Hemirings, Congruences and the Hewitt Realcompactification

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Abstract
The present paper indicates a method of obtaining the Hewitt realcompactification $vX$ of a Tychonoff space $X$, by considering a distinguished family of maximal regular congruences, viz., those which are real, on the hemiring $C_+(X)$ of all the non-negative real valued continuous functions on $X$.

1. Introduction
The structure space $W(R)$ of a hemiring $R$, as the set of all maximal regular congruences on $R$ equipped with the hull-kernel topology, has been introduced in 1990 by Sen and Bandyopadhyay [5], who have shown that $W(R)$ is a $T_1$ topological space and it is $T_2$ only under certain restrictions. In a previous paper [1] the present authors proved that in case $R$ contains the identity, $W(R)$ is compact and for any Tychonoff space $X$, the structure space of the hemiring $C_+(X)$ of all the non-negative real valued continuous functions on $X$ is precisely the Stone-Čech compactifications $\beta X$ of $X$. In this paper we have focused our attention on a particular type of congruences, viz., the real maximal regular congruences on $C_+(X)$. Given any maximal regular congruence $\rho$ on $C_+(X)$, we have shown that a partial ordering `$\leq$' on the quotient hemiring $C_+(X)/\rho$ can be so defined that $C_+(X)/\rho$ becomes a totally ordered hemiring, which further contains an order isomorphic copy of the hemiring $\mathbb{R}_+$ via a canonical map. $\rho$ is called real if $C_+(X)/\rho$ is isomorphic to $\mathbb{R}_+$, otherwise it is called hyper-real. Next we have shown that a real congruence $\rho$ on $C_+(X)$ is charac-

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terized by the property that the set \( \{ \rho(n) : n \in \mathbb{N} \} \) is cofinal in \( C_+(X)/\rho \), where \( \mathbb{N} \) is the set of all natural numbers and for each \( n \) in \( \mathbb{N} \), \( \rho(n) \) denotes the residue class in the hemiring \( C_+(X)/\rho \) which contains the function \( n \) taking value \( n \) constantly on \( X \). This result has further led us to show an intrinsic feature of real congruences on \( C_+(X) \) in terms of their associated \( z \)-filters on \( X \). Using all this result we have finally succeeded in proving that the set of all real maximal regular congruences on \( C_+(X) \) with the hull-kernel topology in \( vX \), the Hewitt realcompactification of \( X \).

2. Partially ordered hemirings

Definition 2.1 Following [4] we define a non-empty set \( R \) with two distinct compositions \( ' + ' \) and \( ' . ' \) a hemiring, if it satisfies all the axioms of a ring except possibly the one that ensures the existence of additive inverses of the members of \( R \); and which satisfies the additional axiom:

\[
a.0 = 0.a = 0 \quad \forall \ a \in R.
\]

Definition 2.2 Following [5] we define a congruence on a hemiring \( R \) to be an equivalence relation \( \rho \) on \( R \) which satisfies the following conditions:

\[
\forall x, y, z \in R, (x, y) \in \rho \Rightarrow (x + z, y + z) \in \rho,
\]

\[
(x.z, y.z) \in \rho \quad \text{and} \quad (z.x, z.y) \in \rho.
\]

The congruence \( \rho \) is called cancellative if,

\[
\forall \ x, y, z \in R, (x + z, y + z) \in \rho \Rightarrow (x, y) \in \rho.
\]

A cancellative congruence \( \rho \) on a hemiring \( R \) is called regular if there exist elements \( e_1, e_2 \) in \( R \) such that

\[
\forall a \in R, (a + e_1.a, e_2.a) \in \rho \quad \text{and} \quad (a + a.e_1, a.e_2) \in \rho.
\]

Evidently each cancellative congruence on a hemiring with unity 1 is regular.

For details of these concepts we refer to [4] and [5]. For further results and notations regarding residue classes of a hemiring modulo maximal regular congruences we refer to [1] because they will frequently be used in this article.

Definition 2.3 A hemiring \((H, +, .)\) equipped with a partial order \( ' \leq ' \) is called a partially ordered hemiring if the following conditions are satisfied: \( \forall a, b, c, d \in H \)

1. \( a \leq b \iff a + c \leq b + c \)

2. \( a \leq c \text{ and } b \leq d \Rightarrow a.d + c.b \leq a.b + c.d \).
Definition 2.4 A congruence $\rho$ on a partially hemiring $H$ is called convex if for all $a, b, c, d$ in $H$,

$$(a, b) \in \rho \text{ and } a \leq c \leq d \leq b \Rightarrow (c, d) \in \rho.$$ 

The following tells precisely when the residue class hemiring of a partially ordered hemiring modulo a regular congruence on it can be partially ordered in some natural way.

Theorem 2.5 Let $H$ be a partially ordered hemiring, $\rho$ be a regular congruence on $H$. In order that $H/\rho$ be a partially ordered hemiring, according to the definition: $\rho(a) \leq \rho(b)$ if and only if there exist $x, y$ in $H$ such that $(x, y) \in \rho$ and $a + x \leq b + y$, it is necessary and sufficient that $\rho$ is convex.

Proof. First assume that $\rho$ is convex. To prove the antisymmetry assume that $\rho(a) \leq \rho(b)$ and $\rho(b) \leq \rho(a)$ where $a, b$ belong to $H$. Then there exist $(x_i, y_i)$ in $\rho$, $i = 1, 2$ such that $a + x_1 \leq b + y_1$ and $b + x_2 \leq a + y_2$. This implies that $a + x_1 + x_2 \leq b + y_1 + x_2 \leq a + y_1 + y_2$. Since $(a + x_1 + x_2, a + y_1 + y_2)$ belongs to $\rho$, in view of the convexity of $\rho$, we have $(a + x_1 + x_2, b + y_1 + x_2)$ belongs to $\rho$. Since $\rho$ is cancellative, this implies that $(a + x_1, b + y_1)$ belongs to $\rho$ which gives $(a + x_1, b + x_1 + y_1)$ belongs to $\rho$ and this yields $(a, b)$ belongs to $\rho$, i.e., $\rho(a) = \rho(b)$. The reflexivity and transitivity of $\leq$ on $\rho$ is trivial and hence their proofs are omitted.

It can easily be verified that for any $a, b, c$ in $H, \rho(a) \leq \rho(b)$ if and only if $\rho(a) + \rho(c) \leq \rho(b) + \rho(c)$. So to complete the proof we need to check only that for $a, b, c, d$ in $H, \rho(a) \leq \rho(c)$ and $\rho(b) \leq \rho(d)$ implies that $\rho(a).\rho(d) + \rho(c).\rho(b)$ $\leq$ $\rho(a).\rho(b) + \rho(c).\rho(d)$. Let us take $a, b, c, d$ in $H$ such that $\rho(a) \leq \rho(c)$ and $\rho(b) \leq \rho(d)$. So there exist $(x_1, y_1), (x_2, y_2)$ in $\rho$ such that $a + x_1 \leq c + y_1$ and $b + x_2 \leq d + y_2$. Then, since $H$ is partially ordered hemiring, we have

$$(a + x_1).(d + y_2) + (c + y_1).(b + x_2) \leq (a + x_1).(b + x_2) + (c + y_1).(d + y_2)$$

i.e.,

$$(a.d + c.b) + (a.y_2b + x_1.d + x_1.y_2 + y_1.x_2 + c.x_2 + y_1.b)$$

$$\leq (a.b + c.d) + (a.x_2 + y_1.d + y_1.y_2 + x_1.x_2 + c.y_2 + x_1.b)$$

Since $(x_1, y_1)$ and $(x_2, y_2)$ belong to $\rho$ we have that all of $(a.y_2, a.x_2), (x_1.d, y_1.d), (x_1.y_2, y_1.y_2), (y_1.x_2, y_1.y_2), (c.x_2, c.y_2)$ and $(y_1.b, x_1.b)$ are members of $\rho$. Thus,

$$(a.y_2 + x_1.d + x_1.y_2 + y_1.x_2 + c.x_2 + y_1.b, a.x_2 + y_1.d + y_1.y_2 + x_1.x_2 + c.y_2 + x_1.b) \in \rho.$$ 

Hence,

$$\rho(a.d + c.b) \leq \rho(a.b + c.d),$$

i.e.,

$$\rho(a).\rho(d) + \rho(c).\rho(b) \leq \rho(a).\rho(b) + \rho(c).\rho(d).$$

Thus $H/\rho$ is a partially ordered hemiring.

Conversely, if $H/\rho$ is a partially ordered hemiring according to the given definition, then it is easy to verify that $\rho$ is convex.
Remark 2.6  For $a, b$ in $H$ we write $\rho(a) < \rho(b)$ if $\rho(a) \leq \rho(b)$ and $\rho(a) \neq \rho(b)$.

3. Congruences on the lattice ordered hemiring $C_+(X)$

In what follows $X$ will stand for a Tychonoff space. $C(X)$ denotes the ring of all real valued continuous functions on $X$. For a real number $r, r^-$ denotes the constant function on $X$ such that $r(x) = r$ for all $x$ in $X$. We take $\mathbb{R}_+$ to be the hemiring of all non-negative real numbers and $C_+(X) = \{ f \in C(X) : f(x) \geq 0 \ \forall x \in X \}$. Then $C_+(X)$ is a lattice ordered hemiring with usual definition of $\cdot^+, \cdot$ and $\cdot^\leq$ and for any two $f, g$ in $C_+(X)$, $f \vee g$ and $f \wedge g$ are defined by,

$$(f \vee g)(x) = \max\{f(x), g(x)\} \text{and}$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\} \ \forall x \in X.$$

Obviously $f \vee g$ and $f \wedge g$ belong to $C_+(X)$.

Convention. Each congruence on $C_+(X)$ considered in this paper will be assumed to be regular and further every such congruence $\rho$ will stand for a proper one i.e., for which $\rho \neq C_+(X) \times C_+(X)$.

We recall some notions and results pertaining to the congruences on the hemiring $C_+(X)$. For a detailed discussion see [1].

Theorem 3.1  If $\rho$ is a congruence on $C_+(X)$ then $E(\rho) = \{ E(f, g) : (f, g) \in \rho \}$ is a $z$-filter on $X$, where $E(f, g) = \{ x \in X : f(x) = g(x) \}$ is the agreement set of $f$ and $g$.

Definition 3.2  A congruence $\rho$ on $C_+(X)$ is called

1. a $z$-congruence if for all $f, g$ in $C_+(X)$, $E(f, g)$ belongs to $E(\rho)$ implies that $(f, g)$ belongs to $\rho$.

2. a prime congruence if for all $f, g, h, k$ in $C_+(X)$, $(f \cdot h + g \cdot k; f \cdot k + g \cdot h) \in \rho$ implies either $(f, g) \in \rho$ or $(h, k) \in \rho$.

3. a maximal congruence if there does not exist any congruence $\sigma$ on $C_+(X)$ which properly contains $\rho$.

Theorem 3.3  If $\mathcal{F}$ is a $z$-filter on $X$, then

$$E^{-1}(\mathcal{F}) = \{ \{ f \}, \{ g \} \in C_+(X) \times C_+(X) : E(\{ f \}, \{ g \}) \in \mathcal{F} \}$$

is a $z$-congruence on $C_+(X)$.

Theorem 3.4  The assignment $\rho \to E(\rho)$ establishes a one-to-one correspondence between the set of all $z$-congruences on $C_+(X)$ and that of all $z$-filters on $X$. 

Theorem 3.5  If \( \rho \) is a maximal congruence on \( C_+(X) \) then \( E(\rho) \) is a \( z \)-ultrafilter on \( X \) and conversely if \( F \) is a \( z \)-ultrafilter on \( X \) then \( E^{-1}(F) \) is a maximal congruence on \( C_+(X) \).

We now state two results which are not included in [1]. Their proofs follow immediately from the following fact:

\[
E(f_1, g_1) \cup E(f_2, g_2) = E(f_1 \cdot f_2 + g_1 \cdot g_2, f_1 \cdot g_2 + f_2 \cdot g_1)
\]

for all \( f_1, f_2, g_1, g_2 \) in \( C_+(X) \).

Theorem 3.6  If \( \rho \) is a prime \( z \)-congruence on \( C_+(X) \), then \( E(\rho) \) is a prime \( z \)-filter on \( X \). Conversely, for any prime \( z \)-filter \( F \) on \( X \), \( E^{-1}(F) \) is a prime \( z \)-congruence on \( C_+(X) \).

Theorem 3.7  Each maximal congruence on \( C_+(X) \) is both a prime congruence and \( z \)-congruence.

4. Order structure on the quotient hemiring of \( C_+(X) \)

Our contemplated main result of this paper demands some study on the order structure of the quotient hemiring of \( C_+(X) \) modulo maximal congruences. The following is the first proposition towards such an end.

Theorem 4.1  A \( z \)-congruence \( \rho \) on \( C_+(X) \) is convex.

Proof. Let \( (f, g) \) belong to \( \rho \) and \( h_1, h_2 \) in \( C_+(X) \) be such that \( f \leq h_1 \leq h_2 \leq g \). Since \( E(f, g) \subset E(h_1, h_2) \) and \( E(f, g) \) belongs to \( E(\rho), E(h_1, h_2) \) belongs to \( E(\rho) \). Clearly then \( (h_1, h_2) \) belong to \( \rho \) because \( \rho \) is a \( z \)-congruence.

The following two results show that the order structure of the quotient hemiring \( C_+(X)/\rho \) has some connection with agreement sets of the members of \( \rho \). (Compare with similar results in the Sec. 5.4 of [3] for the quotient ring \( C(X)/I \), where \( I \) is a \( z \)-ideal in \( C(X) \).)

Theorem 4.2  Let \( \rho \) be a \( z \)-congruence on \( C_+(X) \) and \( f, g \) belong to \( C_+(X) \). Then \( \rho(f) \leq \rho(g) \) if and only if \( f \leq g \) on some member of \( E(\rho) \). On the other hand if \( f < g \) at each point of some member of \( E(\rho) \), then \( \rho(f) < \rho(g) \).

Proof. Let \( \rho(f) \leq \rho(g) \). Then there exists \( (h_1, h_2) \) in \( \rho \) with \( f + h_1 \leq g + h_2 \). Therefore \( f \leq g \) on the set \( E(h_1, h_2) \) in \( E(\rho) \). Conversely, let \( f \leq g \) on \( Z \) where \( Z \) is a member of \( E(\rho) \). Then there exists \( (h_1, h_2) \) in \( \rho \) such that \( Z = E(h_1, h_2) \). Put \( h = (f - g) \lor 0 \). Then \( h \) belongs to \( C_+(X) \) and \( E(h, 0) \) contains \( E(h_1, h_2) \). Since \( \rho \) is a \( z \)-congruence, this implies that \( (0, h) \) belongs to \( \rho \). We assert that \( f + 0 \leq g + h \). Hence \( \rho(f) \leq \rho(g) \).
For the remaining part of this theorem assume that $f < g$ everywhere on some $Z$ in $E(\rho)$. Then $E(f,g) \cap Z = \emptyset$ which implies that $(f,g)$ does not belong to $\rho$. Therefore $\rho(f) \neq \rho(g)$. But by the first part of this theorem, we have $\rho(f) \leq \rho(g)$. Hence $\rho(f) < \rho(g)$. \hfill \Box

**Theorem 4.3** Let $f, g$ belong to $C_+(X)$ and $\rho$ be a maximal congruence on $C_+(X)$ with $\rho(f) < \rho(g)$. Then there exists a set $Z$ in $E(\rho)$ at each point of which $f < g$.

**Proof.** The result follows by using Theorem 4.2 and arguing similarly as in the Proof of 5.4 (b) of [3]. \hfill \Box

A question may be raised - what are the $z$-congruences on $C_+(X)$ which makes the partially ordered hemiring $C_+(X)/\rho$ a totally ordered one? A sufficient condition is provided in the following.

**Theorem 4.4** If $\rho$ is a prime $z$-congruence on $C_+(X)$, then $C_+(X)/\rho$ is a totally ordered hemiring. The same assertion is true in particular therefore for a maximal congruence.

**Proof.** We need to verify only that for arbitrary $f, g$ in $C_+(X), \rho(f)$ and $\rho(g)$ are comparable with respect to the relation ‘$\leq$’. Now $Z_1 = \{x \in X : f(x) \leq g(x)\}$ and $Z_2 = \{x \in X : g(x) \leq f(x)\}$ are zero sets in $X$ such that $Z_1 \cup Z_2 = X$. By Theorem 3.4, $E(\rho)$ is a prime $z$-filter on $X$. Hence either $Z_1$ belongs to $E(\rho)$ or $Z_2$ belongs to $E(\rho)$. But $f \leq g$ on $Z_1$ and $g \leq f$ on $Z_2$. By Theorem 4.2 we have either $\rho(f) \leq \rho(g)$ or $\rho(g) \leq \rho(f)$. \hfill \Box

The following proposition is basic towards the initiation of real and hyper-real congruences on $C_+(X)$. The proof is a routine verification and hence omitted.

**Theorem 4.5** Let $\rho$ be maximal congruence on $C_+(X)$. Then the mapping $\psi : r \rightarrow \rho(r)$ establishes an order preserving isomorphism of the totally ordered hemiring $\mathbb{IR}_+$ into the totally ordered hemiring $C_+(X)/\rho$.

This theorem leads to the following

**Definition 4.6** A maximal congruence $\rho$ on $C_+(X)$ is called

1. real if $\psi(\mathbb{IR}_+) = C_+(X)/\rho$,

2. hyper-real if it not real.

Therefore Theorem 3.7 of [1] can be restated as follows:

**Theorem 4.7** For each point $x$ in $X$, the fixed congruence $\rho_x = \{(f,g) \in C_+(X) \times C_+(X) : f(x) = g(x)\}$ on $C_+(X)$ is real.

The following is criterion for a maximal congruence on $C_+(X)$ to be a real one.
Theorem 4.8 \ A maximal congruence $\rho$ on $C_+(X)$ is real if and only if the set 
\[ \{ \rho(n) : n \in \mathbb{N} \} \] 
is cofinal in the totally ordered hemiring $C_+(X)/\rho$.

To prove this we need the following lemma.

Lemma 4.9 \ For any maximal congruence $\rho$ on $C_+(X)$ each non-zero element in 
$C_+(X)/\rho$ has a multiplicative inverse.

Proof. \ Let $f$ belong to $C_+(X)$ be such that $\rho(f) \neq \rho(\frac{1}{n})$. Since $\rho$ is a $z$-congruence, 
this ensures that $E(f, \frac{1}{n})$ does not belong to $E(\rho)$. Since $E(\rho)$ is $z$-ultrafilter on 
$X$ one can find $(h_1, h_2)$ in $\rho$ with $E(f, \frac{1}{n}) \cap E(h_1, h_2) = \phi$. Let $h = |h_1 - h_2|$ and 
g = $1/(f + h)$. Then $h, g \in C_+(X)$ and $E(f, g, \frac{1}{n}) = E(h_1, h_2)$. Since $(h_1, h_2)$ belongs 
to $\rho$ and $\rho$ is a $z$-congruence, $(f, g, \frac{1}{n})$ belongs to $\rho$. Thus $\rho(f), \rho(g) = \rho(\frac{1}{n})$.

Proof of the theorem. \ Since $n$ is cofinal in the totally ordered hemiring $IR_+$, the 
necessity part of the theorem becomes trivial. Assume therefore that the set 
\[ \{ \rho(n) : n \in \mathbb{N} \} \] is cofinal in the totally ordered hemiring $C_+(X)/\rho$. We first show that the set 
\[ \{ \rho(q) : q \in Q_+ \} \] is dense in the totally ordered hemiring $C_+(X)/\rho$, where $Q_+$ denotes the set of all non-negative rationals. Let $f, g$ belongs to $C_+(X)$ be such that $\rho(f) < \rho(g)$. Then we assert that there is a positive integer $n$ such that $\rho(f) + \rho(1/n) < \rho(g)$. If possible, let for all $n \in \mathbb{N}$ 
\[ \rho(f) + \rho(1/n) \geq \rho(g) \cdots \cdot 4.8.1. \]

Set, 
\[ B = \{ b \in C_+(X)/\rho : \rho(f) + b < \rho(g) \}. \]

Since $\rho(f) \leq \rho(g)$, by Theorem 4.3 one can find $Z$ in $E(\rho)$ such that $f(x) < g(x)$
for each $x$ in $Z$. Put $h = ((g - f) \cup \frac{1}{n})/2$. Then $f(x) < f(x) + h(x) < g(x)$ for all 
x in $Z$. By the second part of Theorem 4.2 we have $\rho(f) < \rho(f) + \rho(h) < \rho(g)$. This shows that $\rho(h) \neq \rho(\frac{1}{n})$ and $\rho(h) \in B$. Thus $B$ contains non-zero elements of 
$C_+(X)/\rho$. Let $b$ be an arbitrary non-zero element of $B$. Then by Lemma 4.9, $b$ has a multiplicative inverse, $b^{-1}$, in $C_+(X)/\rho$. Inequality 4.8.1 gives us 
\[ \rho(f) + b < \rho(g) \leq \rho(f) + \rho(1/n) \forall n \in \mathbb{N}. \]

This shows that $b < \rho(1/n)$ for all $n \in \mathbb{N}$, and hence $b^{-1} \geq \rho(\frac{1}{n})$ for all $n \in \mathbb{N}$ This is contradiction to the assumption that \[ \{ \rho(n) : n \in \mathbb{N} \} \] is cofinal in $C_+(X)/\rho$. Thus there is a positive integer $n$ for which $\rho(f) + \rho(1/n) < \rho(g)$, so that 
\[ \rho(n)\rho(f) + \rho(1) < \rho(n)\rho(g) \cdots \cdot 4.8.2 \]

Let $m$ be the smallest integer such that $\rho(m)\rho(f) < \rho(m)$ and hence in view of 4.8.2 we have 
\[ \rho(n)\rho(f) < \rho(m) < \rho(n)\rho(g). \]
Thus, $\rho(f) < \rho(m/n) < \rho(g)$. Therefore \[ \{ \rho(q) : q \in Q_+ \} \] is dense in $C_+(X)/\rho$.

Let us define a map $\Phi : C_+(X)/\rho \to IR_+$ by the following rule: let $f \in C_+(X)$. If there is a $q \in Q_+$ such that $\rho(f) = \rho(q)$ then we put $\Phi(\rho(f)) = q$. Otherwise set, 
\[ L_f = \{ s \in Q_+ : \rho(s) < \rho(f) \} \cup \{ q : q \text{ is a negative rational} \} \]
\( U_f = \{ s \in Q_+ : \rho(f) < \rho(s) \} \).
Then \((L_f, U_f)\) defines a Dedekind section of the set of rationals and accordingly determines a unique real number \(t\), say, which is clearly non-negative. We put in this case \(\Phi(\rho(f)) = t\).

In order to show that \(\Phi\) is an isomorphism of \(C_+(X)/\rho\) onto \(IR_+\) we choose \(f, g\) in \(C_+(X)\) arbitrarily. Then for any four non-negative rational numbers \(p, q, r, s\), satisfying

\[
\rho(p) \leq \rho(f) < \rho(r) \text{ and } \rho(q) \leq \rho(g) < \rho(s),
\]

one, in view of Theorems 4.2 and 4.3 can easily verify that

\[
p + q \leq \Phi(\rho(f)) + \Phi(\rho(g)) < r + s
\]

and

\[
p + q \leq \Phi(\rho(f) + \rho(g)) < r + s.
\]

The last pair of inequalities together with the denseness of \(\{\rho(q) : q \in Q_+\}\) in \(C_+(X)/\rho\) clearly ensures that

\[
\Phi(\rho(f) + \rho(g)) = \Phi(\rho(f)) + \Phi(\rho(g)).
\]

By an argument similar to one used above we can show that

\[
\Phi(\rho(f), \rho(g)) = \Phi(\rho(f)), \Phi(\rho(g)).
\]

Let \(f, g\) belong to \(C_+(X)\) such that \(\rho(f) < \rho(f)\). Since \(\{\rho(q) : q \in Q_+\}\) is dense in \(C_+(X)/\rho\), in view of the definition of \(\Phi\) it follows that \(\Phi(\rho(f)) < \Phi(\rho(g))\). Thus \(\Phi\) is an order preserving isomorphism of \(C_+(X)/\rho\) onto \(IR_+\) and hence \(\rho\) is a real maximal congruence on \(C_+(X)\). \(\square\)

From the above Theorem we can say that for any hyperreal maximal congruence \(\rho\) on \(C_+(X)\) there exists an \(f \in C_+(X)\) for which \(\rho(f) \geq \rho(n)\) for all \(n \in \mathbb{N}\). We call such a \(f(\rho(f)\) an infinitely large element of \(C_+(X)/\rho\). The multiplicative inverse of an infinitely large element is called an infinitely small element of \(C_+(X)/\rho\). One can check that the multiplicative inverse of an infinitely small element is infinitely large. Thus a hyper-real congruence on \(C_+(X)\) is characterised by the presence of infinitely large (or infinitely small) elements in the residue class hemiring.

The following proposition correlates hyper-real congruences on \(C_+(X)\) with unbounded functions on this hemiring.

**Theorem 4.10** Let \(\rho\) be a maximal congruence on \(C_+(X)\) and \(f \in C_+(X)\) be arbitrary. Then the following statements are equivalent:

1. \(\rho(f)\) is infinitely large.
2. For all \(n \in \mathbb{N}\) the set \(Z_n = \{ x \in X : f(x) \geq n \}\) is a member of \(E(\rho)\).
3. For all \(n \in \mathbb{N}\), \((f \Lambda_n, n)\) belongs to \(\rho\).
4. \(f\) is unbounded on each member of \(E(\rho)\).

(Compare with Result 5.7 (a) of [3]).
Proof (1) ⇒ (2). Let \( \rho(f) \) be infinitely large. Then \( \rho(n) \leq \rho(f) \) for all \( n \in \mathbb{N} \).

Now for an arbitrary \( n \in \mathbb{N} \), in view of Theorem 4.2, \( \rho(\underline{n}) \leq \rho(f) \) implies that there exists \( Z \in E(\rho) \) such that \( n \leq f \) of \( Z \). Thus \( Z \subset Z_n \). Since \( E(\rho) \) is a \( z \)-ultrafilter on \( X \) and \( Z_n \) is a zero set in \( X \), it follows that \( Z_n \) belongs \( E(\rho) \).

(2) ⇒ (3). Since \( Z_n = E(f\underline{\mathcal{A}}, \underline{\mathcal{A}}) \) for all \( n \in \mathbb{N} \) and \( \rho \) is a \( z \)-congruence, the result follows.

(3) ⇒ (2). Trivial.

(2) ⇒ (4). Let (2) holds. Let \( Z \) be an arbitrary member of \( E(\rho) \). Since \( E(\rho) \) is a \( z \)-ultrafilter, \( Z \cap Z_n \neq \phi \) for all \( n \in \mathbb{N} \). So, for any \( x \) in \( Z \cap Z_n \), \( f(x) \geq n \), for all \( n \in \mathbb{N} \). This shows that \( f \) is unbounded on \( Z \). Consequently (4) holds.

(4) ⇒ (1). Let (4) holds. If possible let \( \rho(f) \) be not infinitely large. So there exists \( n \in \mathbb{N} \) such that \( \rho(f) \leq \rho(n) \). Then by Theorem 4.2 there is \( Z \in E(\rho) \) such that \( f \leq n \) on \( Z \), which contradicts our assumption. Thus \( \rho(f) \) is infinitely large.

We conclude this section with a simple but useful characterisation of real congruences.

**Theorem 4.11** A maximal congruence \( \rho \) on \( C_+(X) \) is real if and only if \( E(\rho) \) is closed under countable intersection.

**Proof.** Let \( \rho \) be real. If possible suppose that \( E(\rho) \) is not closed under countable intersection. So there exists a sequence \( \{(f_n, g_n) : n \in \mathbb{N}\} \) in \( \rho \) such that the set \( \cap\{E(f_n, g_n) : n \in \mathbb{N}\} \) does not belong to \( E(\rho) \). Set \( f = \sum_{n=1}^{\infty} (|f_n - g_n| \wedge 2^{-n}) \). Then by Weierstrass M-test it follows that \( f \in C_+(X) \). Now \( E(f, \underline{\mathcal{A}}) = \cap\{E(f, g) : n \in \mathbb{N}\} \) and hence \( (f, \underline{\mathcal{A}}) \not\in \rho \). Therefore \( \rho(\underline{\mathcal{A}}) < \rho(f) \), because \( \underline{\mathcal{A}} \leq f \). For any positive integer \( m \), \( f \leq 2^{-m} \) on the set \( \cap_{n=1}^{m} E(f_n, g_n) \) which is member of \( E(\rho) \). By Theorem 4.2, \( \rho(f) \leq \rho(2^{-m}) \). Since \( m \) is an arbitrary positive integer, \( \rho(f) \) is an infinitely small element of \( C_+(X)/\rho \), whence \( \rho \) becomes hyper-real-a contradiction.

Conversely, let \( E(\rho) \) be closed under countable intersection. If possible suppose that \( \rho \) is not real. Then there exists \( g \) in \( C_+(X) \) such that \( \rho(g) \) is infinitely large. So by Theorem 4.10, for each \( n \in \mathbb{N} \) the set \( Z_n = \{x \in X : n \leq g(x)\} \) is a member of \( E(\rho) \). Obviously \( \cap_{n=1}^{\infty} Z_n = \phi \), - which contradicts our hypothesis. Hence \( \rho \) is real.

\( \Box \)

5. The realcompactification theorem

Let \( W(X) \) be the collection of all maximal congruences on \( C_+(X) \) and \( W_R(X) = \{\rho \in W(X) : \rho \text{ is real}\} \). It is easy to verify that the collection \( \{W(f, g) : f, g \in C_+(X)\} \) is a base for the closed sets of a topology on \( W(X) \) where \( W(f, g) = \{\rho \in W(X) : (f, g) \in \rho\} \). \( W(X) \), equipped with this topology is known as the structure space of \( C_+(X) \). The subspace \( W_R(X) \) of \( W(X) \) is called the real structure space of \( C_+(X) \). It has been established in [1] that \( (\eta_X, W(X)) \) is the Stone-Čech compactification \( \beta X \) of \( X \) where \( \eta_X(x) = \rho_x \) for each \( x \in X \). In this section we propose to state and proof that \( (\eta_X, W_R(X)) \) is the Hewitt realcompactification \( vX \) of \( X \) which is the main result of this article.
In what follows we recall a definition and two results (without proof) of [2] which play a vital role to achieve our goal.

**Definition 5.1** For any subset $A$ of $X$, the set

$$rcl A = \{x \in X : \text{each } G_\delta \text{ set in } X \text{ containing } x \text{ meets } A\}$$

is called the realclosure (or $Q$-closure of) $A$. $A$ is called realclosed (or $Q$-closed) if $A = rcl A$.

It is clear that every closed set in $X$ is realclosed, while any open interval $(a, b)$ of $\mathbb{R}$ is realclosed subset of $\mathbb{R}$ without being closed.

**Theorem 5.2** Every realclosed subset of a realcompact space is realcompact.

**Theorem 5.3** $X$ is realcompact if and only if it is realclosed in $\beta X$.

Now we are in a position to state and prove our main result.

**Theorem 5.4** Let $f : X \to Y$ be a continuous map where $Y$ is a realcompact space. There there exists continuous function $F : W_R(X) \to Y$ such that $F \circ \eta_X = f$ i.e., the following diagram commutes.

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\eta_X & \downarrow & \nearrow F \\
W_R(X) & & 
\end{array}
$$

In order to prove this theorem the following two lemmas are needed.

**Lemma 5.5** The subspace $W_R(X)$ of the space $W(X)$ is realcompact.

**Proof.** Recall that $W(X)$ is compact and hence in particular realcompact. Thus in view of Theorem 5.2, to complete the proof it is sufficient to check that $W_R(X)$ is realclosed subset of $W(X)$.

Let us choose an element $\rho_0$ in $W(X) - W_R(X)$. Since $\rho_0$ is hyper-real, there exists $g \in C_+(X)$ such that $\rho_0(g)$ is infinitely large. Set $f_n = g \lor \underline{n}$ and $h_n = g\underline{n}$ for each $n \in \mathbb{N}$. Then by Theorem 4.10, we get that $(h_n, \underline{n})$ belongs to $\rho_0$ for each $n \in \mathbb{N}$. Since $(f_n, \underline{n}) \cap E(h_{n+1}, n+1) = \phi$ for each $n \in \mathbb{N}$, $(f_n, \underline{n}) \not\in \rho_0$ for each $n \in \mathbb{N}$. Now set $V = W(X) - \bigcup_{n=1}^{\infty} W(f_n, \underline{n})$. Then $V$ is a $G_\delta$-set in $W(X)$ containing $\rho_0$. Let $\rho$ be an arbitrary element in $W_R(X)$. Then by Theorem 4.8, $\rho(g) \leq \rho(m)$ for some $m \in \mathbb{N}$. Also by Theorem 4.2, there is a $Z$ in $E(\rho)$ such that $g \leq \underline{m}$ on $Z$ and hence $Z \subset E(f_n, \underline{m})$. Consequently $(f_n, \underline{m}) \in \rho$ which implies that $\rho \in W(f_n, \underline{m})$. Thus $V \cap W_R(X) = \phi$ and hence $W_R(X)$ is realclosed in $W(X)$.

□
**Lemma 5.6** Let \( f : X \to Y \) be continuous, \( \rho \) be a prime \( z \)-congruence on \( C_+(X) \). Then \( f^*(\rho) \), defined by

\[
f^*(\rho) = \{ E(h, g) : h, g \in C_+(Y), (h \circ f, g \circ f) \in \rho \},
\]
is a prime \( z \)-filter on \( Y \). Moreover if \( \rho \) is real maximal congruence on \( C_+(X) \), then \( f^*(\rho) \) has the countable intersection property.

**Proof.** Obviously \( \phi \) is not a member of \( f^*(\rho) \). Let \( Z \) belong to \( f^*(\rho) \) and \( Z_1 \) be a zero-set in \( Y \) containing \( Z \). Then there exists \( h, g, h_1, g_1 \in C_+(Y) \) such that \( Z = E(h, g) \), \( Z_1 = E(h_1, g_1) \) and \( (h \circ f, g \circ f) \) belongs to \( \rho \). So \( E(h \circ f, g \circ f) \) belongs to \( E(\rho) \). It can easily be verified that \( E(h \circ f, g \circ f) \subset E(h_1 \circ f, g_1 \circ f) \) and hence, \( \rho \) being a \( z \)-congruence, \( (h_1 \circ f, g_1 \circ f) \) belongs to \( \rho \). Consequently \( Z_1 = E(h_1, g_1) \) belongs to \( f^*(\rho) \).

Now suppose that \( Z_1, Z_2 \) be two arbitrary members of \( f^*(\rho) \). So there are \( h_1, g_1, h_2, g_2 \) in \( C_+(Y) \) such that \( Z_i = E(h_i, g_i) \) and \( (h_i \circ f, g_i \circ f) \) are members of \( \rho \) for \( i = 1, 2 \). Since for any \( h, g \) in \( C_+(Y) \), \( (h \circ f, g \circ f) = (h \circ f).g \circ f \) and \( (h + g) \circ f = (h \circ f) + (g \circ f) \), it follows that

\[
E(h_1 \circ f, g_1 \circ f) \cap E(h_2 \circ f, g_2 \circ f) = E((h_1 \circ f)^2 + (g_1 \circ f)^2 + (h_2 \circ f)^2 + (g_2 \circ f)^2, 2((h_1 \circ f).g_1 \circ f) + (h_2 \circ f).g_2 \circ f))
\]

which is a member of \( E(\rho) \). Thus

\[
Z_1 \cap Z_2 = E((h_1^2 + g_1^2 + h_2^2 + g_2^2), 2(h_1 \circ g_1 + h_2 \circ g_2)) \in f^*(\rho).
\]

This shows that \( f^*(\rho) \) is a \( z \)-filter on \( Y \).

Finally, let \( Z_1 \cup Z_2 \) belong to \( f^*(\rho) \) where \( Z_i = E(f_i, g_i) ; f_i, g_i \in C_+(Y) \), \( i = 1, 2 \). Then since \( Z_1 \cup Z_2 = E(f_1.g_2 + f_2.g_1, f_1.f_2 + g_1.g_2) \) and \( \rho \) is prime, by an argument similar to the above we can show that either \( Z_1 \in f^*(\rho) \) or \( Z_2 \in f^*(\rho) \). Thus \( f^*(\rho) \) is a prime \( z \)-filter on \( Y \).

To show, for a real maximal congruence \( \rho \) on \( C_+(X) \), \( f^*(\rho) \) has the countable intersection property, let us take a sequence \( \{ E(h_n, g_n) \} \) in \( f^*(\rho) \). Then for all \( n \in \mathbb{N}, (h_n \circ f, g_n \circ f) \) belongs to \( \rho \) and hence by the Theorem 4.11, \( \cap_{n=1}^{\infty} E(h_n \circ f, g_n \circ f) \) is non-empty. For any \( x \) in \( \cap_{n=1}^{\infty} E(h_n \circ f, g_n \circ f) \), \( f(x) \in \cap_{n=1}^{\infty} E(h_n, g_n) \). Thus \( f^*(\rho) \) has the countable intersection property.

**Proof of the Theorem.** Let \( \rho \) be a member of \( W_R(X) \). Since for a prime \( z \)-filter with countable intersection property on a realcompact space is fixed and since prime \( z \)-filter contains at most one cluster point (see 8.12 and 3.18 of [3]) it follows that there exists a unique \( y \) in \( Y \) such that \( \{ y \} = \cap f^*(\rho) \). For every \( \rho \) in \( W_R(X) \) set \( F(\rho) = y \) where \( \{ y \} = \cap f^*(\rho) \). This defines a map \( F : W_R(X) \to Y \). For each \( x \in X \) it follows that \( F(\rho_x) = f(x) \) because \( f(x) \in \cap f^*(\rho_x) \). Thus \( F(\eta_X(x)) = f(x) \) \( \forall x \in X \) and hence \( F \eta_X = f \).
To prove the continuity of the function $F$, choose any $\rho_0$ in $W_R(X)$ and any open set $V$ in $Y$ such that $F(\rho_0) \in V$. Then there exist $g_1, g_2 \in C_+(Y)$ such that

$$F(\rho_0) \in Y - Z(g_1) \subset Z(g_2) \subset V.$$ 

Clearly then $g_1, g_2 = 0$. Now $F(\rho_0)$ does not belong to $Z(g_1)$ and hence $Z(g_1) = E(g_1, 0)$ does not belong to $f^*(\rho)$. Consequently $(g_1 \circ f, 0)$ does not belong to $\rho_0$ and this implies that the set $U = (W(X) - W(g_1 \circ f, 0)) \cap W_R(X)$ is an open neighbourhood of $\rho_0$ in $W_R(X)$. Now choose any $\rho$ in $U$. Then $(g_1 \circ f, 0) \not\in \rho$. Since $(g_1 \circ f)(g_2 \circ f) = 0$ and $\rho$ is a prime congruence on $C_+(X)$, it follows that $(g_2 \circ f, 0) \in \rho$. Thus $Z(g_2) = E(g_2, 0) \in f^*(\rho)$ and hence $F(\rho) \in E(g_2, 0) = Z(g_2) \subset V$. Thus $F(U) \subset V$. Therefore the map $F: W_R(X) \to Y$ is continuous. \hfill $\Box$

Recall that the Hewitt realcompactification $vX$ of a space $X$ is characterised by the fact that any continuous map of $X$ into an arbitrary realcompact space admits a continuous extension over $vX$. Hence in view of the above theorem we conclude our article with the following

**Corollary 5.7** $(n_X, W_R(X))$ is the Hewitt realcompactification $vX$.

**References**


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