Some Small-Centralizer Properties for Rings

Howard E. Bell∗ Abraham A. Klein

Department of Mathematics, Brock University
St. Catharines, Ontario, Canada L2S 3A1
e-mail: hbell@brocku.ca

School of Mathematical Sciences, Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
e-mail: aaklein@post.tau.ac.il

Abstract. We characterize rings R in which certain elements x have the property that $C_R(x)$ (resp. the set of zero divisors in $C_R(x)$) is finite. We also explore the consequences of an assumption that certain x satisfy $C_R(x) = \langle x \rangle$.

1. Introduction

Let R be a ring with center Z, and let D be the set of zero divisors of R. For $x \in R$, let $C_R(x)$ be the centralizer of x in R. We study rings in which $C_R(x)$ is finite for all $x \in R \setminus Z$ and rings in which $C_R(x) \cap D$ is finite for all $x \in D \setminus Z$. In the first case we show that R is either finite or commutative; in the second case we show that either R is finite or $D \subseteq Z$.

As in [2], we call an element $x \in R$ extremely noncommutative if $C_R(x) = xZ[x]$ – i.e. if $C_R(x)$ is the subring generated by x. Our most difficult result deals with rings such that each element of $D \setminus Z$ is extremely noncommutative.

Let us fix some additional notation and terminology. Let $N = N(R)$ denote the set of nilpotent elements of R, and $T = T(R)$ the set of elements of finite additive
order. For \( x \in R \), let \( \langle x \rangle \) and \( A(x) \) be respectively the subring generated by \( x \) and the two-sided annihilator of \( x \). For a subring \( S \) of \( R \), let \( [R : S] \) denote the index of \((S,+)\) in \((R,+);\) and for a subset \( X \) of \( R \), let \( |X| \) denote the cardinality of \( X \). An element \( x \in R \) is called periodic if there exist distinct positive integers \( m, n \) for which \( x^m = x^n \), and the ring \( R \) is called periodic if each of its elements is periodic.

The following lemmas will be useful.

**Lemma 1.1.** [6] If \( R \) is a periodic ring with \( N \subseteq Z \), then \( R \) is commutative.

**Lemma 1.2.** [3] Let \( R \) be a ring such that for each \( x \in R \) there exist a positive integer \( m \) and a polynomial \( p(X) \in Z[X] \) for which \( x^m = x^{m+1}p(x) \). Then \( R \) is periodic.

**Lemma 1.3.** [7] If \( R \) is infinite and \( x \in N \), then \( |A(x)| = |R| \). In particular, \( A(x) \) is infinite.

2. Finite-centralizer conditions

**Theorem 2.1.** If \( R \) is a ring such that \( C_R(x) \) is finite for all \( x \in R\setminus Z \), then \( R \) is either finite or commutative.

**Proof.** Suppose that \( R \) is infinite. Since \( A(x) \subseteq C_R(x) \), it follows by Lemma 1.3 that \( N \subseteq Z \). Suppose also that \( R \) is not commutative and \( x \in R\setminus Z \). Since \( \langle x \rangle \subseteq C_R(x) \), \( \langle x \rangle \) is finite and hence \( x \) is periodic; and since \( Z \) is clearly finite, central elements are periodic as well. Thus, \( R \) is a noncommutative periodic ring with \( N \subseteq Z \), contrary to Lemma 1.1. Therefore \( R \) must be commutative. \( \square \)

**Theorem 2.2.** Let \( R \) be a ring such that \( C_R(x) \cap D \) is finite for all \( x \in D\setminus Z \). Then either \( R \) is finite or \( D \subseteq Z \).

**Proof.** Note that if \( S \) is any infinite subring of \( R \) such that \( C_S(x) = C_R(x) \cap S \) is finite for all \( x \in S\setminus Z \), \( S \) is commutative by Theorem 2.1 and therefore \( S \subseteq Z \). In particular, if \( S \) is any infinite subring contained in \( D \), \( S \subseteq Z \).

Suppose that \( D\setminus Z \neq \emptyset \), and assume without loss of generality that \( xy = 0 \) with \( x \in D\setminus Z \) and \( y \neq 0 \). If \( A_i(y) \) is infinite, we have \( x \in A_i(y) \subseteq Z \) – a contradiction; therefore \( A_i(y) \) is finite. For each \( w \in A_i(y) \), consider the map \( f_w : R \to A_i(y) \) given by \( f_w(r) = rw \). By applying the first isomorphism theorem for additive groups, we see that \( \ker(f_w) = A_i(w) \) is of finite index in \( R \); hence \( S = A_i(A_i(y)) \) is of finite index and therefore is infinite. Thus \( S \subseteq Z \); and since \( S \subseteq A_i(x) \), we see that \( A(x) \) is an infinite subset of \( C_R(x) \cap D \) – a contradiction. \( \square \)

3. An extreme non-commutativity condition

In [2], the following theorem is proved.

**Theorem 3.1.** If \( R \) is a ring in which all noncentral elements are extremely noncommutative, then \( R \) is either finite or commutative.
In [8], we were led to consider an infinite noncentral subring $A$ with subring $B = A \cap Z$ such that $A^2 \subseteq Z$, $A = \langle a \rangle$ for all $a \in A \setminus B$, and $[A : B]$ is a prime. In the sections headed Proof of Theorem 2.1 and Completion of proof of Theorem 2.1, we showed that such a subring cannot exist. Thus, we proved, but did not explicitly state, the following lemma.

**Lemma 3.2.** Let $R$ be an infinite noncommutative ring. Then $R$ contains no infinite noncentral subring $A$ such that $A^2 \subseteq Z$, $A = \langle a \rangle$ for all $a \in A \setminus Z$, and $[A : A \cap Z]$ is a prime.

The principal theorem of this section, which we now state, is obtained by weakening the extreme noncommutativity hypothesis in Theorem 3.1.

**Theorem 3.3.** Let $R$ be a ring in which every element of $D \setminus Z$ is extremely noncommutative. Then either $R$ is finite or $D \subseteq Z$.

The proof will be presented as a series of lemmas, the first of which is almost obvious. In each lemma, it will be assumed without explicit mention that $R$ is a ring in which every element of $D \setminus Z$ is extremely noncommutative.

**Lemma 3.4.** If $D \not\subseteq Z$, then $R$ is indecomposable. Hence $R$ has no nonzero central idempotent zero divisors.

**Lemma 3.5.** If $N \not\subseteq Z$, then $R$ is finite.

*Proof.* Since $Z$ centralizes $N \setminus Z$, $Z \subseteq N$. We show first that all zero divisors are periodic. This is clearly true for nilpotent elements, so we consider $d \in D \setminus N$. Then $d^2 \not\in N$, so $d^2 \not\in Z$ and hence $d \in \langle d^2 \rangle$. Thus there exists $p(X) \in Z[X]$ such that $d = d^2 p(d)$. Since each element of $D$ is in some subring of zero divisors, Lemma 1.2 shows that zero divisors are periodic.

Next we show that $D \subseteq T(R)$. Let $d \in D$ and $D' = \langle d \rangle$. By [1, Lemma 1(c)], $d = a + u$ with $u \in N$ and $a$ a power of $d$ such that $a^n = a$ for some $n > 1$. Now $e = a^{n-1}$ is an idempotent such that $a = ae$; and since $e$ is in the periodic ring $D'$, $2e$ is periodic, hence $e \in T(R)$ and $a \in T(R)$. We now need to show that $N \subseteq T(R)$; and since $Z = Z \cap N \subseteq (N \setminus Z) - (N \setminus Z)$, it suffices to show that $N \setminus Z \subseteq T(R)$. Let $u \in N \setminus Z$ and suppose $u^k \in T(R)$ for $k \geq 2$. Since $u \not\in Z$, there exists $n \geq 2$ such that $nu \not\in Z$; and it follows that $u \in \langle nu \rangle$, so that there exist $c_1, c_2, \ldots, c_t \in Z$ such that $u = c_1(nu) + c_2(nu)^2 + \cdots + c_t(nu)^t$. Multiplying by $u^{k-2}$ gives $(1 - c_1n)u^{k-1} \in T(R)$ and hence $u^{k-1} \in T(R)$. By backward induction, $u \in T(R)$.

We now know that if $d \in D \setminus Z$, $\langle d \rangle$ is finite and consequently $C_R(d)$ is finite. Thus, $R$ is finite by Theorem 2.2.

**Lemma 3.6.**

(i) If $D \not\subseteq Z$ and $N \subseteq Z$, then $d^n \in Z$ for all $d \in D$ and $n \geq 2$.

(ii) If $D \not\subseteq Z$, there exists a prime $p$ such that $pD \subseteq Z$. 


Proof. By Lemma 3.4, R has no nonzero idempotent zero divisors. Hence, we need only adapt in an obvious way the proof of Lemma 2.8 of [2].

Lemma 3.7. If $N \subseteq Z$, then every subring of zero divisors is commutative.

Proof. Let $H$ be any subring of zero divisors, and let $h \in H \setminus Z(H)$. Then $C_H(h) = \langle h \rangle$, so $H$ is either finite or commutative by Theorem 3.1. Moreover, if $H$ is finite, it is commutative by Lemma 1.1.

Lemma 3.8. Let $R$ be infinite with $D \not\subseteq Z$ and $N \subseteq Z$. Then

(i) $D$ is infinite;
(ii) $D$ is a commutative ideal and hence $D^2 \subseteq Z$;
(iii) $D = \langle d \rangle$ for every $d \in D \setminus Z$;
(iv) $[D : D \cap Z] = p$ for some prime $p$.

Proof. (i) follows immediately from an old theorem of Ganesan [4, 5], which asserts that any ring $R$ with $1 \leq |D \setminus \{0\}| < \infty$ must be finite.
(ii) Use the proof of Lemma 2.4 of [8], which employs Lemma 3.7.
(iii) Let $d \in D \setminus Z$. By (ii), $D \subseteq C_R(d) = \langle d \rangle$; and obviously $\langle d \rangle \subseteq D$.
(iv) Since we now know that $D$ is an additive subgroup, the result follows from (iii) and Lemma 3.6.

Proof of Theorem 3.3. Assume that $D \not\subseteq Z$. By Lemmas 3.8 and 3.2, we cannot have $N \subseteq Z$; hence $R$ is finite by Lemma 3.5.

Theorem 3.3 does not provide a characterization of rings such that all $d \in D \setminus Z$ are extremely noncommutative, since we do not have complete information about the finite examples. We do, however, have partial information.

Theorem 3.9. Let $R$ be a finite ring with $D \not\subseteq Z$ such that each $d \in D \setminus Z$ is extremely noncommutative. Then either $R$ is isomorphic to a matrix ring of form $GF(p)e_{11} + GF(p)e_{12}$ or $GF(p)e_{11} + GF(p)e_{21}$, or $R$ is nil.

Proof. If $R = D$, the result follows by Theorem 2.11 of [2]. Otherwise, if $x \in R \setminus D$, some power of $x$ is a regular idempotent, necessarily 1. Now by Lemma 1.1, there exists $u \in N \setminus Z$; and since $1 + u \in C_R(u)$, $1 + u \in \langle u \rangle$. But this is not possible, since $\langle u \rangle$ is a nil ring and $1 + u$ is invertible.

References


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