = \int |u|^2 dx = E_0(0),$$

$$\begin{aligned} E_1(t) &= \alpha \int (-|u_x|^2 + |u|^4) dx \\ &\quad + \beta \int \left(|u_{2x}|^2 - 8|u|^2|u_x|^2 - 2Re(u^2(u_x^*)^2) + 2|u|^6 \right) dx \\ &\equiv \alpha I_1 + \beta I_2 = E_1(0) \end{aligned}$$

$$\begin{aligned} E_2(t) &= \alpha \int \left(|u_{2x}|^2 - 8|u|^2|u_x|^2 - 2Re(u^2(u_x^*)^2) + 2|u|^6 \right) dx \\ &\quad + \beta \int (-|u_{3x}|^2 + 12|u|^2|u_{2x}|^2 + 2Re(u^2(u_{2x}^*)^2) - 12Re(|u_x|^2 u u_{2x}^*) \\ &\quad + 6Re(|u|^4 u u_{2x}^*) - 4|u|^2((|u|^2)_x)^2 - 24|u|^4|u_x|^2 - 11|u_x|^4 + 5|u|^8) dx \\ &\equiv \alpha I_2 + \beta I_3 = E_2(0), \end{aligned}$$

$$\begin{aligned} E_3(t) &= \alpha I_3 + \beta \int \left| u_{4x} + 8|u|^2 u_{2x} + 2u^2 u_{2x}^* + 4|u_x|^2 u + 6u_x^2 u^* + 6|u|^4 \right|^2 dx \\ &= E_3(0) \end{aligned}$$

Proof. Multiply (1.4) by u^*, integrate in x and t and take the imaginary part to obtain the first identity. Multiply (1.4) by u_t^*, integrate in x and t, and take the real part to obtain the second identity. Multiply (1.4) by $(u_{2x}^* + 2|u|^2 u^*)_t$, $(\frac{\delta I_2}{\delta u})_t$, proceed as before to obtain the third and fourth identities. Here $\frac{\delta I_2}{\delta u}$ is the functional derivative of I_2(for definition see [1, 21]). In fact we can use the properties of complete integrability [1, 21] of NLS equation (1.1) as a Hamiltonian system in the above proof.

Corollary 3.2. *Under the condition of lemma 3.1 we have*

$$\left(|\alpha| Sup_{0 \leq t < \infty} ||u(t)||_3^2 + |\beta| Sup_{0 \leq t < \infty} ||u(t)||_4^2 \right) \leq C(\alpha, \beta, ||\phi||_4)$$

This proposition can be proved by using lemma 3.1, lemma 2.3, and Young's inequality.

Lemma 3.3. *Let $T > 0$ be arbitrarily given. Under the conditions of lemma 3.1, we have*

$$Sup_{0\le t\le T}||u(t)||_s \le C(||\phi||_s, T) \tag{3.2}$$

Proof. In order to show (3.2) we consider the following derivative

$$\frac{d}{dt}\left\{\int \left[|u_{sx}|^2 + ReC_s(u^2(u^*_{(s-1)x})^2) + D_s|u|^2|u_{(s-1)x}|^2\right] dx\right\}$$

where C_s and D_s are constants to be determined later. For the first two terms in the above derivative we have

$$\begin{aligned}
&\frac{d}{dt}\left[\int |u_{sx}|^2 dx + ReC_s\int u^2(u^*_{(s-1)x})^2 dx\right]\\
&= 2Re\int u^*_{sx}u_{sxt}dx + 2C_sRe\int uu_t(u^*_{(s-1)x})^2dx + 2C_sRe\int u^2\partial_x^{s-1}u^*\partial^{s-1}u_t^*dx\\
&= 2Im\int \partial_x^s u^*\partial_x^s\left[\beta\left(8|u|^2u_{2x} + 2u^2u^*_{2x} + 4|u_x|^2u + 6u_x^2u^* + 6|u|^4u\right)\right.\\
&\left.+\alpha(u_{2x} + 2|u|^2u)\right]dx + 2\beta C_sIm\int uu_{4x}(u^*_{(s-1)x})^2dx\\
&+2\beta C_sIm\int u\left(8|u|^2u_{2x} + 2u^2u^*_{2x} + 4|u_x|^2u + 6u_x^2u^* + 6|u|^4u\right)\left(\partial_x^{s-1}u^*\right)^2dx\\
&+2\alpha C_sIm\int u\left(u_{2x} + 2|u|^2u\right)\left(\partial_x^{s-1}u^*\right)^2dx - 2\beta C_sIm\int u^2\partial_x^{s-1}u^*\partial_x^{s-1}u^*_{4x}dx\\
&-2\beta C_sIm\int u^2\partial_x^{s-1}u^*\partial_x^{s-1}\left[8|u|^2u^*_{2x} + 2(u^*)^2u_{2x}\right]dx\\
&\qquad -2C_sIm\int u^2\partial_x^{s-1}u^*\partial_x^{s-1}[\beta\left(4|u_x|^2u^* + 6(u_x^*)^2u + 6|u|^4u^*\right)\\
&\qquad\qquad +\alpha\left(u^*_{2x} + 2|u|^2u^*\right)]dx\\
&= 2\beta Im\int \partial_x^s u^*\partial_x^s(8|u|^2u_{2x} + 2u^2u^*_{2x})dx + 2\beta Im\int \partial_x^s u^*\partial_x^s(4|u_x|^2u + 6u_x^2u^*)dx\\
&+2\beta C_sIm\int uu_{4x}(\partial_x^{s-1}u^*)^2dx - 2\beta C_sIm\int u^2\partial_x^{s-1}u^*\partial_x^{s-1}u^*_{4x}dx\\
&-2\beta C_sIm\int u^2\partial_x^{s-1}u^*\partial_x^{s-1}[8|u|^2u^*_{2x} + 2(u^*)^2u_{2x}]dx + \text{Remaining term}\\
&= \sum_{j=1}^5 L_j + \text{Remaining term}
\end{aligned} \tag{3.3}$$

For the remaining term we have

$$|\text{Remaining term}| \le C(1 + |u_{sx}|_2^2) \tag{3.4}$$

In what follows we estimate each term L_j, Using integration by parts, lemma 1.3 and Young's inequality we have

$$L_1 = 2\beta Im\int \partial_x^s u^* \partial_x^s (8|u|^2 u_{2x} + 2u^2 u_{2x}^*)dx$$

$$\leq 2\beta \Big[sIm\int (8|u|^2)_x \partial_x^s u^* \partial_x^{s+1} u dx + 8Im\int |u|^2 \partial_x^s u^* \partial_x^{s+2} u dx$$

$$+2Im\int u^2 \partial_x^s u^* \partial_x^{s+2} u^* dx + 2sIm\int (u^2)_x \partial_x^s u^* \partial_x^{s+1} u^* dx\Big] + C(1+|u_{sx}|_2^2)$$

$$\leq 2\beta(s-1)Im\int (8|u|^2)_x \partial_x^s u^* \partial_x^{s+1} u dx - 4\beta Im\int u^2 (\partial_x^{s+1} u^*)^2 dx$$

$$+C(1+|u_{sx}|_2^2) \tag{3.5}$$

$$L_2 = 2\beta Im\int \partial_x^s u^* \partial_x^s (4|u_x|^2 u + 6u_x^2 u^*)dx$$

$$\leq 12\beta Im\int u^* \partial_x^s u^* \partial_x^s (u_x)^2 dx + 8\beta Im\int u \partial_x^s u^* \partial_x^s |u_x|^2 dx + C(1+|u_{sx}|_2^2)$$

$$\leq 24\beta Im\int u^* u_x \partial_x^s u^* \partial_x^{s+1} u dx + 8\beta Im\int u u_x^* \partial_x^s u^* \partial_x^{s+1} u dx + C(1+|u_{sx}|_2^2)$$

$$= 2\beta \Big[Im\int (8|u|^2)_x \partial_x^s u^* \partial_x^{s+1} u dx + 4Im\int (u_x^* u)_x |\partial_x^s u|^2 dx\Big] + C(1+|u_{sx}|_2^2)$$

$$\leq 2\beta Im\int (8|u|^2)_x \partial_x^s u^* \partial_x^{s+1} u dx + C(1+|u_{sx}|_2^2) \tag{3.6}$$

$$L_3 = 2\beta C_s Im\int u u_{4x} (\partial_x^{s-1} u^*)^2 dx$$

$$= -2\beta C_s Im\Big\{\int u_x u_{3x} (\partial_x^{s-1} u^*)^2 + 2\int u u_{3x} \partial_x^{s-1} u^* \partial_x^s u^* dx\Big\}$$

$$\leq C(1+|u_{sx}|_2^2) \tag{3.7}$$

$$L_4 = -2\beta C_s Im\int u^2 \partial_x^{s-1} u^* \partial_x^{s+3} u^* dx$$

$$\leq -2\beta C_s \int u^2 (\partial_x^{s+1} u^*)^2 dx + C(1+|u_{sx}|_2^2) \tag{3.8}$$

For L_5 we can easily obtain

$$L_5 \leq C(1+|u_{sx}|_2^2) \tag{3.9}$$

Now consider

$$\frac{d}{dt} D_s \int |u|^2 |\partial_x^{s-1} u|^2 dx$$

$$= 2D_s Re\int |\partial_x^{s-1} u|^2 u^* u_t dx + 2D_s Re\int |u|^2 \partial_x^{s-1} u^* \partial_x^{s-1} u_t dx$$

$$= 2D_s Re\left\{-i\int |\partial_x^{s-1}u|^2u^*\left[\beta(u_{4x}+8|u|^2u_{2x}+2u^2u_{2x}^*+4|u_x|^2u\right.\right.$$

$$\left.\left.+6(u_x)^2u^*+6|u|^4u)+\alpha(u_{2x}+2|u|^2u)\right]dx\right\}$$

$$+2D_s Re\left\{-i\int |u|^2\partial_x^{s-1}u^*\partial_x^{s-1}\left[\beta(u_{4x}+8|u|^2u_{2x}+2u^2u_{2x}^*\right.\right.$$

$$\left.\left.+4|u_x|^2u+6(u_x)^2u^*+6|u|^4u)+\alpha(u_{2x}+2|u|^2u)\right]dx\right\}$$

$$\leq 2\beta D_s Im\int |\partial_x^{s-1}u|^2u^*u_{4x}dx+2\beta D_s Im\int |u|^2\partial_x^{s-1}u^*\partial_x^{s+3}udx+C(1+|u_{sx}|_2^2)$$

$$\leq \frac{\beta}{2}D_s Im\int(8|u|^2)_x\partial_x^s u^*\partial_x^{s+1}udx+C(1+|u_{sx}|_2^2) \tag{3.10}$$

Considering (3.3)-(3.10) and taking $C_s=-2,\quad D_s=-4s$, we have

$$\frac{d}{dt}\left\{\int |u_{sx}|^2dx-4s\int |u|^2|\partial_x^{s-1}u|^2dx-2Re\int u^2(\partial_x^{s-1}u^*)^2dx\right\}$$

$$\leq C(1+\|u_{sx}\|_2^2) \tag{3.11}$$

Thus, integrating the above inequality with respect to t we get

$$|u_{sx}|_2^2\leq C+C\int_0^t|u_{sx}(\tau)|_2^2d\tau,\quad t\in[0,T] \tag{3.12}$$

Therefore, applying Gronwall's inequality to (3.12) gives

$$|u_{sx}|_2^2\leq C,\quad t\in[0,T]$$

which completes the proof of the lemma.

4 Proof of Theorem 2.5

The proof depends on a series of lemmas. We first establish the existence of a unique global solution $u^\epsilon(x,t)$ to the initial value problem

$$u_t+\epsilon(-u_{6x}+u_{4x}-u_{2x})=-i(\alpha K_1(u)+\beta K_2(u)),\quad 0<\epsilon<1 \tag{4.1}$$

with the initial data $u(x,0)=\phi(x)$, where

$$K_1(u)=u_{2x}+2|u|^2u,$$

$$K_2(u)=u_{4x}+8|u|^2u_{2x}+2u^2u_2^*x+6u^*(u_x)^2+4|u_x|^2u+6|u|^4u.$$

We reduce (4.1) to a problem of finding fixed points of completely continuous maps by conidering the family of nonlinear problems

$$u_t+\epsilon(-u_{6x}+u_{4x}-u_{2x})=-i\tau(\alpha K_1(u)+\beta K_2(u)),\quad 0<\epsilon<1,\quad 0\leq\tau\leq 1 \tag{4.2}$$

and the related linear problem

$$u_t + \epsilon(-u_{6x} + u_{4x} - u_{2x}) = -i\tau(\alpha K_1(v) + \beta K_2(v)), \quad 0 < \epsilon < 1, \quad 0 \leq \tau \leq 1 \quad (4.3)$$

We introduce two sets $X(n,T)$, $Y(n,T)$ of functions on $R \times [0,T]$ with finite norms:

$$<< v >>^2_{X(n,T)} = \int_0^T |v_t|_2^2 dt + \int_0^T \|v(t)\|^2_{n+3} dt + Sup_{0\leq t\leq T}\|v(t)\|_n^2$$

and

$$<< v >>^2_{X(n,T)} = \int_0^T \|v(t)\|^2_{n+1} dt + Sup_{0\leq t\leq T}\|v(t)\|^2_{n-2}$$

respectively. Let $n \geq 4$. If $v \in Y(n,T)$ there exists a global solution $u(x,t)$ in $X(n,T)$ to (4.3) which is uniquely determined by its initial values. Thus problem (4.3) defines a nonlinear operator $u^\tau = \Phi(v;\tau)$, whose fixed points are solutions of (4.2). The fixed points of Φ for $\tau = 1$ are solutions of (4.1). In the following we first apply the Leray-Schauder theorem [13] todetermine the fixed points of Φ. Then we show that the solution u^ϵ to problem (4.1) converges to a solution of (1.4). Now this limiting procedure is well-known, Hence, in this part we are mainly concerned with the solvability of (4.1) and the global estimates of its solutions. The following statements are aimed at verifying the conditions of Leray-Schauder theorem.

Lemma 4.1. *Let $n \geq 4$. If $u^\tau(x,t)$ is a solution in $X(n,T)$ of (4.2), then*

$$\int_0^T |\partial_t u^\tau(t)|_2^2 dt \quad and \quad \epsilon \int_0^t \|u^\tau(t)\|^2_{n+3} dt + Sup_{0\leq t\leq T}\|u^\tau(t)\|_n^2 \qquad (4.4)$$

are majored by a constant depending only on $\|\phi\|_n$ and $T < \infty$, most importantly it is independent of τ.

Proof. It should be pointed out that in the process of the proof we use the property of complete integrability of NLS equation (1.1) as a Hamiltonian system [1,21].

1) Multiply equation (4.2) by u^*, integrate over R and take the real part of the resulting expression to obtain

$$\frac{d}{dt}|u(t)|_2^2 + 2\epsilon \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 = 0$$

which leads to

$$|u(t)|_2^2 + 2\epsilon \int_0^t \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 dt = |\phi|_2^2 \qquad (4.5)$$

2) Procced as in lemma 3.1 to obtain

$$\frac{dE_j}{dt} + 2\epsilon Re \int (-u_{6x} + u_{4x} - u_{2x}) \frac{\delta E_j}{\delta u} dx = 0, \quad j = 1, \quad 2, \quad 3 \qquad (4.6)$$

By Sobolev embedding theorem, (4.5) and (4.6) one obtains

$$Sup_{0\leq t\leq\infty}(|\alpha|\|u\|_3^2 + |\beta|\|u\|_4^2) + \epsilon \int_0^\infty (|\alpha|\|u\|_6^2 + |\beta|\|u\|_7^2) dt \leq C(\alpha, \beta, \|\phi\|_4) \quad (4.7)$$

3) Assume that the second result in (4.4) is proved for all values less than or equal to $s-1$. We now prove it for s. To this end, use (3.11) in proposition 3.2 to obtain

$$\frac{dE_s}{dt} + 2\epsilon Re \int (-u_{6x} + u_{4x} - u_{2x}) \frac{\delta E_s}{\delta u} dx \le C(1 + |u_{sx}|_2^2) \tag{4.8}$$

where $E_s(t) = \int \{|u_{sx}|^2 - 4s|u|^2|\partial_x^{s-1}u|^2 - 2Re(u^2(\partial_x^{s-1}u^*)^2)\}\, dx$. By definition we have

$$\frac{\delta E_s}{\delta u} = (-1)^s u^*_{2sx} - 4s|\partial_x^{s-1}u|^2 u^* + (-1)^s 4s\partial_x^{s-1}\left(|u|^2\partial_x^{s-1}u^*\right)$$
$$-4u(\partial_x^{s-1}u^*)^2 + (-1)^s 4\partial_x^{s-1}\left((u^*)^2\partial_x^{s-1}u\right)$$

Now we have

$$2\epsilon Re \int (-u_{6x} + u_{4x} - u_{2x}) \frac{\delta E_s}{\delta u} dx$$
$$= 2\epsilon \sum_{j=1}^{3} |\partial_x^{j+s}u|_2^2 + 2\epsilon Re \int (-u_{6x} + u_{4x} - u_{2x})\{(-1)^s 4s\partial_x^{s-1}(|u|^2\partial_x^{s-1}u^*)$$
$$+(-1)^s 4\partial_x^{s-1}((u^*)^2\partial_x^{s-1}u) - 4s|\partial_x^{s-1}u|^2u^* - 4u(\partial_x^{s-1}u^*)^2\}dx \tag{4.9}$$

Using Sobolev embedding theorem and Young's inequality we can obtain the following estimates

$$8s\epsilon|Re \int (-u_{6x} + u_{4x} - u_{2x})\partial_x^{s-1}\left(|u|^2\partial_x^{s-1}u^*\right) dx$$
$$\le \frac{\epsilon}{2}\left(|\partial_x^{s+3}u(t)|_2^2 + |\partial_x^{s+1}u(t)|_2^2\right) + C(\|\phi\|_3)\epsilon\left(|\partial_x^{s-1}u|_2^2 + |\partial_x^{s}u|_2^2 + |\partial_x^{s+1}u|_2^2\right) \tag{4.10}$$

$$8\epsilon\left|Re \int (-u_{6x} + u_{4x} - u_{2x})\partial_x^{s-1}\left((u^*)^2\partial_x^{s-1}u\right) dx\right|$$
$$\le \frac{\epsilon}{2}\left(|\partial_x^{s+3}u|_2^2 + |\partial_x^{s+1}u(t)|_2^2\right) + C(\|\phi\|_3)\epsilon\left(|\partial_x^{s-1}u|_2^2 + |\partial_x^{s}u|_2^2 + |\partial_x^{s+1}u|_2^2\right) \tag{4.11}$$

$$8\epsilon\left|Re \int (-u_{6x} + u_{4x} - u_{2x})u^*|\partial_x^{s-1}u|^2dx\right|$$
$$\le \epsilon\left(|u_{6x}|_2^2 + |u_{4x}|_2^2 + |u_{2x}|_2^2\right) + C(|\phi|_2)\epsilon|\partial_x^{s-1}u|_2^2|\partial_x^{s}u|_2^2, \tag{4.12}$$

$$8\epsilon\left|Re \int (-u_{6x} + u_{4x} - u_{2x}u(\partial_x^{s-1}u^*)^2dx\right|$$
$$\le \epsilon(|u_{6x}|_2^2 + |u_{4x}|_2^2 + |u_{2x}|_2^2) + C(|\phi|_2)\epsilon|\partial_x^{s-1}u|_2^2|\partial_x^{s}u|_2^2 \tag{4.13}$$

¿From (4.9) to (4.13) there appears that

$$E_s(t) + \epsilon\int_0^t \sum_{j=1}^{3} |\partial_x^{s+j}u(t)|_2^2dt \le C(\|\phi\|_s, T) + C\int_0^T (\epsilon|\partial_x^{s-1}u|_2^2 + 1)|\partial_x^{s}u|_2^2dt \tag{4.14}$$

By the expression of $E_s(t)$ and Young's inequality, from (4.14) it follows that

$$|u_{sx}|_2^2 + \epsilon \int_0^t \sum_{j=1}^{3} |\partial_x^{s+j} u(t)|_2^2 dt \le C(\|\phi\|_s, T) + C \int_0^T (\epsilon |\partial_x^{s-1} u|_2^2 + 1)|\partial_x^s u|_2^2 dt \quad (4.15)$$

Now applying Gronwall's inequality to (4.15) to complete the proof of the second inequality. The first inequality in (4.4) can be easily obtained by multiplying (4.2) by u_t^*, integrating in x and t and taking the real part. This finishes the proof.

Lemma 4.2. *If* $n \ge 4$ *and* $u \in X(n,T)$ *solves (4.3), then*

$$|u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 dt \le C(<< v >>_{Y(n,T)}^2 + |\phi|_2^2),$$

$$\int_0^T |u_t|_2^2 dt + \epsilon \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 \le C(<< v >>_{Y(n,T)}^2 + ||\phi||_3^2),$$

$$|\partial_x^h u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^{h+j} u(t)|_2^2 dt \le C(<< v >>_{Y(n,T)}^2 + ||\phi||_h^2), \quad h \le n$$

where C *depends on* ϵ, T, *and the size of* ϕ.

Proof. Multiply equation (4.3) by u^*, integrate and take the real part of the resultant expression to obtain

$$|u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 dt \le |\phi|_2^2 + \frac{1}{2} \int_0^T |u(t)|_2^2 + \frac{1}{2} \int_0^T |K(v)|_2^2 dt$$

which implies (by Gronwall's inequality)

$$|u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 dt \le C(|\phi|_2^2 + \int_0^T |K(v)|_2^2 dt)$$

where $K(v) = \alpha K_1(v) + \beta K_2(v)$, By the standard estimates and the definition of $Y(n,T)$ we have

$$|u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 dt \le C(|\phi|_2^2 + << v >>_{Y(n,T)}^2) \quad (4.16)$$

To obtain the second inequality of the lemma, multiply equation (4.3) by u_t^*, integrate in x and t, take the real part and proceed as before to finish the proof.

The final estimate is established by first taking ∂_{hx} of (4.3), multiply by u_{hx}^*. Then integrate the resulting expression in x and t and take the real part to obtain

$$|\partial_x^h u(t)|_2^2 + 2\epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^{h+j} u(t)|_2^2 dt \le |\partial_x^h \phi|_2^2 + \left| \int_0^T \int \partial_x^h u^* \partial_x^h K(v) dx dt \right| \quad (4.17)$$

where $K(v)$ is the same as above. Obviously,

$$|\beta|\left|\int_0^T\int \partial_x^h u^*\partial_x^{h+4} v dxdt\right| = |\beta|\left|\int_0^T\int \partial_x^{h+3} u^*\partial_x^{h+1} v dxdt\right|$$

$$\leq \frac{\epsilon}{2}\int_0^T |\partial_x^{h+3}u(t)|_2^2 dt + C_\epsilon \int_0^T |\partial_x^{h+1}v(t)|_2^2 dt \tag{4.18}$$

For $1 \leq h \leq 2$ we have

$$\left|\int_0^T\int \partial_x^h u^*\partial_x^h \widehat{K_1}(v) dxdt\right| = \left|\int_0^T\int \partial_x^{h+1} u^*\partial_x^{h-1} \widehat{K_1}(v) dxdt\right|$$

$$\leq \epsilon\int_0^T |\partial_x^{h+1}u(t)|_2^2 dt + C << v >>^2_{Y(n,T)} \tag{4.19}$$

where $\widehat{K_1}(v) = \alpha K_1(v) + \beta(8|v|^2 v_{2x} + 2v^2 v_{2x}^* + 6v^*(v_x)^2 + 4|v_x|^2 v + 6|v|^2 v)$.

For $h \geq 3$ we have

$$\left|\int_0^T\int \partial_x^h u^*\partial_x^h \widehat{K_1}(v) dxdt\right| = \left|\int_0^T\int \partial_x^{h+3} u^*\partial_x^{h-3} \widehat{K_1}(v) dxdt\right|$$

$$\leq \frac{\epsilon}{2}\int_0^T |\partial_x^{h+3}u(t)|_2^2 dt + C_\epsilon << v >>^2_{Y(n,T)} \tag{4.20}$$

Now the final result follows from (4.17)-(4.20).

Now we turn to the proof of the solvability of equation (4.1). If $n \geq 4$ and $\phi \in H^n(R)$, then it is known that there exists a unique solution $u \in X(n,T)$ to the linear parabolic equation (4.3) for each $\tau \in [0,1]$ and $v \in Y(n,T)$. This defines a nonlinear operator $u = \Phi(v;\tau)$, which for each $\tau \in [0,1]$ determines the solution to (4.3). To obtain the existence of fixed points of Φ we apply Leray-Schauder theorem. In what follows we proceed as in [13, 18] to verify the conditions of Leray-Schauder theorem.

Consider $\Phi(v;\tau)$ on the set B of $Y(n,T)$ consisting of funcions $v \in Y(n,T)$ satistying the inequalities (4.4) with the bound increased by adding the positive number $\gamma \geq 0$. We shall verify that $\Phi(v;\tau)$ on $B \times [0,1]$ is

a) continuous in v, uniformly in τ,

b) continuous in τ , uniformly in v,

c) completely continuous on $B \times [0,1]$ We also show that all fixed points of Φ lie strictly in the interior of the set B and that $v - \Phi(v;0)$ has nonzero degree. We then apply Leray-Schauder theorem for existence of fixed points of Φ.

I) $\Phi(v;\tau)$ is continuous with respect to v in $Y(n,T)$ uniformly for $\tau \in [0,1]$. Begin with two elements v, $\hat{v}$ from B so that $u = \Phi(v;\tau)$ and $\hat{u} = \Phi(\hat{v};\tau)$. Then

$U = u - \hat{u}$ and $V = v - \hat{v}$ satisfy

$$U_t + \epsilon(-U_{6x} + U_{4x} - U_{2x}) = -i\tau[K(v) - K(\hat{v})] \tag{4.21}$$

$$U(x,0) = 0$$

Proceeding as in the proof of lemma 4.2 we can obtain

$$|U(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j U(t)|_2^2 dt \le C << V >>^2_{Y(n,T)}$$

$$\int_0^T |U_t|_2^2 dt + \epsilon \sum_{j=1}^{3} |\partial_x^j U(t)|_3^2 \le C << V >>^2_{Y(n,T)},$$

$$|\partial_x^h U(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^{h+j} U(t)|_2^2 dt \le C << V >>^2_{Y(n,T)} .$$

where the C does not depend on $\tau \in [0,1]$. The above three equalities give the proof of a).

II) $\Phi(v;\tau)$ is continuous in τ, uniformly in v on $B \times [0,1]$. Let $u^{\tau+\Delta\tau} = \Phi(v;\tau + \Delta\tau)$, $u^\tau = \Phi(v;\tau)$ and $u = u^{\tau+\Delta\tau} - u^\tau$. By equation (4.3) we obtain

$$u_t + \epsilon(-u_{6x} + u_{4x} - u_{2x}) = -i\Delta\tau K(v)$$

By standard estimates and proceeding as in the proof of lemma 3.2 it is not difficult to know that

$$|u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^j u|_2^2 dt \le C\Delta\tau,$$

$$\int_0^T |u(t)|_2^2 dt + \epsilon \sum_{j=1}^{3} |\partial_x^j u(t)|_2^2 \le C\Delta\tau,$$

$$|\partial_x^h u(t)|_2^2 + \epsilon \int_0^T \sum_{j=1}^{3} |\partial_x^{h+j} u(t)|_2^2 dt \le C\Delta\tau,$$

which give

$$<< u(t) >>^2_{X(n,T)} \le C\Delta\tau$$

This completes the proof of b).

III) Φ on B is completely continuous. The preceding results show that Φ is continuous at every point in $B \times [0,1]$. By lemma 4.2 $\Phi(v;\tau)$ for each $\tau \in [0,1]$ maps a set B in $Y(n,T)$ into a set $\{u\}$ with

$$<< u >>_{X(n,T)} \le C << v >>_{Y(n,T)}$$

where C is independent of $\tau \in [0,1]$. From Lions [15], Schwartz [18] or an easy direct proof we know that a bounded set in $X(n,T)$ is compact in $Y(n,T)$. Thus, Φ maps an arbitrary set $(v;\tau)$ in $Y(n,T) \times [0,1]$ into a set that is compact in $Y(n,T)$. This shows that the set of values of $\Phi(v;\tau)$ on $B \times [0,1]$ is compact.

IV) all possible fixed points u lie strictly in the interior of the set B. The a priori estimates for the solution to (4.2) in lemma 4.1, and the definition of B show that all possible fixed points u of Φ lie strictly in the interior of the set B.

V) $v - \Phi(v;\tau)$ has nonzero degree. For $\tau = 0$, Φ maps B into a single point u, the unique solution to $u_t + \epsilon(-u_{6x} + u_{4x} - u_{2x}) = 0$. Thus $v - \Phi(v;\tau)$ is invertible and has nonvanishing degree.

Now the appalication of Leray-Schauder theorem tells us that for each $\tau \in [0,1]$ there exists at least one fixed point $u(x,t)$ for Φ , which for $\tau = 1$ is a solution to (4.1). The uniqueness of solutions to (4.1) can be proved by the standard energy estimates. Obviously, For $\epsilon \in (0,1)$ fixed and $\phi \in H^\infty$ there exists a unique solution $u \in C^\infty([0,T]; H^\infty)$ to (4.1), which satisfies the estimate (4.4) for every integer n.

Proof of Theorem 2.5. Here, only an outline is given. From the arguments above we know that for $\phi \in H^s, \quad s \geq 4$, there exists a unique solution $u^\epsilon(x,t)$ to (4.1) for each $\epsilon \in (0,1)$, which satisfies the estimate (4.4) for $n = s$ and the bound is independent of $\epsilon \in (0,1)$. By the standard limiting process $\epsilon \longrightarrow 0$ we know that there exists a subsequence $u^{\epsilon'}$ converging weakly star in $L^\infty(0,T;H^s)$ to some u, which is a desired solution to (1). In view of equation (1) one sees that $u_t \in L^\infty(0,T;H^{s-4})$. In this way we can obtain $u \in \bigcap_{k+4h\leq s} W^h_\infty(0,T;H^k(R))$.

5 Proof of Theorem 2.6

It is of interest to know that equation (1.4) has solutions decreasing faster than $H^s(R)$ convergence as x tends to infinity, in particular, solutions in the Schwartz space $S(R)$, for each t provided that its initial data is in $S(R)$. This can be realized by considering the initial value problem (1.4), (1.5) in the weighted Sobolev space $J^s_r(R)$. For the $J^s_r(R)$ convergence of solutions to problem (1.4),(1.5) we have Theorem 2.6 and Corollary 2.7. In the proof of our result we employ the same method as in Tsutsumi[19] or the present author [3]. In fact, the proof of this part can be reproduced from [3]. In order for the paper to be self-contained we give it out in detail.

Since $S(R)$ is dense in $J^s_r(R)$, there exists a sequence $\{\phi^k\} \in S(R)$ such that

$$\{\phi^k\} \longrightarrow \phi \quad \text{strongly} \quad \text{in} \quad J^s_r(R) \quad as \quad k \longrightarrow \infty \tag{5.1}$$

We first consider the parabolic regularization of equation (1): for $k \in Z \setminus 0$,

$$iu_t = \frac{i}{k}[u_{6x} - u_{4x} + u_{2x}] + \alpha(u_{2x} + 2|u|^2 u)$$

$$+\beta(u_{4x} + 8|u|^2 u_{2x} + 2u^2 u^*_{2x} + 6u^*(u_x)^2 + 4|u_x|^2 u + 6|u|^4 u) \tag{5.2}$$

with

$$u(x,0) = \phi^k(x) \tag{5.3}$$

For problem (5.2), (5.3) we have

Lemma 5.1. *For every fixed $k \in Z \setminus 0$, problem (5.2), (5.3) has a unique global solution $u^k \in C^\infty([0,T]; S(R)), \quad T > 0$.*

Proof: By the results of section 4 we know that problem (5.2), (5.3) has a unique global solution $u^k \in C^\infty([0,T]; H^\infty)$ and

$$Sup_{0\leq t\leq T}||u^k(t)||^2_s + \frac{1}{k}\int_0^T \left(\sum_{j=1}^{3} |\partial_x^{j+s} u^k(t)|^2_2\right) dt \leq C \tag{5.4}$$

where C is independent of k but depends on the size of ϕ and T. In order to prove the assertion of this lemma, it suffices to show that

$$u^k \in L^\infty(0,T;J_r^0(R)) \tag{5.5}$$

for every $r \in Z$. The proof of (5.5) together with

$$\partial_x^j u^k \in L^2(0,T;J_r^0(R)), \quad j=1, \quad 2, \quad 3 \tag{5.6}$$

is done by induction on r. when $r=0$, it is obvious. Assume that the result is known for all values less than or equal to $r-1 (r \geq 1)$. We prove it for r. Let $\mu(x), \quad \mu_\epsilon(x)$ be as given in Lemma 1.2. For simplicity we sometimes suppress k in u^k. Differentiating $||u^k(t)||_{r,0}^2$ with respect to t, using equation (5.2) and integrating by parts we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\mu_\epsilon u(t)||_{r,0}^2 + \frac{1}{k}\sum_{m=1}^{3}||\mu_\epsilon \partial_x^m u(t)||_{r,0}^2 \\
&+\frac{1}{k}\sum_{m=1}^{3} Re\int \partial_x^m u \left(\sum_{\substack{d+j+n+h=m \\ h<m}} \frac{m!}{d!j!n!h!}\partial_x^d \omega^{2r}\partial_x^j \mu_\epsilon \partial_x^n \mu_\epsilon \partial_x^h u^* \right) dx \\
&+Im\int (\mu_\epsilon)^2 \omega^{2r} u^* K(u) dx = 0
\end{aligned} \tag{5.7}$$

Since $|\partial_x^d \omega^{2r}| \leq C(d)(\omega(x))^{2r-d}$ and $|\partial_x^j \mu_\epsilon(x)| \leq C(j)(\omega(x))^{-j}$ we obtain

$$\begin{aligned}
&\frac{1}{k}\left| \sum_{m=1}^{3} Re\int \partial_x^m u \left(\sum_{\substack{d+j+n+h<m \\ h<m}} \frac{m!}{d!j!n!h!}\partial_x^d \omega^{2r}\partial_x^j \mu_\epsilon \partial_x^n \mu_\epsilon \partial_x^h u^* \right) dx \right| \\
&\leq \frac{1}{6k}\sum_{m=1}^{3}||\mu_\epsilon \partial_x^m u||_{r,0}^2 + C\sum_{j=0}^{3}||\partial_x^j u||_{r-1,0}^2
\end{aligned} \tag{5.8}$$

Now consider

$$Im\int (\mu_\epsilon)^2 \omega^{2r} u^* K(u) dx = \beta Im\int (\mu_\epsilon)^2 \omega^{2r} u^* u_{4x} dx + Im\int (\mu_\epsilon)^2 \omega^{2r} u^* \widehat{K}_1(u) dx \tag{5.9}$$

For the second term of (5.9) we have

$$|\text{the second term}| \leq \frac{1}{6k}\sum_{m=1}^{2}||\mu_\epsilon \partial_x^m u||_{r,0}^2 + C(k)||\mu_\epsilon u||_{r,0}^2 \tag{5.10}$$

For the first term of (5.9) we have

$$\beta Im\int (\mu_\epsilon)^2 \omega^{2r} u^* u_{4x} dx$$

$$= \beta Im \int u_{2x} \left(\sum_{\substack{d+j+n+h=2 \\ h<2}} \frac{2}{d!j!n!h!} \partial_x^d \mu_\epsilon \partial_x^j \mu_\epsilon \partial_x^n \omega^{2r} \partial_x^h u^* \right) dx$$

$$\leq \frac{1}{6k} \sum_{m=1}^{2} ||\mu_\epsilon \partial_x^m u||_{r,0}^2 + C(k) \sum_{m=0}^{2} ||\partial_x^m u||_{r-1,0}^2 \tag{5.11}$$

Considering (5.7)–(5.11) we get

$$\frac{d}{dt} ||\mu_\epsilon u(t)||_{r,0}^2 + \frac{1}{k} \sum_{j=1}^{3} ||\mu_\epsilon \partial_x^j u(t)||_{r,0}^2 \leq C(k) ||\mu_\epsilon u||_{r,0}^2 + C(k) \sum_{j=0}^{3} ||\partial_x^j u||_{r-1,0}^2 \tag{5.12}$$

where $C(k)$ is a positive constant independent of ϵ. We integrate (5.12) with respect to t and use the assumptions of the induction. Then Gronwall's inequality yields that

$$Sup_{0 \leq t \leq T} ||\mu_\epsilon u^k(t)||_{r,0}^2 \leq C \tag{5.13}$$

$$\int_0^T ||\mu_\epsilon \partial_x^j u^k(t)||_{r,0}^2 dt \leq C \quad j = 1, \quad 2, \quad 3 \tag{5.14}$$

where C is a positive constant independent of ϵ. Therefore, $\{\mu_\epsilon u^k\}$ remains in a bounded set of $L^\infty(0, T; J_r^0(R))$. So, taking the limit as $\epsilon \longrightarrow 0$, we see that $\mu_\epsilon u^k \longrightarrow u^k$ weakly star in $L^\infty(0, T; J_r^0(R))$ and the assertions (5.5), (5.6) hold for r since $L^\infty(0, T; J_r^0) = (L^1(0, T; H_{-r}^0 + L^2))'$. This ends the proof of the lemma.

Proof of Theorem 2.6. With lemma 5.1 we now consider the convergence of u^k as $k \longrightarrow \infty$. From lemma 5.1 we know that for any given $T > 0$,

$$Sup_{0 \leq t \leq T} ||u^k(t)||_{0,s} \leq C \tag{5.15}$$

holds for all integers s. From this it follows that $\{u^k\}$ forms a bounded set of $L^\infty(0, T; J_0^s(R))$. We next show that for $r > 0$, $\{u^k\}$ remains bounded in $L^\infty(0, T; J_r^0(R))$. In fact, we have

$$\frac{1}{2} \frac{d}{dt} ||u^k(t)||_{r,0}^2 = Re \int \omega^{2r} u^* u_t dx = \frac{1}{k} Re \int \omega^{2r} u^* (u_{6x} - u_{4x} + u_{2x}) dx$$

$$-\beta Im \int \omega^{2r} u^* u_{4x} dx - Im \int \omega^{2r} u^* \widehat{K_1}(u) dx \equiv B_1 + B_2 + B_3 + B_4 + B_5 \tag{5.16}$$

In what follows we bound each term B_j. Using Lemma 1.3 and (5.15) we obtain

$$|B_5| \leq C ||u||_{r,0}^2 \tag{5.17}$$

Integrating by parts and using Lemma 1.3 and (5.15) we have

$$|B_4| = |\beta| \left| Im \int u_{2x} (2r\omega^{2(r-1)} u^* + 4r(r-1)\omega^{2(r-2)} x^2 u^* + 4rx\omega^{2(r-1)} u_x^*) dx \right|$$

$$= |\beta| |Im \int u_{2x} (2r\omega^{2(r-1)} u^* + 4r(r-1)\omega^{2(r-2)} x^2 u^*) dx$$

$$-Im\int(4r\omega^{2(r-1)}u^*u_{2x}+4rx\omega^{2(r-1)}u^*u_{3x}+8r(r-1)x^2\omega^{2(r-2)}u^*u_{2x})dx|$$

$$=|\beta|\left|Im\int(2r\omega^{2(r-1)}u^*u_{2x}+4rx\omega^{2(r-1)}u^*u_{3x}+4r(r-1)x^2\omega^{2(r-2)}u^*u_{2x})dx\right|$$

$$\leq C\left(||u||_{r,0}^2+||u_{2x}||_{r-1,0}^2+||u_{3x}||_{r-1,0}^2\right) \tag{5.18}$$

$$B_3=\frac{1}{k}Re\int\omega^{2r}u^*u_{2x}dx=-\frac{1}{k}Re\int(\omega^{2r}u_x^*+2r\omega^{2(r-1)}xu^*)u_xdx$$

$$\leq\frac{1}{2k}||u_x||_{r,0}^2+C||u||_{r-1,0}^2 \tag{5.19}$$

$$B_2=-\frac{1}{k}Re\int\omega^{2r}u^*u_{4x}dx=-\frac{1}{k}||u_{2x}||_{r,0}^2$$

$$-\frac{1}{k}Re\int\left(2r\omega^{2(r-1)}u^*u_{2x}+4rx\omega^{2(r-1)}u^*u_{3x}+4r(r-1)\omega^{2(r-2)}x^2u^*u_{2x}\right)dx$$

$$\leq-\frac{1}{2k}||u_{2x}||_{r,0}^2+C(||u||_{r,0}^2+||u_{2x}||_{r-1,0}^2+||u_{3x}||_{r-1,0}^2) \tag{5.20}$$

$$B_1=\frac{1}{k}Re\int\omega^{2r}u^*u_{6x}dx=-\frac{1}{k}||u_{3x}||_{r,0}^2$$

$$-\frac{1}{k}Re\int[(\omega^{2r})_{2x}u^*+3(\omega^{2r})_xu_{2x}^*+3(\omega^{2r})_{2x}u_x^*)u_{3x}]dx$$

$$\leq-\frac{1}{2k}||u_{3x}||_{r,0}^2+C\sum_{j=0}^{3}||\partial_x^ju||_{j-1,0}^2 \tag{5.21}$$

¿From (5.16)-(5.21) there appears that since $s\geq max(3r,4)$ and $||u||_{0,s}^2$ is bounded on $[0,T]$, by using f) of lemma 1.1 we have

$$\frac{d}{dt}||u^k(t)||_{r,0}^2+\frac{1}{k}\sum_{j=1}^{3}||\partial_x^ju^k(t)||_{r,0}^2\leq C||u^k||_{r,0}^2+C\sum_{j=0}^{3}||\partial_x^ju^k||_{r-1,0}^2$$

$$\leq C||u^k||_{r,0}^2+C\sum_{j=0}^{3}|||u^k|||_{r,rj}^2\leq C||u^k||_{r,0}^2+C||u^k||_{0,s}^2\leq C(||u^k||_{r,0}^2+1) \tag{5.22}$$

where C is independent of the natural number k. Integrating (5.22) with respect to t and using Gronwall's inequality give

$$Sup_{0\leq t\leq T}||u^k(t)||_{r,0}\leq C \tag{5.23}$$

with the constant C independent of k. From (5.15) and (5.23) there follows that $\{u^k\}$ forms a bounded sequence in $L^\infty(0,T;J_r^s(R))$. Hence, there exists a subsequence of $\{u^k\}$ (also denoted by $\{u^k\}$) and $u\in L^\infty(0,T;J_r^s(R))$ such that

$$u^k\longrightarrow u\ \text{ weakly star in }\ L^\infty(0,T;J_r^s(R))$$

Then, it can be easily seen by the standard argument that u is a desired solution of (1.4), (1.5) (see Lions [15]). From g) of lemma 1.1 it is shown that

$$u_{4x} \in L^{\infty}(0, T; J_{r'}^{s'}(R))$$

where $s' = s - 4$ and $r' = \frac{r(s-4)}{s}$. Here, in view of equation (1.4) we can conclude that

$$u_t \in L^{\infty}(0, T; J_{r'}^{s'}(R))$$

with s' and r' as above. Continuing in this way and using equation (1.4) we obtain the conclusion of Theorem 2.6.

Corollary 2.7 is just a result of Theorem 2.6 and d) of lemma 1.1 combined.

Corollary 2.8 can be seen from the process of the proof of theorems 2.5 and 2.6.

Remark: The main results of this paper are Corollaries 2.7 and 2.8, which show that the convergence of solutions for the natural extension of 4th order correction of the Heisenberg spin ferromagnetic chain equation to its own ones is, to some extent, valid. The methods here can follow from those of the present author's paper [3].

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