A Dialogue Between Two Lifting Theorems

Rodrigo Arocena *

Abstract. The relation between the lifting theorems due to Nagy-Foias and Cotlar-Sadosky is discussed.

PRESENTATION.
The Nagy-Foias commutant lifting theorem is a basic result in Operator Theory and its applications to interpolation problems. Its scope is shown in a fundamental book due to Foias and Frazho where we can read that “the work on the general framework of the commutant lifting theorem continued to grow mainly in Romania, the U.S.A. and Venezuela.” [FF, p. viii]

Now, the “Southamerican” contribution to the subject stems from the purpose of understanding the relations between the Nagy-Foias theorem and the Cotlar-Sadosky theorem on “weakly positive” matrices of measures.

The aim of this note is to recall some aspects of a “dialogue” between those two theorems that ends by showing that they can be seen as alternative ways of describing the same facts: see below, theorems (4) and (7).

*Universidad de La República, Montevideo, Uruguay.
THE COTLAR-SADOSKY THEOREM.

We shall use the following notation:
\[ e_n(t) = e^{int}, n \in \mathbb{Z} \text{ and } t \in \mathbb{R}, \]
\[ P = \text{the space of trigonometric polynomials}, \]
\[ P = \{ \sum a_n e_n : a_n \in \mathbb{C} \}, \]
\[ \mathbb{T} = \text{the unit circle on the complex plane}, \]
\[ \mathbb{C}(\mathbb{T}) = \text{the Banach space of complex continuous functions on } \mathbb{T}, \]
\[ M(\mathbb{T}) = \text{its dual}, \]
\[ H^p = \{ f \in L^p : \hat{f}(n) = 0 \text{ if } n < 0 \}, \]
\[ \langle \mu f, f \rangle = \sum \langle \int_{\mathbb{T}} f_j \hat{f}_k d\mu_{jk} : j, k = 1, 2 \rangle. \]

Then \( \langle \mu f, f \rangle \geq 0, \forall f = (f_1, f_2) \in \mathbb{C}(\mathbb{T}) \times \mathbb{C}(\mathbb{T}), \) iff \( \mu \) is a positive matrix measure, i.e., \( \{ \mu_{jk}(\Delta) \} \) is a positive matrix for any Borel set \( \Delta \subset \mathbb{T}, \) and the Cotlar-Sadosky theorem [CS. 1] can be stated as follows.

(1) Theorem. If the matrix measure \( \mu = \{ \mu_{jk} \}_{j,k=1,2} \) is such that \( \langle \mu f, f \rangle \geq 0, \) \( \forall f = (f_1, f_2) \in P_+ \times P_-, \) there exists a positive matrix measure \( \sigma = \{ \sigma_{jk} \}_{j,k=1,2} \) such that \( \langle \sigma f, f \rangle = \langle \mu f, f \rangle, \) \( \forall f \in P_+ \times P_. \)

The above statement implies that
\[ \sigma_{11} = \mu_{11}, \sigma_{22} = \mu_{22}, \sigma_{12} = \sigma_{21} = \mu_{11} + h dt, \]
where \( dt \) is the Lebesgue measure in \( \mathbb{T} \) and \( h \in H^1. \)

The matrix \( \mu \) such that \( \langle \mu f, f \rangle \geq 0 \) for every \( f \in P_+ \times P_- \) is called “weakly positive” and the theorem says that the weakly positive form defined by \( \mu \) on \( P_+ \times P_- \) can be “lifted” (or, more precisely, extended) to the positive form defined by \( \sigma \) on \( \mathbb{C}(\mathbb{T}) \times \mathbb{C}(\mathbb{T}). \)

THE NAGY-FOIAS THEOREM IMPLIES THE COTLAR-SADOSKY THEOREM

When theorem (1) was proved in 1979, Cotlar said that it was related to the abstract version of Sarason’s generalized interpolation theorem [S.1], i.e., the famous Nagy-Foias commutant lifting theorem proved in 1968 ([NF.1]; see also [NF.2] and [FF]).

In order to recall its statement we fix the following notation. If \( G, H \) are Hilbert spaces, \( \mathcal{L}(G, H) \) is the set of bounded linear operators from \( G \) to \( H \) and \( \mathcal{L}(G) = \mathcal{L}(G, G) \); if \( K \) is a closed subspace of \( G, \) \( P_K \) denotes the orthogonal projection of \( G \) onto \( K, \) \( i_K \) the injection of \( K \) in \( G \) and \( G \ominus K \) the orthogonal complement of \( K \) in \( G. \) Also, \( \mathcal{V} \) means “closed linear span of”. Unless otherwise stated, all spaces are Hilbert spaces and all subspaces are closed subspaces.
If \( X \in \mathcal{L}(E_1, E_2) \) and \( E_j \) is a subspace of the space \( G_j, j = 1, 2, \) then \( B \in \mathcal{L}(G_1, G_2) \) is a \textit{lifting} of \( X \) if \( P_{E_2}B = XP_{E_1} \). Nagy’s dilation theorem ([NF. 2], [FF]) says that if \( T \in \mathcal{L}(E) \) is a contraction there exists an essentially unique unitary operator \( U \in \mathcal{L}(F) \) such that \( E \subset T, T^n = P_EU^n|_E \) for every \( n \geq 0 \) and \( F = \bigvee\{U^nE : n \in \mathbb{Z}\} \); \( U \) is called the \textit{minimal unitary dilation} of \( T \); set \( G = \bigvee\{U^nE : n \geq 0\} \) and \( W = UG \), then \( W \in \mathcal{L}(G) \) is the essentially unique \textit{minimal isometric dilation} of \( T \): \( W \) is an isometry that lifts \( T, P_EW = TP_E \), and \( G = \bigvee\{W^nE : n \geq 0\} \). Then:

**Theorem.** For \( j = 1, 2 \) let \( T_j \in \mathcal{L}(E_j) \) be a contraction in a Hilbert space, \( W_j \in \mathcal{L}(G_j) \) its minimal isometric dilation and \( U_j \in \mathcal{L}(F_j) \) its minimal unitary dilation. If \( X \in \mathcal{L}(E_1, E_2) \) and \( XT_1 = T_2X \), then:

i) \exists B \in \mathcal{L}(G_1, G_2) \) such that \( BW_1 = W_2B, P_{E_2}B = XP_{E_1} \); \( ||B|| = ||X||; \)

ii) \( \exists Y \in \mathcal{L}(F_1, F_2) \) such that \( YU_1 = U_2Y, P_{E_2}Y|_{E_1} = X, ||Y|| = ||X||; \)

In fact, (1) can be proved by means of (2) in the way we now sketch. Let the shift \( S \) be given by \((Sf)(z) \equiv zf(z)\). Set \( F_j = L^2(\mu_{jj}), U_j \) the shift in \( F_j, j = 1, 2, \) \( E_1 (E_2) \) the closure of \( \mathcal{P}_+ (\mathcal{P}_-) \) in \( F_1 (F_2) \), \( T_1 = U_1|_{E_1} \) and \( T_2 = P_{E_2}U_{21|E_2} \). Define \( X \in \mathcal{L}(E_1, E_2) \) by

\[
\langle Xf_1, f_2 \rangle = \int f_1 \tilde{f}_2 d\mu_{12}, \forall (f_1, f_2) \in \mathcal{P}_+ \times \mathcal{P}_-.
\]

Then \( U_j \) is the minimal unitary dilation of \( T_j, j = 1, 2, ||X|| \leq 1 \) and \( XT_1 = T_2X \). Any \( Y \) as in (2ii) is given by the multiplication by a function \( u = Ye_0 \) so \( \langle Yf_1, f_2 \rangle = \int f_1 \tilde{f}_2 u d\mu_{22}, \forall (f_1, f_2) \in \mathcal{P}_+ \times \mathcal{P}_- \). Since \( ||Y|| = ||X|| \leq 1 \), the matrix measure \( \sigma \) given by \( \sigma_{11} = \mu_{11}, \sigma_{22} = \mu_{22}, \sigma_{12} = \sigma_{21} = ud\mu_{22} \) is as stated.

**Remark** We obtained the function \( u \) because any operator \( Y \) that intertwines the shifts, i.e., such that \( YS_1 = S_2Y \), is a multiplication. In this way, the commutant lifting theorem extends Sarason’s method and gives all the solutions of several interpolation problems [FF].

THE EXTENDED COTLAR-SADOSKY THEOREM IMPLIES THE NAGY-FOIAS THEOREM

The following [CS.2] is an extension of theorem (1).

**Theorem.** For \( j = 1, 2 \) let \( V_j \) be a vector space, \( L_j \) a subspace and \( \tau_j : V_j \rightarrow V_1 \) a linear isomorphism such that \( \tau_1 L_1 \subset L_1 \) and \( \tau_2^{-1}L_2 \subset L_2, \alpha_j : V_j \times V_j \rightarrow \mathbb{C} \) is a positive form such that \( \alpha_j(\tau_j v, \tau_j w) = \alpha_j(v, w) \), and \( \beta : L_1 \times L_2 \rightarrow \mathbb{C} \) a sesquilinear form such that \( \beta(\tau_1 v_1, w_2) = \beta_0(w_1, \tau_2^{-1}w_2) \) and \( |\beta(w_1, w_2)|^2 \leq \alpha_1(w_1, w_1)\alpha_2(w_2, w_2), \forall (w_1, w_2) \in L_1 \times L_2 \). Then \( \beta' \) can be extended to a sesquilinear form \( \beta : V_1 \times V_2 \rightarrow \mathbb{C} \) such that \( \beta(\tau_1 v_1, \tau_2 v_2) = \beta(v_1, v_2) \) and \( |\beta(v_1, v_2)|^2 \leq \alpha_1(v_1, v_1)\alpha_2(v_2, v_2), \forall (v_1, v_2) \in V_1 \times V_2 \).
Set $V_j = L^2(\mu_{jj})$, $\tau_j$ the shift in $V_j$, $\alpha_j(v, w) = \int_{\mathbb{T}} v\overline{w}d\mu_{jj}$, $L_1 = \mathcal{P}_+$, $L_2 = \mathcal{P}_-$ and $\beta' = \int_{\mathbb{T}} v\overline{w}d\mu_{12}$; apply (3), then $\beta$ is given by an operator that intertwines the shifts and (1) follows as above.

When $\beta'(\tau_1 w_1, w_2) = \beta'(w_1, \tau_2^{-1} w_2)$ it is said that $\beta'$ is a generalized Hankel form, and when $\beta(\tau_1 v_1, \tau_2 v_2) = \beta(v_1, v_2)$ it is said that $\beta$ is a generalized Toeplitz form; thus, (3) is a result concerning the extension of Hankel forms to Toeplitz forms.

Theorem (1) was presented as a property of a class of “modified Toeplitz kernels” [CS.3]. That result was extended to vector valued “generalized Toeplitz kernels” in [AC], where an extension of the famous Naimark dilation for Toeplitz kernels was proved by the method of unitary extensions of isometries. The dilation theorem for generalized Toeplitz kernels gives a proof of the Nagy-Foias theorem ([A.1]: see also [FF], VII.8) so the last and (an extension of) theorem (1) are in fact closely related.

But the story of the dialogue between these two lifting theorems is much longer. For example, theorem (3) was first proved as a consequence of the Nagy-Foias theorem and then an independent proof was given by the method of unitary extensions of isometries ([CS.3]), a method by means of which a direct proof of the Nagy-Foias theorem can be given ([A.2]; see also [S.2] and [F]).

We shall now show that theorem (3) implies (2). With notations as before, assume $||X|| = 1$ and set $V_j = F_j$, $\tau_j = U_j$, $L_1 = G_1$, $L_2 = G_2 := \vee\{U_n^kE_2 : n \leq 0\}$, $\alpha_j$ the scalar product in $F_j$; let $\beta' : L_1 \times L_2 \to \mathbb{C}$ be given by $\beta'(w_1, w_2) = (XP_{E_1}w_1, w_2)$. Thus

$$\beta'(U_1w_1, w_2) = \langle XP_{E_1}W_1w_1, w_2 \rangle = \langle T_2XP_{E_1}w_1, w_2 \rangle = \beta'(w_1, U_2^{-1}w_2)$$

and $||\beta'(v_1, v_2)||^2 \leq \langle w_1, \langle w_2, v_2 \rangle \rangle \forall (w_1, w_2) \in L_1 \times L_2$. Then (3) says that there exists an extension $\beta$ of $\beta'$ such that $||\beta|| \leq 1$ and $\beta(U_1v_1, U_2v_2) = \beta(v_1, v_2)$. Consequently, there exists $Y \in \mathcal{L}(F_1, F_2)$ such that $\beta(v_1, v_2) \equiv \langle Yv_1, v_2 \rangle$ and that (2.ii) holds. Moreover, $P_{G_2^*}Y|G_1 = XP_{E_1}$; since $G_2^*\theta E_2 = F_2\theta G_2$, we see that $YG_1 \subset G_2$; setting $B = Y|G_1$, (2.1) follows.

### The Fixed Point Proof of an Extended Nagy-Foias Theorem

As an illustration of the approach to lifting problems developed in [AADM.1,2] and [G], and related with [TV], we shall sketch the proof of a particular case of the results obtained by means of a fixed point theorem.

It is said that $W \in \mathcal{L}(G)$ is an expansive operator if $||v|| \leq ||Wv||$ for every $v \in G$. The following is a slightly extended version of the Nagy-Foias theorem.

**Theorem.** Let $W_1 \in \mathcal{L}(G_1)$ be an expansive lifting of $T_1 \in \mathcal{L}(E_1)$ and $W_2 \in \mathcal{L}(G_2)$ be a contracting lifting of $T_2 \in \mathcal{L}(E_2)$; if $X \in \mathcal{L}(E_1, E_2)$ and $XT_1 = T_2X$ there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BW_1 = W_2B$, $P_{E_1}B = XP_{E_1}$ and $||B|| = ||X||$. 

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The theorem can be proved in two steps which we now sketch. We may assume $\|X\| = 1$.

**Assertion** (i) Set $\beta = \{B \in \mathcal{L}(G_1, G_2) : P_{E_2}B = XP_{E_1}, \|B\| = \|X\|\}$; for any $B \in \beta$ there exists $B^2 \in \beta$ such that $B^2W_1 = W_2B$.

Set $X' = XPE_1 \in \mathcal{L}(G_1, G_2)$; then $X' \in \beta$ and $T_2X' = X'W_1$. Thus, $B \in \beta$ iff $B = X' + K(I - X'^*X')^{1/2}$ with $K$ a contraction in $\mathcal{L}(G_1, G_2\theta E_2)$. With obvious notation, $B^2W_1 = W_2B$ iff $K^4(I - X'^*X')^{1/2}W_1 = P_{G_2\theta E_2}W_2B$. Now,

$$
\|P_{G_2\theta E_2}W_2Bw\|^2 = \|W_2Bw\|^2 - \|T_2X'w\|^2 \leq \|w\|^2 - \|X'W_1w\|^2 \\
\leq \|W_1w\|^2 - \|X'W_1w\|^2 = \|\lambda - X'^*X'\|^2
$$

for every $w \in G_1$: let $E$ be the closure of $(I - X'^*X')^{1/2}W_1G_1$; a unique contraction $K^4 \in \mathcal{L}(G_1, G_2\theta E_2)$ is defined by $K^4 = K^4P_L$ and $K^4(I - X'^*X')^{1/2}W_1w \equiv P_{G_2\theta E_2}W_2Bw$. Assertion (i) follows.

**Assertion** (ii) Set $\Sigma = \{K \in \mathcal{L}(G_1, G_2\theta E_2) : \|K\| \leq 1\}$; the map $\lambda : \Sigma \rightarrow \Sigma$ given by $\lambda(K) \equiv K^4$ has a fixed point.

With the operator topology in $\mathcal{L}(G_1, G_2\theta E_2)$, $\Sigma$ is compact and $\lambda$ is continuous: if $K_1 \rightarrow K$ in $\Sigma$ then, for every $w \in G_1$ and $w \in G_2\theta E_2$,

$$
\langle \lambda(K_1)(I - X'^*X')^{1/2}W_1w, v \rangle = \langle W_2[X' + K_1(I - X'^*X')^{1/2}]w, x \rangle \\
\rightarrow \langle W_2[X' + K(I - X'^*X')^{1/2}]w, x \rangle = \langle \lambda(K)(I - X'^*X')^{1/2}W_1w, v \rangle
$$

so $\lambda(K_1) \rightarrow \lambda(K)$. Thus, (ii) follows from the Schauder-Tychonov fixed point theorem [DS].

Clearly, if $\lambda(K) = K, B = X' + K(I - X'^*X')^{1/2}$ is as in (II.1).

**Remark** The lifting problem can have no solution: set ([FF], p.100) $E_1 = E_2 = \mathbb{C}, T_1 = T_2 = 0, X = 1, G_1 = G_2 = \mathbb{C}^2, W_1 = [w_{1j}^{(1)}]$ with $w_{11} = w_{12} = w_{21} = 0, w_{22} = 1, W_2 = [w_{2j}^{(2)}]$ with $w_{11} = w_{12} = w_{22} = 0, w_{21} = 1$. Then an operator $B$ as in (II.1) does not exist. Note that $W_2$ is a contractive lifting of $T_2$ and that $W_1$ is a lifting of $T_1$ but $W_1$ is not expansive.

(5) **Corollary.** For $j = 1, 2$ let $S_j \in \mathcal{L}(E_j)$ be a contraction with minimal isometric dilation $V_j \in \mathcal{L}(G_j)$ such that $P_{E_j}R_j = R_jP_{E_j}$. If $R_1$ is expansive, $R_2$ is contractive and $X \in \mathcal{L}(E_1, E_2)$ is such that $X S_1R_1E_1 = R_2S_2X$, then, there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BV_1R_1 = R_2V_2B$, $P_{E_2}B = XP_{E_1}$ and $\|B\| = \|X\|$.

**Proof.** Set $W_1 = V_1R_1, T_1 = S_1R_1E_1, W_2 = R_2V_2, T_2 = R_2S_2$. Then $W_1$ is expansive, $W_2$ is contractive, $P_{E_1}W_1 = T_1P_{E_1}, P_{E_2}W_2 = T_2P_{E_2}$ and $XT_1 = T_2X$. The result follows from (4).

The corollary above was suggested by the following result due to Sebestyén [Se].
(6) Theorem. Let $S \in \mathcal{L}(E)$ be a contraction with minimal isometric dilation $V \in \mathcal{L}(G)$ and $R \in \mathcal{L}(G)$ a contraction that commutes with the orthogonal projection $P_n$ of $G$ onto $\mathcal{V} = \{V^jE : 0 \leq j \leq n\}$ for $n = 0, 1, \ldots$. If $X \in \mathcal{L}(E)$ satisfies $XS = RSX$ there exists $B \in \mathcal{L}(G)$ such that $BV = RVB$, $P_{E_2}B = xP_{E_1}$ and $||B|| = ||X||$.

Since $P_0 = P_e$, (6) is a particular case of (5).

A REFORMULATION A LA COTLAR-SADOSKY OF THE EXTENDED NAGY-FOIAS THEOREM

As we shall see, the following result is not only quite similar but also equivalent to the extended Nagy-Foias theorem (4) and it gives an extension of the Cotlar-Sadosky theorem (7).

(6) Theorem. Let $E'_j$ be a Hilbert space and $T_j \in \mathcal{L}(E'_j)$, $j = 1, 2$ and $\gamma : E_1 \times E_2 \to \mathbb{C}$ a sesquilinear bounded form such that $\gamma(T_1e_1, e_2) \equiv \gamma(e_1, T_2e_2)$. If $W_1 \in \mathcal{L}(G_1)$ is an expansive lifting of $T_1$ and $W_2 \in \mathcal{L}(G_2)$ a contractive extension of $T_2$, there exists a sesquilinear bounded extension $\lambda : G_1 \times G_2 \to \mathbb{C}$ of $\gamma$ such that:

\[
\lambda(W_1g_1, g_2) = \lambda(g_1, W_2g_2) \quad (1)
\]
\[
\lambda(g_1, e_2) = \gamma(P_{E_1}g_1, e_2) \quad \text{for every } g_1 \in G_1 \text{ and } e_2 \in E_2 \quad (2)
\]
\[
||\lambda|| = ||\gamma|| \quad (3)
\]

Assertion (i) Theorem 4 implies theorem 7.

Let $X \in \mathcal{L}(E_1, E_2)$ be such that $\gamma(e_1, e_2) \equiv \langle Xe_1, e_2 \rangle$, then $XT_1 = T_1^*X$ and $W_2^*$ is a contractive lifting of $T_2^*$, so there exists $B \in \mathcal{L}(G_1, G_2)$ such that $BW_1 = W_2B$, $P_{E_2}B = XP_{E_1}$ and $||B|| = ||X||$. Setting $\lambda(g_1, g_2) \equiv \langle Bg_1, g_2 \rangle$ the result follows.

Assertion (ii) Theorem 7 implies theorem 4.

Set $\gamma(e_1, e_2) \equiv \langle Xe_1, e_2 \rangle$, then $\gamma(T_1e_1, e_2) \equiv \gamma(e_1, T_2^*e_2)$ and $W_2^*$ is a contractive extension of $T_2^*$, so there exists $\lambda$ as in (7); let $B \in \mathcal{L}(G_1, G_2)$ be such that $\lambda(g_1, g_2) \equiv \langle Bg_1, g_2 \rangle$. Then $||B|| = ||\lambda|| = ||\gamma|| = ||X||$; also,

\[
\langle BW_1g_1, g_2 \rangle \equiv \lambda(W_1g_1, g_2) \equiv \lambda(g_1, W_2^*g_2) \equiv \langle Bg_1, W_2^*g_2 \rangle,
\]

so $BW_1 = W_2B$; finally,

\[
\langle P_{E_2}Bg_1, e_2 \rangle \equiv \lambda(g_1, e_2) \equiv \gamma(P_{E_1}g_1, e_2) \equiv \langle XP_{E_1}g_1, e_2 \rangle
\]

so $P_{E_2}B = XP_{E_1}$.

Assertion (iii) Theorem 7 implies theorem 3.
For \( j = 1, 2 \) let \( F_j \) be the Hilbert space generated by the vector space \( V_j \) and the positive form \( \alpha_j \): there exists a linear operator \( \pi_j : V_j \rightarrow F_j \) such that \( \pi_j(V_j) \) is dense in \( F_j \) and \( \langle \pi_j v, \pi_j v' \rangle = \alpha_j(v, v') \) for every \( v, v' \in V_j \). Let \( U_j \in \mathcal{L}(F_j) \) be the unitary operator given by \( U_j \pi_j = \pi_j \tau_j \) and \( G_j \) be the closure in \( F_j \) of \( \pi_j L_j \). Let \( \gamma \) be a sesquilinear form \( \gamma : G_1 \times G_2 \rightarrow \mathbb{C} \) such that \( \gamma(U_1 g_1, g_2) = \gamma(g_1, U_2^{-1} g_2) \) and \( ||\gamma|| \leq 1 \) is defined by setting \( \gamma(\pi_1 v_1, \pi_2 v_2) = \beta'(v_1, v_2) \) for every \( (v_1, v_2) \in L_1 \times L_2 \).

Since \( U_2^{-1} \in \mathcal{L}(F_2) \) is a contractive extension of \( U_2^{-1} |_{G_2} \), there exists a sesquilinear form \( \lambda : G_1 \times F_2 \rightarrow \mathbb{C} \) that extends \( \gamma \) and is such that \( ||\lambda|| \leq 1 \) and that \( \lambda(U_1 g_1, U_2 f_2) = \lambda(g_1, f_2) \) holds for every \( (g_1, f_2) \in G_1 \times F_2 \).

Now set \( F_1' = \sqrt{\{U_1^{-n} G_1 : n \geq 0\}} \) and extend \( \lambda \) to a sesquilinear form \( \lambda_1 : F_1' \times F_2 \rightarrow \mathbb{C} \) by setting, for any \( n \geq 0 \) and \( (g_1, f_2) \in G_1 \times F_2 \), \( \lambda_1(U_1^{-n} g_1, f_2) = \lambda(g_1, U_2^n f_2) \), then \( ||\lambda_1|| = ||\lambda|| \) and \( \lambda_1(f_1, f_2) \equiv \lambda_1(U_1 f_1, U_2 f_2) \).

Setting \( \beta'(v_1, v_2) = \lambda_1(F_j \pi_1 v_1, \pi_2 v_2) \) for every \( (v_1, v_2) \in V_1 \times V_2 \), the result follows.

**Final Remark.**

Summing up, as Cotlar anticipated, theorems 1 and 2 are in fact very closely related.

**References.**


Rodrigo Arocena, Centro de Matemática, Facultad de Ciencias, Universidad de la República. José M. Montero 3006 Montevideo URUGUAY. Fax: (5982)402954 E-mail: rarocena@cmat.edu.uy