Some KKM type, intersection and minimax theorems in spaces with abstract convexities

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Abstract. In this paper we obtain KKM type theorems for $G$-spaces, $M$-spaces and $L$-spaces which are spaces with no linear structure, these theorems are used to obtain some minimax results for these spaces. Also an intersection theorem for $M$-spaces is presented.

Resumen. En este trabajo obtenemos teoremas de tipo KKM para $G$-espacios, $M$-espacios y $L$-espacios que son espacios sin una estructura lineal, estos teoremas se utilizan para obtener unos resultados minimax para estos espacios. También se presenta un teorema de intersección para $M$-espacios.

1 Introduction

In this paper we obtain some KKM type theorems for $G$-spaces. These are Theorems 2.3, 2.6 and 2.11. These latter two results generalize Theorems 1 and Theorem 2 of Bardaro and Cepitelli [1]. We then apply our results to obtain some minimax theorems, including a generalization to $G$-spaces of an inequality of Fan [4]. This is our Corollary 3.3.

Then, using theorem 3.2 and theorem 3.4 of [2], we obtain a collection of similar results for $M$-spaces and for $L$-spaces.


2 Some KKM type theorems for $G$-spaces

In this section we present some KKM type theorem for $G$-spaces. KKM type theorems are intersection theorems for multifunctions which satisfy a condition known as the KKM condition. We begin by recalling the definition of a $G$-space and the concept of a multifunction of KKM type.
Definition 2.1 We call a triple \((X, D, \Gamma)\) a G-space if \(X\) is a topological space, \(D\) is a nonempty subset of \(X\) and \(\Gamma : <D> \to 2^X\) is a multifunction from the set \(<D>\) of nonempty finite subsets of \(D\) into \(X\) such that

1. \(\Gamma(A) \subset \Gamma(B)\) whenever \(A \subset B\)

2. For each \(A = \{a_1, ..., a_{n+1}\} \in <D>\), there is a continuous function \(\phi_A : \Delta_n \to \Gamma(A)\) such that for any subset \(B = \{a_1, ..., a_m\} \subset A\), we have \(\phi_A([e_{i1}, ..., e_{im}]) \subset B\) where \(\Delta_n\) denotes the standard closed n-simplex.

Definition 2.2 Let \((X, D, \Gamma)\) be a G-space. A multifunction \(F : D \to 2^X\) such that \(\Gamma(A) \subset F(A)\) for every \(A \in <D>\) is called a G-KKM multifunction.

The following theorem was proved in [3]

Theorem 2.3 Let \((X, D, \Gamma)\) be a compact G-space. Let \(F : D \to 2^X\) be a closed valued G-KKM multifunction. Then \(\bigcap \{F(x) : x \in D\} \neq \emptyset\).

Next, we generalize Theorem 2.3 to the case where \(X\) is not compact; however, before doing so some definitions are required.

Definition 2.4 Let \((X, D, \Gamma)\) be a G-space. A subset \(S\) of \(X\) is G-convex if \(\Gamma(A) \subset S\) whenever \(A \in <D \cap S>\).

Definition 2.5 Let \((X, D, \Gamma)\) be an G-space, a set \(K \subset X\) is G-compact if for every \(A \in <X>\) there is a compact, G-convex set \(Y\) such that \(K \cup A \subset Y\).

To present the following theorem let us recall that a set \(H\) is compactly closed if \(H \cap B\) is closed in \(B\) for every compact set \(B\).

Theorem 2.6 Let \((X, \Gamma)\) be an G-space, and let \(F : X \to 2^X\) be a closed valued G-KKM multifunction such that:

1. For each \(x \in X\) \(F(x)\) is compactly closed.

2. There is a compact set \(L \subset X\) and an G-compact set \(K \subset X\) such that for each compact G-convex set \(Y\) with \(K \subset Y \subset X\) we have that \(\bigcap \{(F(x) \cap Y) : x \in Y\} \subset L\).

Then \(\bigcap \{F(x) : x \in X\} \neq \emptyset\).
Proof:
It will suffice to show that \( \bigcap \{ (F(x) \cap L) : x \in X \} \neq \emptyset \). From condition (1) it follows that \( \{ F(x) \cap L : x \in X \} \) is a family of closed sets in the compact set \( L \). Thus, it suffices to show that this family has the finite intersection property.

Suppose \( A \in \langle X \rangle \). By condition (2) there is a compact, G-convex set \( Y_0 \) such that \( K \cup A \subset Y_0 \) and \( \bigcap \{ (F(x) \cap Y_0) : x \in Y_0 \} \subset L \). But, \( \bigcap \{ (F(x) \cap Y_0) : x \in Y_0 \} \subset \bigcap \{ (F(x) \cap L) : x \in A \} \neq \emptyset \), so, to show that \( \bigcap \{ (F(x) \cap L) : x \in A \} \neq \emptyset \), it suffices to prove that \( \bigcap \{ (F(x) \cap Y_0) : x \in Y_0 \} \neq \emptyset \).

Now, because \( Y_0 \) is G-convex, the pair \( \langle Y_0, \Gamma \rangle \) is itself a compact G-space, and the multifunction \( H : Y_0 \to 2^{Y_0} \) given by \( H(x) = F(x) \cap Y_0 \), is a G-KKM multifunction.

Indeed, let \( B \in \langle Y_0 \rangle \). Then,
\[
\Gamma(B) = \Gamma(B) \cap Y_0 \\
\subset (\bigcup \{ (F(x) : x \in B) \}) \cap Y_0 \\
= \bigcup \{ F(x) \cap Y_0 : x \in B \} \\
= \bigcup \{ H(x) : x \in B \} = H(B).
\]

Therefore, \( H \) is a G-KKM multifunction for the compact G-space \( \langle Y_0, \Gamma \rangle \). Thus by Theorem 2.3, it follows that \( \bigcap \{ (F(x) \cap Y_0) : x \in Y_0 \} = \bigcap \{ H(x) : x \in Y_0 \} \neq \emptyset \). ♦

Now we will introduce a definition which describe a weaker condition for a multifunction than that of G-KKM, and we will use it later. Before doing that we need the following concept.

**Definition 2.7** Let \( (X, D, \Gamma) \) be a G-space. Let \( A \) be a subset of \( X \). We define the G-convex hull of \( A \), denoted by \( \text{co}^G(A) \), as
\[
\text{co}^G(A) = \bigcap \{ S \subset X : S \text{ is G-convex, and } A \subset S \}
\]

**Definition 2.8** Let \( (X, D, \Gamma) \) be a G-space. A multifunction \( F : D \to 2^X \) such that \( \text{co}^G(A) \subset F(A) \) for every \( A \in \langle D \rangle \) is called a \( \text{G}^* \)-KKM multifunction.

The next proposition and its corollary were proved in [3].

**Proposition 2.9** Let \( (X, D, \Gamma) \) be an G-space. Suppose \( F : D \to 2^X \) is a \( \text{G}^* \)-KKM multifunction, then it is a G-KKM multifunction.

**Corollary 2.10** Let \( (X, D, \Gamma) \) be a compact G-space. Let \( F : D \to 2^X \) be a closed valued \( \text{G}^* \)-KKM multifunction. Then \( \bigcap \{ F(x) : x \in D \} \neq \emptyset \).
Theorem 2.11 Let \((X, \Gamma)\) be a G-space, and let \(F, H : X \to 2^X\) be two multi-functions such that:

1. For all \(x \in X\), \(H(x)\) is compactly closed, and \(F(x) \subset H(x)\);
2. \(x \in F(x)\) for every \(x \in X\);
3. For all \(x \in X\), \(F^*(x)\) is G-convex;
4. \(H\) satisfies condition (2) of Theorem 2.6.

Then \(\bigcap \{H(x) : x \in X\} \neq \emptyset\).

Proof:
By Corollary 2.10 it will suffice to show that the multifunction \(H\) is a G*-KKM multifunction.

Suppose that \(H\) is not a G*-KKM multifunction, then there is a subset \(A \in \mathcal{D}\) such that \(\text{co}^G(A) \not\subset H(A)\).

Thus, there exists \(y \in \text{co}^G(A)\) such that \(y \notin H(A)\), which means that, \(y \notin H(x)\) for all \(x \in A\), that is, \(x \in H^*(y)\) for all \(x \in A\). Thus, \(A \subset H^*(y)\).

On the other hand, condition (1) implies \(H^*(y) \subset F^*(y)\). Thus, \(F^*(y)\) is a G-convex subset containing \(A\), which implies that, \(\text{co}^G(A) \subset F^*(y)\), but \(y \in \text{co}^G(A)\). Then \(y \in F^*(y)\), which is equivalent to \(y \notin F(y)\), in contradiction with condition (2).

Hence \(H\) is a G*-KKM multifunction and so \(\bigcap \{H(x) : x \in X\} \neq \emptyset\). ◊

Thus, theorems 2.6 and 2.11 generalize to G-spaces, theorems 1 and 2 in [1].

Corollary 2.12 Let \((X, \Gamma)\) be a compact G-space. Let \(F : X \to 2^X\) be a multifunction and let \(H : X \to 2^X\) be a closed valued multifunction such that:

1. For all \(x \in X\), \(F(x) \subset H(x)\);
2. \(x \in F(x)\) for every \(x \in X\);
3. For all \(x \in X\), \(F^*(x)\) is G-convex.

Then \(\bigcap \{H(x) : x \in X\} \neq \emptyset\).

3 Some Minimax theorems for G-spaces
In this section we present a minimax inequality which is a generalization to G-spaces of an inequality previously proved by K. Fan in [4].
Theorem 3.1  Let \((X, \Gamma)\) be a compact G-space, let \(f : X \times X \to R\) and \(h : X \times X \to R\) be two functions such that:

1. \(h(x, y) \leq f(x, y)\) for all \((x, y) \in X \times X\).

2. The function \(h_{x} : X \to R\) given by \(h_{x}(y) = h(x, y)\) is lower semicontinuous.

3. Given any \(\lambda \in R\) and any \(y \in X\) the set \(\{x \in X : f(x, y) > \lambda\}\) is G-convex.

Then for any \(\lambda \in R\) either there exists \(y_{0} \in X\) such that \(h(x, y_{0}) \leq \lambda\) for all \(x \in X\), or there exists \(y_{0} \in X\) such that \(f(y_{0}, y_{0}) > \lambda\).

Proof:
Let us set \(H(x) = \{y \in X : h(x, y) \leq \lambda\}\) and \(F(x) = \{y \in X : f(x, y) \leq \lambda\}\). Since \(h_{x}\) is lower semicontinuous, \(H(x)\) is a closed set, so in the terminology of multifunctions, we have a multifunction \(F : X \to 2^{X}\), and a closed valued multifunction \(H : X \to 2^{X}\), such that \(F(x) \subset H(x)\) for all \(x \in X\) because of condition (1).

Now for the multifunction \(F\), we have two possibilities:

Either there is an \(x_{0} \in X\), such that \(x_{0} \notin F(x_{0})\), in which case we have that \(f(x_{0}, x_{0}) > \lambda\), that is, the second part of the alternative is true.

Or, for all \(x \in X\), \(x \in F(x)\). Now \(F^{*}(y) = \{x \in X : y \notin F(x)\}\) is a closed valued multifunction \(H(x) \to 2^{X}\), such that \(F(x) \subset H(x)\) for all \(x \in X\) because of condition (3).

Therefore \(F\) and \(H\) are two multifunctions satisfying the hypotheses of Corollary 2.12, so we have that, \(\bigcap \{H(x) : x \in X\} \neq \emptyset\).

Thus if \(x_{0} \in \bigcap \{H(x) : x \in X\}\) we have that \(h(x_{0}, y) \leq \lambda\) for all \(y \in X\), that is the first part of the alternative is true. 

\(\Box\)

Corollary 3.2  With the hypotheses of Theorem 3.1 we obtain the following minimax inequality.

\[
\min_{y \in X} \sup_{x \in X} h(x, y) \leq \sup_{x \in X} f(x, x).
\]

Proof:
Let \(\lambda = \sup_{x \in X} f(x, x)\), then either \(\lambda = \infty\), in which case the inequality is obvious or \(\lambda\) is finite. Then because of definition of \(\lambda\), the first part of the alternative in Theorem 3.1 is true. Therefore exists \(y_{0} \in X\) such that:

\[
h(x, y_{0}) \leq \sup_{x \in X} f(x, x) \quad \text{forall} \quad x \in X.
\]

Then

\[
\sup_{x \in X} h(x, y) \leq \sup_{x \in X} f(x, x) \quad \text{forall} \quad y \in X.
\]
that is,
\[ \sup_{x \in X} h_x(y) \leq \sup_{x \in X} f(x, x) \quad \text{for all } y \in X. \]
Thus
\[ \inf_{y \in X} \sup_{x \in X} h_x(y) \leq \sup_{x \in X} f(x, x); \]
but \( \sup_{x \in X} h_x \) is lower semicontinuous, and it is well known that in this case this infimum is a minimum therefore we have that
\[ \min_{y \in X} \sup_{x \in X} h(x, y) \leq \sup_{x \in X} f(x, x). \]

Based on this, the inequality proved by Fan in [4] can be generalized to G-spaces by the following corollary.

**Corollary 3.3** Let \((X, \Gamma)\) be a compact G-space and let \( f : X \times X \to \mathbb{R} \) be a function such that:

1. The function \( f_x : X \to \mathbb{R} \) given by \( f_x(y) = f(x, y) \) is lower semicontinuous.
2. Given any \( \lambda \in \mathbb{R} \) and any \( y \in X \) the set \( \{ x \in X : f(x, y) > \lambda \} \) is G-convex.

Then the following inequality is true
\[ \min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x). \]

**Proof:**
Take \( h(x, y) = f(x, y) \) in Corollary 3.2. \( \Diamond \)

**4 Some KKM and Minimax Theorems for M-spaces and L-spaces**

Theorem 3.2 of [2], shows that if \((X, M, k)\) is an M-space, and \( D \subset X \) is an admissible subset, then there exists the corresponding M-space \((X, D, \Gamma)\), such that the collection of M-convex subsets with respect to \( D \) in \((X, M, k)\) coincides with the collection of G-convex sets in \((X, D, \Gamma)\). We will use this result to obtain from the KKM and minimax theorems proved for G-spaces, similar results for M-spaces.

On the other hand, Theorem 3.4 of [2] states that given an L-space \((X, D, P)\), there is an M-space \((X, M, k)\) for which \( D \) is an admissible subset, and the collection of L-convex subsets in \((X, D, P)\) coincides with the collection of M-convex subsets with respect to \( D \) in \((X, M, k)\). Based on this theorem some KKM and minimax theorems for L-spaces will be obtained.
Let us begin by recalling the concepts of M-space and M-convex subset, to introduce next the concept of M*-KKM multifunction.

**Notation.** Given any integer \( m \geq 2 \) and \( 1 \leq i \leq m \), let \( \delta_i : \mathbb{R}^n \to \mathbb{R}^n \) denote the function defined by 
\[ \delta_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n). \]

**Definition 4.1** An M-space is a triple \((X, M, k)\), where 
- \( X \) is a topological space, 
- \( M = M_n : \) n integer, \( n \leq 1 \) is a collection of sets where \( M_n \subset X^n \) for all \( n \geq 1 \), and 
- \( k = kn : \) n integer, \( n \leq 1 \) is a collection of functions satisfying

1. \( k_{n+1} : M_{n+1} \times \Delta_n \to X \).
2. If \( x \in M_{n+1}(n \geq 1) \) and \( i \leq n + 1 \), then \( \delta_i(x) \in M_n \) and for any \( t \in \Delta_n \) with \( t_i = 0 \), \( k_{n+1}(x, t) = k_n(\delta_i(x), \delta_i(t)) \).
3. If \( x \in M_{n+1} \), then the map \( t \to k_{n+1}(x, t) \), from \( \Delta_n \) to \( X \), is continuous.

**Definition 4.2** Let \((X, M, k)\) be an M-space. A nonempty subset \( D \subset X \) is said to be admissible if \( D^n \subset M_n \) for all \( n \).

**Definition 4.3** Let \((X, M, k)\) be an M-space, let \( D \subset X \) be an admissible subset. We say that a subset \( S \subset X \) is M-convex with respect to \( D \), if for each subset \( A \in \langle D \rangle \cap D \rangle \) and any indexing of \( A = \{a_1, ..., a_{n+1}\} \), we have that 
\[ k_{n+1}((a_1, ..., a_{n+1}), \Delta_n) \subset S. \]
If \( D = X \) we say M-convex.

**Definition 4.4** Let \((X, M, k)\) be an M-space, let \( D \subset X \) be an admissible subset. Let \( K \) be subset of \( X \). We define the M-convex hull of \( K \) with respect to \( D \), denoted by \( \text{co}^M_D \) as:
\[ \text{co}^M_D = \bigcap \{S \subset X : S \text{ is M-convex with respect to } D, K \subset S\}. \]
In case \( D = X \), the M-convex hull of \( K \) with respect to \( X \) will be denoted by \( \text{co}^M \).

**Definition 4.5** Let \((X, M, k)\) be an M-space and let \( D \subset X \) be an admissible subset. A multifunction \( F : D \to 2^X \) is said to be M*-KKM, if for each \( A \in \langle D \rangle \), \( \text{co}^M_D(A) \subset F(A) \).
**Proposition 4.6** Let \((X, M, k)\) be a compact M-space, and let \(D \subset X\) be an admissible subset. Let \(F : D \to 2^X\) be a closed valued M*-KKM multifunction. Then \(\bigcap\{F(x) : x \in D\} \neq \emptyset\).

**Proof:**
By Theorem 3.2 of [2], the collection of M-convex subsets with respect to \(D\) in the space \((X, M, k)\), coincide with the collection of G-convex subsets in the corresponding G-space \((X, D, \Gamma)\). Therefore \(F : D \to 2^X\) is a G*-KKM multifunction in the G-space \((X, D, \Gamma)\). Thus, by Corollary 2.9 we have that \(\bigcap\{F(x) : x \in D\} \neq \emptyset\). ♦

As consequences of our next proposition we obtain minimax results for M-spaces, all these proofs are omitted because they are similar to those corresponding to G-spaces.

**Proposition 4.7** Let \((X, M, k)\) be a compact M-space, such that \(X\) is admissible. Let \(F : X \to 2^X\) be a multifunction and let \(H : X \to 2^X\) be a closed valued multifunction such that:
1. For all \(x \in X\), \(F(x) \subset H(x)\);
2. \(x \in F(x)\) for every \(x \in X\);
3. For all \(x \in X\), \(F^*(x)\) is M-convex.
Then \(\bigcap\{H(x) : x \in X\} \neq \emptyset\).

**Proposition 4.8** Let \((X, M, k)\) be a compact M-space, such that \(X\) is admissible. Let \(f : X \times X \to \mathbb{R}\) and \(h : X \times X \to \mathbb{R}\) be two functions such that:
1. \(h(x, y) \leq f(x, y)\) for all \((x, y) \in X \times X\).
2. The function \(h_x : X \to \mathbb{R}\) given by \(h_x(y) = h(x, y)\) is lower semicontinuous.
3. Given any \(\lambda \in \mathbb{R}\) and any \(y \in X\) the set \(\{x \in X : f(x, y) > \lambda\}\) is M-convex.
Then for any \(\lambda \in \mathbb{R}\) either there exist \(y_0 \in X\) such that for all \(x \in X\), \(h(x, y_0) \leq \lambda\), or there exists \(y_0 \in X\) such that \(f(y_0, y_0) > \lambda\).

**Proposition 4.9** With the hypotheses of Proposition 4.8 we obtain the following minimax inequality.
\[
\min_{y \in X} \sup_{x \in X} h(x, y) \leq \sup_{x \in X} f(x, x).
\]
Proposition 4.10 Let \((X, M, k)\) be a compact \(M\)-space, such that \(X\) is admissible and let \(f : X \times X \to \mathbb{R}\) be a function such that:

1. The function \(f_x : X \to \mathbb{R}\) given by \(f_x(y) = f(x, y)\) is lower semicontinuous.

2. Given any \(\lambda \in \mathbb{R}\) and any \(y \in X\) the set \(\{x \in X : f(x, y) > \lambda\}\) is \(M\)-convex.

Then the following inequality is true

\[
\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x).
\]

This proposition generalizes to \(M\)-spaces an inequality proved by Fan in [4].

Now, we give the definition of an \(L^*\)-KKM multifunction, and then by employing Theorem 3.4 of [2], we state some KKM and minimax theorems for \(L\)-spaces. We begin by recalling the concepts of an \(L\)-space, an \(L\)-convex subset and the \(L\)-convex hull of a subset.

Definition 4.11 An \(L\)-space is a triple \((X, D, P)\), where \(X\) is a topological space, \(D\) is a nonempty subspace of \(X\) and \(P = \{P_a : a \in X\}\) is a collection of functions \(P_a : D \times [0, 1] \to D\), such that \(P_a(x, 0) = x\), \(P_a(x, 1) = a\), and \(P_a\) is continuous respect to \(t \in [0, 1]\). When \(D = X\), we write \((X, P)\).

Definition 4.12 Suppose \((X, D, P)\) is an \(L\)-space. Given \(A \in \langle D \rangle\), let \(A = \{a_0, ..., a_n\}\) be any indexing of \(A\) by \(\{0, ..., n\}\). Define the multifunction \(G_A : [0, 1]^n \to D\) by

\[
G_A(t_0, ..., t_n) = P_{a_0}(P_{a_1}(P_{a_2}(...P_{a_{n-1}}(a_n, t_{n-1})...t_1)...t_0))
\]

For \(A = \{a\}\), we define \(G_{\{a\}} = \{a\}\). We say that a subset \(S \subset X\) is \(L\)-convex if for every \(A \in \langle A \cap D \rangle\), and every indexing of \(A = \{a_0, ..., a_n\}\), it follows that \(G_A([0, 1]^n) \subset S\).

Definition 4.13 Let \((X, D, P)\) be an \(L\)-space. Let \(A\) be a subset of \(X\). We define the \(L\)-convex hull of \(A\) by

\[
\text{co}^L(A) = \bigcap \{S \subset X : S \text{ is } L\text{-convex and } A \subset S\}
\]

Definition 4.14 Let \((X, D, P)\) be an \(L\)-space. A multifunction \(F : D \to 2^X\) such that \(\text{co}^L(A) \subset F(A)\) for every \(A \in \langle D \rangle\) is called an \(L^*\)-KKM multifunction.
Proposition 4.15 Let \((X,D,P)\) be a compact L-space. Let \(F : D \rightarrow 2^X\) be a closed valued L*-KKM multifunction. Then \(\bigcap\{F(x) : x \in D\} \neq \emptyset\).

Proof:
The proof follows from Theorem 3.4 of [2] and Proposition 4.6 in similar way to the proof of Proposition 4.6.

The followings propositions together with Proposition 3.4 of [2] allow us to present some minimax results for L-spaces, whose proofs are omitted because of their similarities with the corresponding for M-spaces.

Proposition 4.16 Let \((X,P)\) be a compact L-space. Let \(F : X \rightarrow 2^X\) be a multifunction and let \(H : X \rightarrow 2^X\) be a closed valued multifunction such that:

1. For all \(x \in X\), \(F(x) \subset H(x)\);
2. \(x \in F(x)\) for every \(x \in X\);
3. For all \(x \in X\), \(F^*(x)\) is L-convex.

Then \(\bigcap\{H(x) : x \in X\} \neq \emptyset\).

Proposition 4.17 Let \((X,P)\) be a compact L-space, let \(f : X \times X \rightarrow \mathbb{R}\) and \(h : X \times X \rightarrow \mathbb{R}\) be two functions such that:

1. \(h(x,y) \leq f(x,y)\) for all \((x,y) \in X \times X\).
2. The function \(h_x : X \rightarrow \mathbb{R}\) given by \(h_x(y) = h(x,y)\) is lower semicontinuous.
3. Given any \(\lambda \in \mathbb{R}\) and any \(y \in X\) the set \(\{x \in X : f(x,y) > \lambda\}\) is L-convex.

Then for any \(\lambda \in \mathbb{R}\) either there exist \(y_0 \in X\) such that for all \(x \in X\), \(h(x,y_0) \leq \lambda\), or there exists \(y_0 \in X\) such that \(f(y_0,y_0) > \lambda\).

Corollary 4.18 With the hypotheses of Proposition 4.17 we obtain the following minimax inequality.

\[
\min_{y \in X} \sup_{x \in X} h(x,y) \leq \sup_{x \in X} f(x,x).
\]

Corollary 4.19 Let \((X,P)\) be a compact L-space and let \(f : X \times X \rightarrow \mathbb{R}\) be a function such that:

1. The function \(f_x : X \rightarrow \mathbb{R}\) given by \(f_x(y) = f(x,y)\) is lower semicontinuous.
2. Given any \(\lambda \in \mathbb{R}\) and any \(y \in X\) the set \(\{x \in X : f(x,y) > \lambda\}\) is L-convex.

Then the following inequality is true

\[
\min_{y \in X} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} f(x,x).
\]
5 An intersection Theorem for M-spaces

In this section, by employing an intersection theorem due to J. Kindler [5], proved without using the Theorem of Knaster-Kuratowski-Mazurkiewicz, we show another type of intersection theorem for M-spaces.

Theorem 5.1 For a multifunction \( F : X \to 2^Y \) the following are equivalent.

1. \( \bigcap \{ F(x) : x \in X \} \neq \emptyset \).

2. There exist topologies on \( X \) and \( Y \) such that
   (a) \( Y \) is compact.
   (b) Every value \( F(x), x \in X \) is closed.
   (c) For all \( A \in \langle X \rangle \) the subset \( \bigcap \{ F(x) : x \in A \} \) is connected.
   (d) For all \( B \subset Y \) the subset \( \bigcap \{ F^*(y) : y \in B \} \) is connected.

Theorem 5.2 Let \((X, M, k)\) be an M-space such that \( X \) is admissible, and such that \( k_1(x, 1) = x \) for all \( x \in X \). Let \( Y \) be a compact topological space and \( F : X \to 2^Y \) an upper semicontinuous multifunction such that

1. \( F(\Gamma_{x_1, x_2}) = F(x_1) \cup F(x_2) \) for all \( x_1, x_2 \in X \).

2. \( \bigcap \{ F(x) : x \in A \} \) is connected for all \( A \in \langle X \rangle \).

Then \( \{ F(x) : x \in X \} \neq \emptyset \).

Proof:
Due to Theorem 5.1 it suffices to prove that for all \( B \subset Y \) the subset \( \bigcap \{ F^*(y) : y \in B \} \) is connected, so let \( B \subset Y \) and let us prove that \( \bigcap \{ F^*(y) : y \in B \} \) is connected.

To this end we will show that given \( x_1, x_2 \in \bigcap \{ F^*(y) : y \in B \} \) there is a connected set \( C \) such that \( \{ x_1, x_2 \} \subset C \subset \bigcap \{ F^*(y) : y \in B \} \).

Now \( x_1, x_2 \in \bigcap \{ F^*(y) : y \in B \} \) means that \( B \cap F(x_1) = \emptyset \) and \( B \cap F(x_2) = \emptyset \), then \( B \cap (F(x_1) \cup F(x_2)) = B \cap F(\Gamma_{x_1, x_2}) = \emptyset \). Therefore \( x_1, x_2 \in \Gamma_{x_1, x_2} \subset \bigcap \{ F^*(y) : y \in B \} \).

On the other hand \( \Gamma_{x_1, x_2} = \{ \{ k_2((x_2, x_2), t) : t \in \Delta_1 \} \cup \{ \{ k_2((x_2, x_1), t) : t \in \Delta_1 \} \) is path-connected.

In fact, let \( x, y \in \Gamma_{x_1, x_2} \). We will show that there is a path joining \( x \) and \( y \). Assume that \( x = k_2((x_1, x_2), (t_1, t_2)) \) with \( (t_1, t_2) \in \Delta_1 \) and consider the path \( \phi : [0, 1] \to X \) defined by \( \phi(t) = k_2((x_1, x_2), (t_1 + t - tt_1, t_2 - tt_2)) \). By definition of M-space it follows that \( \phi \) is continuous function such that \( \phi(0) = k_2((x_1, x_2), (t_1, t_2)) \) and \( \phi(1) = k_2((x_1, x_2), (1, 0)) = k_1(x_1, 1) = x_1 \). Therefore \( \phi \) is a path joining \( x \) and \( x_1 \).
In a similar way we can construct a path joining $y$ and $x_1$. Thus any pair $x, y \in \Gamma_{\{x_1, x_2\}}$ can be joined by a path, which means that $\Gamma_{\{x_1, x_2\}}$ is path connected.

Therefore, given two points $\{x_1, x_2\} \in \bigcap \{F^*(y) : y \in B\}$ we have found a connected set $C = \Gamma_{\{x_1, x_2\}}$ containing these two points and contained in $\bigcap \{F^*(y) : y \in B\}$, this means that $\bigcap \{F^*(y) : y \in B\}$ is connected. ♦

References


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