Covering a Disk by Disks

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Abstract. For a convex body $C$ in $\mathbb{R}^d$, what is the smallest number $f = f_d(C)$ such that any sequence of smaller homothetic copies of $C$ whose total volume is at least $f$ times the volume of $C$ permits a translative covering of $C$? László Fejes Tóth conjectured in 1984 that $f_2(C) \leq 3$ for any convex body $C$ in the plane. This conjecture has been only confirmed for parallelograms and triangles: Moon and Moser had shown in 1967 that $f_2(C) = 3$ for a square $C$. Since $f_d(C)$ is invariant under affine transformation of $C$, it follows from Moon and Moser’s result that $f_2(C) = 3$ for any parallelogram $C$. In 2003, Füredi settled the case of triangles with a sharper bound, by showing that $f_2(C) = 2$ for a triangle $C$, and thus confirming a stronger conjecture of A. Bezdek and K. Bezdek for this case. For an arbitrary planar convex body $C$, the current best bound is $f_2(C) \leq 6.5$, due to Januszewski. In this paper, we prove that $f_2(D) < 3$ for a disk $D$, and thereby confirm the conjecture of László Fejes Tóth for disks. We also present the first non-trivial bound for covering a disk by disks in the online model. Our methods lead to very efficient algorithms for both offline and online disk covering.

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1. Introduction

Covering a convex body by its smaller homothetic copies is a classic problem in geometry, that has generated a lot of interest over the years, initially in finding optimal structural patterns for packing and covering, and more recently in designing efficient online algorithms.

Let $C$ be a convex body in $\mathbb{R}^d$, that is, a compact convex set with nonempty interior in the $d$-dimensional Euclidean space. Let $C = \{C_1, C_2, \ldots\}$ be a (possibly infinite) sequence of convex bodies in $\mathbb{R}^d$. We say that $C$ permits a covering of $C$ if there exist rigid motions $\sigma_i$ such that $C \subseteq \bigcup_i \sigma_i(C_i)$. We say that $C$ permits a translative covering of $C$ if there exist translations $\tau_i$ such that $C \subseteq \bigcup_i \tau_i(C_i)$.

Define $f_d(C)$ as the smallest number $f$ with the following property:

- Any sequence of smaller homothetic copies of $C$ whose total volume is at least $f$ times the volume of $C$ permits a translative covering of $C$.

Very recently, Naszódi [14] showed that $f_d(C) \leq 6^d$ for any convex body $C$ in $\mathbb{R}^d$; for the planar case, the current best bound, $f_2(C) \leq 6.5$, is due to Januszewski [7]. It is also known that $f_2(C) = 3$ for a square $C$ (indeed for any parallelogram $C$ too since $f_d(C)$ is invariant under affine transformation of $C$) [13] and that $f_2(C) = 2$ for a triangle $C$ [3].

László Fejes Tóth conjectured [1, Page 131, Conjecture 1] in 1984 that for any planar convex body $C$, $f_2(C) \leq 3$.

Besides this offline setting, the above problem has also been studied in the online model [11], [8], [9], [10], [12], where each homothetic copy in the sequence is revealed only after the placement of the preceding copy in the sequence. Define $g_d(C)$ for the online model, analogous to $f_d(C)$ for the offline model. Obviously $f_d(C) \leq g_d(C)$ holds for any $C$ and $d$. It is known that $g_2(C) \leq 28$ for any convex body $C$ [8]. It is also known that $g_2(C) \leq \frac{7}{4} \sqrt{3} + \frac{13}{8} = 5.265 \ldots$ for a square $C$ [9], $g_3(C) \leq 9.843 \ldots$ for a cube $C$ [12], and $g_d(C) \leq 2^d + (5/3)(1 + 2^{-d})$ for a hypercube $C$ in $\mathbb{R}^d$ [12]. For a survey of this and many other related problems, we refer the reader to the book by Braß et al. [1, Chapter 3].

All previous results on $f_2(C)$ and $g_2(C)$, except the two general results $f_2(C) \leq 6.5$ [7] and $g_2(C) \leq 28$ [8] for any convex body in the plane, are for simple convex bodies with straight boundaries, such as squares and triangles. While the conjecture of László Fejes Tóth that $f_2(C) \leq 3$ for any planar convex body has been confirmed for squares [13] and triangles [3], it was not confirmed until now for any natural convex body with a curved boundary, where the analysis is more difficult. In this paper, we prove the conjecture of László Fejes Tóth for disks, the most natural convex bodies with curved boundaries.

Let $D$ be a unit disk. Write $\rho = f_2(D)$. That is, $\rho$ is the smallest number

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1A method by Januszewski [5] achieves a bound of 15, but the covering is not translative because it uses 90° rotations. Braß et al. [1, Page 126, Problem 1] incorrectly state that 15 is the current best bound for translative covering.

2Whether $D$ has unit radius or unit diameter will be made clear in the respective sections. For convenience, we will use different definitions of unit disk in our analysis for $f_2(D)$ and $g_2(D)$; the bounds are not affected by this difference because they are ratios.
such that any sequence of disks with total area at least $\rho$ times the area of a unit disk permits an offline covering of the unit disk (note that any disk covering is automatically translative). For $k \geq 1$, also define $\rho_k$, similar to $\rho$, but with the additional constraint that the sequence contains at most $k$ disks. It is clear that $\rho \geq \rho_{k+1} \geq \rho_k$ for any $k \geq 1$, and that $\rho = \lim_{k \to \infty} \rho_k$. Write $\eta = g_2(D)$ for the online model.  

Our main results are summarized in the following three theorems. Theorem 1 and Theorem 3 are both obtained by analytical proofs, while Theorem 2 is obtained by a computer-assisted proof.  

**Theorem 1.** Any sequence of disks with total area at least $3.25$ times the area of a unit disk permits an offline covering of the unit disk. That is, $\rho \leq 3.25$. Moreover, $\rho_1 = 1$, $\rho_2 = 2$, and $\rho_3 = \rho_4 = 2.25$.  

**Theorem 2.** Any sequence of disks with total area at least $2.97$ times the area of a unit disk permits an offline covering of the unit disk. That is, $\rho \leq 2.97$.  

**Theorem 3.** Any sequence of disks with total area at least $9.7633$ times the area of a unit disk permits an online covering of the unit disk. That is, $\eta \leq 9.7633$.  

Note that Theorem 2 confirms the conjecture of László Fejes Tóth for the disk case. Our methods for obtaining these bounds are constructive and lead to very efficient algorithms for disk covering. In particular, Theorems 1 and 2 lead to $O(n)$ time algorithms for offline covering a unit disk by a sequence of $n$ disks ordered by non-increasing radius, and Theorem 3 leads to an $O(n \log n)$ time algorithm for online covering a unit disk by a sequence of $n$ disks.  

In this paper, we cover the unit disk $D$ by a sequence $D = \langle D_1, D_2, \ldots \rangle$ of disks. The sequence $D$ can be finite or infinite. If the sequence is finite and a reference is made to a disk $D_i$ whose index $i$ is larger than the total number of disks, we assume for convenience that $D_i$ exists but has zero radius. While for offline covering we will assume that the disks in the input sequence are sorted by radius (largest disk first), no such assumption is made for online covering.  

### 2. Offline covering  

In this section we prove Theorems 1 and 2. Let the *unit disk* $D$ be a disk of unit radius. Denote by $x_i$ the radius of the $i$th disk $D_i$ in the sequence $D$. Assume that $1 > x_1 \geq x_2 \geq \cdots$. The largest four or five disks in the sequence play especially important roles in our proofs because they will be used first, in a greedy manner, to cover either the whole unit disk or two large cap regions of it. To simplify the case analysis for the relative disk sizes in our proofs, we will transform the largest four disks while maintaining the non-increasing order of the disk radii in the sequence.
2.1. Proof of Theorem 1

We first introduce two covering tools, next give bounds for the special case of covering the unit disk by at most four disks, and then prove a general bound of $\rho \leq 3.25$.

2.1.1. Covering a rectangle by disks

Define the following function $A$ of three variables $w, h, x \in \mathbb{R}$:

$$A(w, h, x) = \min \{w(h + x) + hx, w(h + x) + h^2\}.$$  

For $w \geq h > 0$ and $x > 0$, $A(w, h, x)$ is an area measure used in the following lemma by Januszewski [6], which is an extension of the classical result by Moon and Moser [13] on translative coverings of the unit square by smaller homothetic squares.

**Lemma 1.** (Januszewski [6]). Given an axis-parallel rectangle with width $w$ and height $h$ ($h \leq w$) and a sequence of axis-parallel squares with side length at most $x$, if the total area of the squares is at least $A(w, h, x)$, then the rectangle permits a translative covering by the squares.

Observe that a disk $D_k$ of radius $x_k$ contains a square of side length $\sqrt{2}x_k$ in any orientation. This leads to the following corollary:

**Corollary 1.** Given a rectangle with width $w$ and height $h$ ($h \leq w$) and a sequence of disks with radius at most $x$, if the total area of the disks is at least $\frac{\pi}{2}A(w, h, \sqrt{2}x)$, then the rectangle can be covered by the disks.

2.1.2. Covering a cap by two disks

For any $i \geq 1$, put $a_i = \arcsin x_i$. For any $i \neq j$, put $h_{ij} = \cos(a_i + a_j)$. We first prove the following lemma.

**Lemma 2.** For any $i \neq j$, $D_i$ and $D_j$ can be placed to cover a cap of angle $2a_i + 2a_j$ and height $1 - h_{ij}$ of the unit disk $D$.

**Proof.** Refer to Figure 1, where the shaded triangle $\triangle spq$ is inscribed in the unit disk, and has two sides $sp$ and $sq$ of lengths $2x_i$ and $2x_j$, respectively. Place $D_i$ and $D_j$ such that the two sides $sp$ and $sq$ are their diameters. Then the third side $pq$ of the triangle intersects the boundaries of both $D_i$ and $D_j$ at exactly the same point $t$, where $st \perp pq$. $D_i$ and $D_j$ together cover a cap of the unit disk bounded by $pq$, which subtends an angle of $2a_i + 2a_j$ from the unit disk center. The height of the cap is $1 - h_{ij}$, where $h_{ij} = \cos(a_i + a_j)$ is the signed distance from $pq$ to the unit disk center ($h_{ij}$ is negative if $2a_i + 2a_j > \pi$; in this case the unit disk center lies inside the triangle $\triangle spq$). \qed

We now prove some useful properties for covering a cap by two disks.
Lemma 3. Let $i < j$ (thus $x_i \geq x_j$ and $a_i \geq a_j$). Suppose that $0 \leq 2a_j \leq 2a_i \leq \pi$ and that $2a_i + 2a_j$ is fixed (thus $h_{ij}$ is also fixed). Then we have:

(i) If $2a_i + 2a_j \leq \pi$ ($h_{ij} \geq 0$), then $x_i^2 + x_j^2$ is non-decreasing when $x_i$ increases and correspondingly $x_j$ decreases.

(ii) If $2a_i + 2a_j \geq \pi$ ($h_{ij} \leq 0$), then $x_i^2 + x_j^2$ is non-decreasing when $x_i$ decreases and correspondingly $x_j$ increases until $x_i = x_j$.

Proof. Since $2a_i + 2a_j$ is fixed, we have $\frac{da_j}{da_i} = -1$. Then,

$$\frac{d(x_i^2 + x_j^2)}{da_i} = \frac{d(sin^2 a_i + sin^2 a_j)}{da_i} = sin 2a_i - sin 2a_j.$$

Consider two cases:

(i) $2a_i + 2a_j \leq \pi$ ($h_{ij} \geq 0$).

If $0 \leq 2a_i \leq \pi/2$, then $0 \leq 2a_j \leq 2a_i \leq \pi/2$. If $\pi/2 \leq 2a_i \leq \pi$, then $0 \leq 2a_j \leq \pi - 2a_i \leq \pi/2$. We always have $sin 2a_i - sin 2a_j \geq 0$. Therefore $x_i^2 + x_j^2$ is non-decreasing when $a_i$ (hence $x_i$) increases and correspondingly $a_j$ (hence $x_j$) decreases.

(ii) $2a_i + 2a_j \geq \pi$ ($h_{ij} \leq 0$).

If $0 \leq 2a_j \leq \pi/2$, then $0 \leq \pi - 2a_i \leq 2a_j \leq \pi/2$. If $\pi/2 \leq 2a_j \leq \pi$, then $\pi/2 \leq 2a_i \leq 2a_j \leq \pi$. We always have $sin 2a_i - sin 2a_j \leq 0$. Therefore $x_i^2 + x_j^2$ is non-decreasing when $a_i$ (hence $x_i$) decreases and correspondingly $a_j$ (hence $x_j$) increases.

The intersection of the two cases is: $2a_i + 2a_j = \pi$ ($h_{ij} = 0$). In this case, $2a_i = \pi - 2a_j$, sin $2a_i - sin 2a_j = 0$, and $x_i^2 + x_j^2$ is constant. \hfill \Box

2.1.3. The special case of covering the unit disk by at most four disks

It is a simple exercise to prove that $\rho_1 = 1$ and $\rho_2 = 2$. To show that $\rho_3 = \rho_4 = 2.25$, we first prove the following lemma.
Lemma 4. If \( 2a_1 + 2a_2 + 2a_3 + 2a_4 < 2\pi \), then \( x_1^2 + x_2^2 + x_3^2 + x_4^2 < 2.25 \).

Proof. Consider four disks \( D_1, D_2, D_3, \) and \( D_4 \) with a fixed value of \( 2a_1 + 2a_2 + 2a_3 + 2a_4 \) such that \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \) is maximized. Recall that \( 1 > x_1 \geq x_2 \geq x_3 \geq x_4 \). Since \( 2a_1 + 2a_2 + 2a_3 + 2a_4 < 2\pi \), we must have \( 2a_3 + 2a_4 < \pi \). By Lemma 4, we can increase \( x_3 \) and decrease \( x_4 \) with \( 2a_3 + 2a_4 \) fixed until either \( x_2 = x_3 \) or \( x_4 = 0 \). If \( x_4 > 0 \) after this transformation, then \( 2a_2 + 2a_4 = 2a_3 + 2a_4 < \pi \), and again by Lemma 3(i) we can increase \( x_2 \) and decrease \( x_4 \) until either \( x_1 = x_2 \) or \( x_4 = 0 \). If we still have \( x_4 > 0 \), then perform one more such transformation to increase \( x_1 \) and decrease \( x_4 \) until \( x_4 = 0 \). Finally, we have \( a_1 \geq a_2 \geq a_3 \geq a_4 = 0 \) and \( a_1 + a_2 + a_3 < \pi \). Since \( x_i = \sin^2 a_i \) is an increasing function of \( a_i \) for \( 0 \leq a_i \leq \pi/2 \), we can find three angles \( \alpha, \beta, \) and \( \gamma \) of a triangle such that \( a_1 \leq \alpha, a_2 \leq \beta, a_3 \leq \gamma, \) and \( \sin^2 a_1 + \sin^2 a_2 + \sin^2 a_3 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \). Then \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1^2 + x_2^2 + x_3^2 = \sin^2 a_1 + \sin^2 a_2 + \sin^2 a_3 < \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \leq 2.25 \), where the last step is a well-known inequality in the geometry of triangles [16].

If \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq 2.25 \), then it follows by Lemma 4 that \( 2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi \). Then, by Lemma 2, the two pairs of disks \( (D_1, D_2) \) and \( (D_3, D_4) \) can be placed to cover two caps whose union is the unit disk. Therefore \( \rho_4 \leq 2.25 \). On the other hand, it is easy to see that \( \rho_3 \geq 2.25 \), as given by the configuration of three equal disks whose diameters form an equilateral triangle inscribed in the unit disk. In fact, Füredi [4] conjectured that this configuration is the overall worst case and that \( \rho = 2.25 \). Since \( \rho_4 \geq \rho_3 \), we obtain the tight bounds \( \rho_3 = \rho_4 = 2.25 \). This proves the second part of Theorem 1.

2.1.4. A general bound of \( \rho \leq 3.25 \)

We now prove the first part of Theorem 1 by considering four cases (it is easy to check that they cover all possibilities):

Case 0 that \( 2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi \):
- \( D_1, D_2, D_3, \) and \( D_4 \) together cover the whole unit disk \( D \).

Case 1 that \( 2a_1 + 2a_2 \geq \pi \) and \( 2a_3 < \pi/2 \):
- \( D_1 \) and \( D_2 \) together cover a half of \( D \); \( D_3 \) covers less than a quarter of \( D \).

Case 2 that \( 2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2 \):
- \( D_1, D_2, \) and \( D_3 \) together cover three quarters of \( D \).

Case 3 that \( 2a_1 + 2a_2 < \pi \):
- \( D_1 \) and \( D_2 \) together cover less than a half of \( D \).

For Cases 0, 1, and 3, we use our main method illustrated in Figure 2(a): \( D_1 \) and \( D_2 \) cover a cap of height \( 1 - h_{12} \); \( D_3 \) and \( D_4 \) cover a cap of height \( 1 - h_{34} \), so that the chords of the two caps are parallel. The other disks cover a rectangle \( R \) of width 2 and height \( h_{12} + h_{34} \) (if \( h_{12} + h_{34} > 0 \)).

For Case 2, we use an alternative method illustrated in Figure 2(b): \( D_1, D_3, \) and \( D_3 \) cover three sectors of angles \( 2a_1, 2a_2, \) and \( 2a_3 \), so that the diameters of the
three disks are consecutive chords of the unit disk $D$. The other disks cover a rectangle $R$ of width 1 and height $\sin(2\pi - 2a_1 - 2a_2 - 2a_3)$.

In all four cases, we use a few large disks $D_1, \ldots, D_k$, $k = 3$ or 4, to cover part of the unit disk, then use the remaining small disks $D_i$, $i \geq k + 1$, to cover a rectangle $R$ of width $w$ and height $h$. By Corollary 1, the rectangle $R$ can be covered by the remaining disks if their total area is at least $\frac{\pi}{2} A(w, h, \sqrt{2} x_{k+1})$.

Define

$$r = \frac{\pi x_1^2 + \cdots + \pi x_k^2 + \frac{\pi}{2} A(w, h, \sqrt{2} x_{k+1})}{\pi} = x_1^2 + \cdots + x_k^2 + A(w, h, \sqrt{2} x_{k+1})/2.$$  \hfill (1)

Then $\rho \leq r$. It remains to bound the value of $r$ in each of the four cases.

Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$: Refer to Figure 1. $D_1$ and $D_2$ cover a cap of angle $2a_1 + 2a_2$, while $D_3$ and $D_4$ cover a (parallel) cap of angle $2a_3 + 2a_4$. Since $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$, the two caps overlap hence together they cover the unit disk $D$. We can assume without loss of generality that $x_5 = 0$, and suppose that $\sum_{i=1}^{\infty} x_i^2 = \sum_{i=1}^{4} x_i^2 \geq 2.25$. Then Lemma 4 implies a bound of $\rho \leq 2.25 < 3.25$ for Case 0.

Case 1 that $2a_1 + 2a_2 \geq \pi$ and $2a_3 < \pi/2$: We will reduce Case 1 to either Case 2 or Case 3. By Lemma 3(ii), we can assume that $a_1 = a_2$. Then,

$$x_1^2 + x_2^2 = \sin^2 a_1 + \sin^2 a_2 = 2 \sin^2 a_1 = 1 - \cos 2a_1 = 1 - \cos(a_1 + a_2) = 1 - h_{12},$$

$$\frac{d(x_1^2 + x_2^2)}{dh_{12}} = -1.$$
Fix $x_4$. Then,

\[ x_3^2 + x_4^2 = \sin^2 a_3 + \sin^2 a_4 \implies \frac{d(x_3^2 + x_4^2)}{da_3} = \sin 2a_3, \]

\[ h_{34} = \cos(a_3 + a_4) \implies \frac{da_3}{dh_{34}} = -\frac{1}{\sin(a_3 + a_4)}. \]

\[ 0 \leq a_3 + a_4 \leq 2a_3 < \pi/2 \implies 0 \leq \sin(a_3 + a_4) \leq \sin(2a_3) \leq 1, \]

\[ \frac{d(x_3^2 + x_4^2)}{dh_{34}} = \frac{d(x_3^2 + x_4^2)}{da_3} \cdot \frac{da_3}{dh_{34}} = -\frac{\sin 2a_3}{\sin(a_3 + a_4)} \leq -1. \]

Therefore,

\[ \frac{d(x_3^2 + x_4^2)}{dh_{34}} \leq \frac{d(x_1^2 + x_2^2)}{dh_{12}}, \]

which implies that, when keeping $h_{12} + h_{34}$ fixed, the sum $x_1^2 + x_2^2 + x_3^2 + x_4^2$ does not decrease if we decrease $h_{34}$ (increase $x_3$ and fix $x_4$) and correspondingly increase $h_{12}$ (decrease $x_1$ and $x_2$ together). Since

\[ 2a_1 = 2a_2 \geq \pi/2 > 2a_3, \]

as we decrease $2a_1 = 2a_2$, and correspondingly increase $2a_3$, either $2a_3$ will become larger than $\pi/2$, or $2a_1 = 2a_2$ will become smaller than $\pi/2$. Case 1 is therefore reduced to either Case 2 that $2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2$ or Case 3 that $2a_1 + 2a_2 < \pi$.

Case 2 that $2a_1 \geq 2a_2 \geq 2a_3 \geq \pi/2$: $D_1$, $D_2$, and $D_3$ cover three sectors of angles $2a_1$, $2a_2$, and $2a_3$ of the unit disk. Put $\theta = 2\pi - (2a_1 + 2a_2 + 2a_3)$. If $\theta \leq 0$, then we would have Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$. So assume that $0 < \theta \leq \pi/2$. The sector of angle $\theta$ is contained in a rectangle $R$ of width $w = 1$ and height $h = \sin \theta$; see Figure 2(b).

By Lemma 3(ii), we can assume that $a_1 = a_2 = a_3 = (2\pi - \theta)/6$. Therefore,

\[ x_1^2 + x_2^2 + x_3^2 = 3\sin^2 \frac{2\pi - \theta}{6} = \frac{3}{2} \left(1 - \cos \frac{2\pi - \theta}{3}\right) = \frac{3}{2} \left(1 + \cos \frac{\theta + \pi}{3}\right). \]

If $2a_4 \geq \pi/2$, then we would again have Case 0 that $2a_1 + 2a_2 + 2a_3 + 2a_4 \geq 2\pi$. So assume otherwise. Then $x_4 < \sqrt{2}/2$, and we have

\[ A(w, h, \sqrt{2} x_4)/2 \leq A(1, \sin \theta, 1)/2 \leq (1 \cdot (\sin \theta + 1) + \sin \theta \cdot 1)/2 \]

\[ = 1/2 + \sin \theta = 1/2 - \sin(\theta + \pi). \]

Therefore,

\[ r = x_1^2 + x_2^2 + x_3^2 + A(w, h, \sqrt{2} x_4)/2 \leq 2 + \frac{3}{2} \cos \frac{\theta + \pi}{3} - \sin(\theta + \pi). \]

Put $\gamma = (\theta + \pi)/3$. Then

\[ r = r(\gamma) = 2 + (3/2) \cos \gamma - \sin 3\gamma, \quad \pi/3 < \gamma \leq \pi/2. \]
Setting $\frac{dr(\gamma)}{d\gamma}$ to zero to maximize $r(\gamma)$, we have

\[-(3/2) \sin \gamma - 3 \cos 3\gamma = 0\]
\[\sin^2 \gamma = 4\cos^2 3\gamma\]
\[1 - \cos^2 \gamma = 4(4\cos^2 \gamma - 3)^2 \cos^2 \gamma.\]

Put $x = \cos^2 \gamma$, and get the following cubic equation:

\[64x^3 - 96x^2 + 37x - 1 = 0, \quad 0 < x \leq 1/4.\]  \hspace{1cm} (3)

Equation (3) has only one real root $0.02919\ldots$ between $0$ and $1/4$. Correspondingly, $r(\gamma)$ attains the maximum value $3.126\ldots$ at $\gamma = 80.16\ldots^\circ$. We have obtained a bound of $\rho \leq 3.126\ldots < 3.25$ for Case 2.

Case 3 that $2a_1 + 2a_2 < \pi$: The condition implies $h_{12} > 0$. Likewise we also have $2a_3 + 2a_4 < \pi$, and correspondingly $h_{34} > 0$. Since $2a_3 + 2a_4 < \pi$, we have $a_4 < \pi/4$. $D_1$, $D_2$, $D_3$, and $D_4$ cover two caps of total height $2 - h_{12} - h_{34}$. The remaining uncovered area of the unit disk is contained in a rectangle of width $w = 2$ and height $h = h_{12} + h_{34}$.

For two variables $x$ and $y$, $0 \leq x \leq y \leq \pi/2$, such that $x + y$ is fixed, we have

\[\frac{d(\cos x + \cos y)}{dx} = -\sin x + \sin y \geq 0.\]  

Therefore,

\[h = h_{12} + h_{34} = \cos(a_1 + a_2) + \cos(a_3 + a_4) \leq \cos(a_1 + a_4) + \cos(a_2 + a_3).\]  \hspace{1cm} (4)

Note that $a_5 \leq a_4 < \pi/4$ hence $x_5 < \sqrt{2}/2$. We have

\[A(w, h, \sqrt{2}x_5)/2 \leq h + \sqrt{2}x_5 + (h \sqrt{2}x_5)/2 < h + \sqrt{2}x_5 + h/2 \leq 1.5h + \sqrt{2}x_4.\]  \hspace{1cm} (5)

By Lemma 3(i), we can enlarge $D_1$ and correspondingly shrink $D_2$ until $x_2 = x_3$. So assume that $a_2 = a_3$. Now it follows from (5) and (4) that

\[r = x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(2, h, \sqrt{2}x_5)/2\]
\[< \sin^2 a_1 + 2 \sin^2 a_2 + \sin^2 a_4 + 1.5(\cos(a_1 + a_4) + \cos 2a_2) + \sqrt{2}\sin a_4\]
\[= \sin^2 a_1 + 1.5 \cos(a_1 + a_4) + \sqrt{2}\sin a_4 + \sin^2 a_4 + 2 \sin^2 a_2 + 1.5 \cos 2a_2\]
\[= \sin^2 a_1 + 1.5 \cos(a_1 + a_4) + \sqrt{2}\sin a_4 + \sin^2 a_4 - \sin^2 a_2 + 1.5\]
\[\leq \sin^2 a_1 + 1.5 \cos(a_1 + a_4) + \sqrt{2}\sin a_4 + 1.5.\]

Fix $a_1 + a_4$. Then,

\[\frac{dr}{da_1} = \sin 2a_1 - \sqrt{2}\cos a_4 < \sin 2a_1 - 1 \leq 0,\]

where the first inequality follows from $a_4 < \pi/4$. Therefore we can assume that $a_1 = a_4$. 

\hspace{1cm} A. Dumitrescu, M. Jiang: Covering a Disk by Disks
Put $\theta = a_1 = a_4$. Then,

$$
\begin{align*}
\rho &< \sin^2 \theta + 1.5 \cos \theta + \sqrt{2} \sin \theta + 1.5 \\
&= -2 \sin^2 \theta + \sqrt{2} \sin \theta + 3 \\
&= -2(\sin \theta - \sqrt{2}/4)^2 + 3.25 \\
&\leq 3.25.
\end{align*}
$$

We have obtained a bound of $\rho \leq 3.25$ for Case 3, and the proof of Theorem 1 is now complete.

**Extremal configuration.** It is illuminating to take a closer look at the extremal configuration in Case 3 for the 3.25 bound, where $x_1 = x_2 = x_3 = x_4 = x_5 = \sqrt{2}/4 = 0.3535\ldots$ A calculation shows that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.5$, $h = 2 \cos 2a_2 = 2(1 - 2 \sin^2 a_2) = 2(1 - 1/4) = 1.5$, and

$$
A(w, h, \sqrt{2} x_5)/2 = A(2, 1.5, 0.5)/2 = 1.5 + 0.5 + 0.375 = 2.375.
$$

So the real bound for the extremal configuration should be

$$
x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(w, h, \sqrt{2} x_5)/2 = 0.5 + 2.375 = 2.875.
$$

However, to simplify the analysis, we have used a rather conservative estimate $x_5 < \sqrt{2}/2 = 0.7071\ldots$ in the second inequality in (5), which led to a looser bound:

$$
A(w, h, \sqrt{2} x_5)/2 < 1.5 + 0.5 + 0.75 = 2.75, \\
x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(w, h, \sqrt{2} x_5)/2 < 0.5 + 2.75 = 3.25.
$$

We found it difficult to obtain a better bound than 3.25 with an analytical proof (note that it is not trivial even to determine the minimum radius of five equal disks that cover a unit disk [15]). However with the help of a computer program we have obtained a bound less than 3 (in the next subsection).

### 2.2. Proof of Theorem 2

We will use two more covering tools to obtain a bound of $\rho \leq 2.97$ with a computer- assisted proof.

#### 2.2.1. Two more covering tools

Let $D_i$, $D_j$, $D_k$ be three disks such that $i < j < k$ (thus $x_i \geq x_j \geq x_k$). Note that $D_j$ contains a copy of $D_k$. Refer to Figure 3, where the shaded trapezoid is inscribed in the unit disk. Place the large disk $D_i$ and two copies of the small disk $D_k$ such that (i) the centers of the three disks are collinear, (ii) the diameters of the two copies of $D_k$ are the left and right sides of the trapezoid, and (iii) the boundary of $D_i$ passes through the two vertices of the upper side of the trapezoid. Then the lower side of the trapezoid intersects the boundaries of $D_i$ and each copy
Let $h_{ijk}$ be the signed distance from the lower side of the trapezoid to the unit disk center. Let $2a$ and $2b$, respectively, be the upper side length and the height of the trapezoid. Let $2\alpha$ and $2\beta$, respectively, be the two angles subtended by the lower and upper sides of the trapezoid from the unit disk center. The five parameters $h_{ijk}, a, b, \alpha, \beta$ are determined by $x_i$ and $x_k$ according to the following five equations:

$$\cos \alpha = h_{ijk}, \quad \cos \beta = h_{ijk} + 2b, \quad \alpha - \beta = 2 \arcsin x_k, \quad a = \sin \beta, \quad a^2 + b^2 = x_i^2.$$ 

We have the following lemma by construction:

**Lemma 5.** $D_i, D_j,$ and $D_k$ can be placed to cover a cap of height $1 - h_{ijk}$ of the unit disk.

We will also use the following lemma by Neville [15] which provides the solution to a popular problem from the 19th century:

**Lemma 6.** (Neville [15]). A unit disk can be covered by five equal disks of radius 0.609383... .

### 2.2.2. A bound of $\rho \leq 2.97$ with a computer-assisted proof

Put $\hat{x}_5 = 0.6094$ and $\hat{r} = 2.97$. Recall our definition of $r$ in (1). Now define $r_4$ and $r_5$ as follows:

$$r_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + A(2, h_{12} + h_{34}, \sqrt{2} x_5)/2, \quad (6)$$

$$r_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + A(2, h_{12} + h_{345}, \sqrt{2} x_5)/2. \quad (7)$$

\[2, \text{Problem D3}]: “The problem of completely covering a circular region by placing over it, one at a time, five smaller equal circular disks was familiar to frequenters of English fairs a century ago.”
Our next two lemmas are about the four conditions $h_{12} + h_{34} \leq 0$, $x_5 \geq \hat{x}_5$, $\hat{r} \geq r_4$, and $\hat{r} \geq r_5$.

**Lemma 7.** If the total area of the disks in $D$ is at least $\hat{r}$ times the area of the unit disk $D$, and if any one of the four conditions is satisfied, then the unit disk $D$ can be covered by the disks in $D$.

**Proof.** We give a covering method for each condition:

1. $h_{12} + h_{34} \leq 0$: By Lemma 2, the unit disk can be covered as follows:
   (a) $D_1$ and $D_2$ cover a cap of height $1 - h_{12}$;
   (b) $D_3$ and $D_4$ cover a cap of height $1 - h_{34}$.

2. $x_5 \geq \hat{x}_5$: By Lemma 6, the unit disk can be covered by the five disks $D_1$, $D_2$, $D_3$, $D_4$, and $D_5$.

3. $\hat{r} \geq r_4$: By Lemma 2 and Corollary 1, the unit disk can be covered as follows:
   (a) $D_1$ and $D_2$ cover a cap of height $1 - h_{12}$;
   (b) $D_3$ and $D_4$ cover a cap of height $1 - h_{34}$.
   (c) If $h_{12} + h_{34} > 0$, the other disks cover a rectangle of width 2 and height $h_{12} + h_{34}$.

4. $\hat{r} \geq r_5$: By Lemma 2, Lemma 5, and Corollary 1, the unit disk can be covered as follows:
   (a) $D_1$ and $D_2$ cover a cap of height $1 - h_{12}$;
   (b) $D_3$, $D_4$, and $D_5$ cover a cap of height $1 - h_{345}$.
   (c) If $h_{12} + h_{345} > 0$, the other disks cover a rectangle of width 2 and height $h_{12} + h_{345}$. □

**Lemma 8.** If the total area of the disks in $D$ is at least $\hat{r}$ times the area of the unit disk $D$, then at least one of the four conditions is satisfied.

**Proof.** We were unable to find a simple analytical proof of Lemma 8, but have verified it by a computer program (Appendix A). The program enumerates all discrete combinations of $(x_1, x_2, x_3, x_4, x_5)$ where $1 > x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 \geq 0$ with the step size\(^4\) $\delta = 0.005$. For each discrete combination $(x_1, x_2, x_3, x_4, x_5)$, the program uses closed-form formulas to calculate

\[
\begin{align*}
h_{12} &= \cos(\arcsin(x_1) + \arcsin(x_2)) \\
h_{34} &= \cos(\arcsin(x_3) + \arcsin(x_4))
\end{align*}
\]

\(^4\)With the step size $\delta = 0.005$, the program takes less than one minute on a low-end desktop computer (tested on an Apple iMac computer with a 2GHz PowerPC G5 processor running Mac OS X 10.4.11). A smaller step size (with a longer running time) gives a bound better than 2.97, but not below 2.9.
and uses a binary search to find a value $\hat{h}_{345}$ such that $h_{345}(x_3, x_4, x_5) \leq \hat{h}_{345}$. To account for the sampling error, the program uses the enlarged values $x_i + \delta$ instead of $x_i$ in (6) and (7) to calculate
\[
\hat{r}_4 = (x_1 + \delta)^2 + (x_2 + \delta)^2 + (x_3 + \delta)^2 + (x_4 + \delta)^2 \\
+ A(2, h_{12} + h_{34}, \sqrt{2}(x_5 + \delta))/2
\]
\[
\hat{r}_5 = (x_1 + \delta)^2 + (x_2 + \delta)^2 + (x_3 + \delta)^2 + (x_4 + \delta)^2 + (x_5 + \delta)^2 \\
+ A(2, h_{12} + \hat{h}_{345}, \sqrt{2}(x_5 + \delta))/2.
\]

The program then verifies that at least one of the four conditions is satisfied.

Note that $h_{12}$ is a decreasing function of $x_1$ and $x_2$, and $h_{34}$ is a decreasing function of $x_3$ and $x_4$. Although we don’t have a closed-form formula for $h_{345}$, it is clear from our construction in Figure 3 that $h_{345}$ is a non-increasing function of $x_3$, $x_4$, and $x_5$. Also note that $A(w, h, x)$ is a non-decreasing function of $w$, $h$, and $x$. Therefore, for any (not necessarily discrete) combination $(x'_1, x'_2, x'_3, x'_4, x'_5)$ such that $x_i \leq x'_i \leq x_i + \delta$, $1 \leq i \leq 5$, we have
\[
h_{12}(x_1, x_2) + h_{34}(x_3, x_4) \leq 0 \quad \implies \quad h_{12}(x'_1, x'_2) + h_{34}(x'_3, x'_4) \leq 0
\]
\[
x_5 \geq \hat{x}_5 \quad \implies \quad x'_5 \geq \hat{x}_5
\]
\[
\hat{r} \geq \hat{r}_4(x_1, x_2, x_3, x_4, x_5) \quad \implies \quad \hat{r} \geq \hat{r}_4(x'_1, x'_2, x'_3, x'_4, x'_5)
\]
\[
\hat{r} \geq \hat{r}_5(x_1, x_2, x_3, x_4, x_5) \quad \implies \quad \hat{r} \geq \hat{r}_5(x'_1, x'_2, x'_3, x'_4, x'_5).
\]

Since all discrete combinations are checked by the program, it follows that all possible combinations are also verified.

By Lemma 7 and Lemma 8, it follows that $\rho \leq \hat{r} = 2.97$. The proof of Theorem 2 is now complete. Given a sequence $\mathcal{D}$ of $n$ disks ordered by non-increasing radius, a covering as in Lemma 1 can be obtained in $O(n)$ time. As a consequence, Theorems 1 and 2 lead to $O(n)$ time algorithms for offline covering under the same disk order assumption.

## 3. Online covering

In this section we prove Theorem 3. Let the unit disk $D$ be a disk of unit diameter\(^5\). Denote by $d_i$ the diameter of the $i$th disk $D_i$ in the sequence $\mathcal{D}$. Denote by $|C|$ the area of a convex body $C$ in the plane. Let $\mathbb{N} = \{0, 1, 2, \ldots \}$ denote the set of natural numbers, and let $\mathbb{N}^+ = \{1, 2, 3, \ldots \}$ denote the set of positive integers.

The unit disk $D$ is contained in a unit square $S$. Each disk $D_i$ of diameter $d_i$ contains a square $S_i$ of side length $s_i = d_i/\sqrt{2}$. Note that $|S|/|D| = 4/\pi$ and $|D_i|/|S_i| = \pi/2$. Therefore, using the current best bound for online covering a unit square $S$ by squares $S_i$, namely $7\sqrt{3}/4 + 13/8 = 5.265 \ldots [9]$, we immediately obtain a bound of
\[
\eta \leq \frac{4}{\pi} \cdot \frac{\pi}{2} \cdot \left(\frac{7\sqrt{3}}{4} + \frac{13}{8}\right) = 10.5302 \ldots
\]

\(^5\)The unit disk was defined as a disk of unit radius in Section 2. Here we use a different definition for convenience in analysis.
By using an efficient adaptation of a method by Januszewski and Lassak [8, 9], we obtain a better bound of \( \eta < 9.7633 \). The idea is to use an inscribed rectangle \( R_i \) in each disk \( D_i \), instead of an inscribed square \( S_i \), to cover the unit square \( S \).

We first review some basic techniques for online covering [8, 9]. Suppose we want to cover the unit square \( S = [0, 1]^2 \) by a sequence \( S \) of axis-parallel squares. And suppose that each square \( S_i \in S \) is normalized: its side length has the form \( 2^{-r} \), \( r \in \mathbb{N}^+ \). The method of the current bottom [8] places each square \( S_i \) as follows: First find the largest number \( b_i \) such that every point of \( S \) with \( y \)-coordinate at most \( b_i \) has been covered by some square \( S_j \), \( j < i \). The set of points of \( S \) with \( y \)-coordinate equal to \( b_i \) is called the \( i \)th bottom. A point of the \( i \)th bottom is called a surface point if no point of \( S \) with the same \( x \)-coordinate and with a larger \( y \)-coordinate has been covered by the preceding squares. Now place \( S_i \) at the bottom, that is, find a translation \( \tau_i \) such that \( \tau_i(S_i) \) contains a surface point and has the form

\[
\left\{ (x, y) \mid m 2^{-r} \leq x \leq (m+1)2^{-r} \text{ and } b_i \leq y \leq b_i + 2^{-r} \right\},
\]

where \( m \in \{0, \ldots, 2^r - 1\} \).

Since \( \tau_i(S_i) \) contains a surface point on its lower side, it does not overlap with the preceding squares that are larger or of equal size. Hence the upper half of \( \tau_i(S_i) \) is not covered by the preceding squares. The lower half of \( \tau_i(S_i) \) consists of the lower-left quarter and the lower-right quarter; at least one of the two quarters contains a surface point on its lower side. Apply the same argument recursively to this quarter of \( \tau_i(S_i) \), and it follows that the fraction of the area of \( \tau_i(S_i) \) not covered by the preceding squares is at least

\[
\frac{1}{2} \left( 1 + \frac{1}{4} + \frac{1}{4 \cdot 4} + \frac{1}{4 \cdot 4 \cdot 4} + \cdots \right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}.
\]

That is, \( 2/3 \) of the area of \( \tau_i(S_i) \) is covered for the first time. Similarly observe that, above the current bottom \( b_i \), an area of at most \( 1/3 \) (the total area of a collection of squares, one of each side length \( 2^{-k} \), \( k = 1, 2, \ldots \)) is covered by the squares \( S_j \), \( j < i \). Therefore, the total area of the squares preceding \( S_i \) is at most

\[
\frac{3}{2} \left( b_i + \frac{1}{3} \right).
\]

The unit square \( S \) becomes completely covered when \( b_i \) reaches 1. Thus a total area of \( (3/2)(1 + 1/3) = 2 \) is sufficient for online covering a unit square by normalized squares. Since every square contains a normalized square of at least \( 1/2 \) of its side length and hence at least \( 1/4 \) of its area, it follows that \( g_2(C) \leq 8 \) for a square \( C \). The method of the current bottom has been extended to the method of the current bottom and top [8, 9], in which a square \( S_i \) may be placed at either the current bottom \( b_i \) or the current top \( t_i \) (which is defined analogously). Initially, \( b_1 = 0 \) and \( t_1 = 1 \). The unit square \( S \) becomes completely covered when \( b_i \geq t_i \) for some \( i \). This extended method yields the current best bound of \( g_2(C) \leq \frac{7}{4} \sqrt{9} + \frac{12}{5} = 5.265 \ldots \) for a square \( C \) [9].
We now outline another way to extend the method of the current bottom. Observe that, when the sequence $S$ of squares is replaced by a sequence $B$ of similar rectangles with width and height of the form $2^{-r}$ and $u \cdot 2^{-r}$, $r \in \mathbb{N}^+$, the previous argument for the ratio 2/3 remains valid. The total area of the rectangles preceding rectangle $B_i \in B$ becomes
\[
\frac{3}{2} \left( b_i + \frac{u}{3} \right).
\]
Now suppose that we have another sequence $T$ of similar rectangles with width and height of the form $2^{-r}/3$ and $v \cdot 2^{-r}/3$, $r \in \mathbb{N}$. To cover the unit square $S$ from the top, place each rectangle $T_i \in T$ such that $\tau_i(T_i)$ has the form
\[
\{(x, y) \mid m \cdot 2^{-r}/3 \leq x \leq (m + 1)2^{-r}/3 \text{ and } t_i - 2^{-r}/3 \leq y \leq t_i\},
\]
where $m \in \{0, \ldots, 2^r - 1\}$.
Then the total covered area below the current top becomes
\[
\frac{v}{9} + \frac{v}{9} + \frac{v}{9} \cdot 4 + \frac{v}{9} \cdot 4 \cdot 4 + \cdots = \frac{v}{9} + \frac{v}{1 - \frac{1}{3}} = \frac{7v}{27}.
\]
The total area of the rectangles preceding $T_i$ becomes
\[
\frac{3}{2} \left( 1 - t_i + \frac{7v}{27} \right).
\]
We now present a method that covers the unit disk $D$ by a sequence $D$ of disks. We show that each disk $D_i \in D$ of diameter $d_i$ contains a normalized rectangle $R_i$ of width $w_i$ and height $h_i$ (defined below), and use these rectangles $R_i$ to cover the unit square $S$ containing $D$. Let $1 < c < 2$. The exact value of $c$ will be determined later. Consider two cases:

1. $\frac{1}{c} \cdot 2^{-k} \leq d_i < 2^{-k}, k \in \mathbb{N}$. Then $D_i$ contains a rectangle $R_i$ of width $w_i = \frac{1}{3} \cdot 2^{-k}$ and height $h_i = \frac{v}{3} \cdot 2^{-k}$, where $v = \sqrt{4/c^2 - 1}$. Place $R_i$ to cover $S$ from the bottom. Define
\[
f(c) = \frac{\pi}{u}.
\]
We have
\[
\frac{|D_i|}{|R_i|} = \frac{\pi(d_i/2)^2}{w_i h_i} \leq \frac{\pi/4}{u/4} = f(c).
\]

2. $\frac{1}{c} \cdot 2^{-k} \leq d_i < \frac{1}{c} \cdot 2^{-k}, k \in \mathbb{N}$. Then $D_i$ contains a rectangle $R_i$ of width $w_i = \frac{1}{3} \cdot 2^{-k}$ and height $h_i = \frac{v}{3} \cdot 2^{-k}$, where $v = \sqrt{5}/2$. Place $R_i$ to cover $S$ from the top. Define
\[
g(c) = \frac{9\pi}{4c^2v}.
\]
We have
\[
\frac{|D_i|}{|R_i|} = \frac{\pi(d_i/2)^2}{w_i h_i} \leq \frac{\pi/(4c^2)}{v/9} = g(c).
\]
We now show that our method achieves a bound of $\eta < 9.7633$. Define
\[ b(z) = \frac{3}{2} \left( z + \frac{u}{3} \right), \quad t(z) = \frac{3}{2} \left( 1 - z + \frac{7v}{27} \right), \]
\[ r(z, c) = \frac{b(z) \cdot f(c) + t(z) \cdot g(c)}{\pi/4}. \]

Note that $b(z) \cdot f(c)$ and $t(z) \cdot g(c)$ bound the total areas of the disks that cover the unit square $S$ from the bottom and from the top, respectively, and that $\pi/4$ is the area of the unit disk $D$. Then we have
\[ \eta \leq \max_{0 \leq z \leq 1} r(z, c). \]

Now,
\[
r(z, c) = \frac{4}{\pi} \cdot \frac{3}{2} \left( z + \frac{u}{3} \right) \cdot \pi + \frac{4}{\pi} \cdot \frac{3}{2} \left( 1 - z + \frac{7v}{27} \right) \cdot \frac{9\pi}{4c^2v} \\
= \left( \frac{6}{u} - \frac{27}{2c^2v} \right) \cdot z + 2 + \frac{27}{2c^2v} + \frac{7}{2c^2}.
\]

Let $c$ be the solution of the following equation:
\[
\frac{6}{u} - \frac{27}{2c^2v} = 0.
\]

Then $r(z, c)$ does not depend on $z$. A calculation shows that $c = 1.4164\ldots$ and $r(z, c) = 9.7632\ldots$. Therefore $\eta \leq 9.7633$. This completes the proof of Theorem 3.

An $O(n \log n)$-time algorithm can be achieved by using a linked list to represent the “coastline” of horizontal segments bounding from above the current covered area at the bottom of the unit square, and by maintaining these segments in a priority queue. The segments bounding from below the current covered area at the top of the unit square are maintained in a similar way. The amortized cost for processing a disk is $O(\log n)$.

A. Source code

```c
#include <math.h>
#include <stdio.h>
#define RATIO 2.97
#define STEP 0.005

double find_hijk(double xi, double xj, double xk);

int main() {
    double x1, x2, x3, x4, x5;
    double x1_, x2_, x3_, x4_, x5_;
    double s1_, s2_, s3_, s4_, s5_;
```
double a1, a2, a3, a4;
double h12, h34, h345;
double w, h, x, x_; 
double r4_, r5_; 
double start = 0.0, end = 1.0;

printf("Testing ratio %g with step size %g ...\n", RATIO, STEP);
for (x1 = start; x1 <= end; x1 += STEP) {
    x1_ = x1 + STEP; s1_ = x1_ * x1_;
    a1 = asin(x1);
    for (x2 = start; x2 <= x1; x2 += STEP) {
        x2_ = x2 + STEP; s2_ = s1_ + x2_ * x2_;
        a2 = asin(x2);
        h12 = cos(a1 + a2);
        for (x3 = start; x3 <= x2; x3 += STEP) {
            x3_ = x3 + STEP; s3_ = s2_ + x3_ * x3_;
            a3 = asin(x3);
            for (x4 = start; x4 <= x3; x4 += STEP) {
                x4_ = x4 + STEP; s4_ = s3_ + x4_ * x4_;
                a4 = asin(x4);
                h34 = cos(a3 + a4);
                if (h12 + h34 <= 0.0) /* condition 1 */
                    break;
                for (x5 = start; x5 <= x4; x5 += STEP) {
                    if (x5 >= 0.6094) /* condition 2 */
                        break;
                    x5_ = x5 + STEP; s5_ = s4_ + x5_ * x5_;
                    x = x5_ * M_SQRT2;
                    w = 2.0;
                    h = h12 + h34;
                    x_ = x < h ? x : h;
                    r4_ = s4_ + (w * (h + x) + h * x_) / 2.0;
                    if (RATIO >= r4_) /* condition 3 */
                        continue;
                    h345 = find_hijk(x3, x4, x5);
                    h = h12 + h345;
                    x_ = x < h ? x : h;
                    r5_ = s5_ + (w * (h + x) + h * x_) / 2.0;
                    if (RATIO >= r5_) /* condition 4 */
                        continue;
                }
            }
        }
    }
}

/* difficult case */
printf("%5.3f %5.3f %5.3f %5.3f %5.3f %5.3f\n", x1, x2, x3, x4, x5, r4_, r5_);
printf(" x %5.3f h12+h34 %5.3f h12+h345 %5.3f\n", x, h12 + h34, h12 + h345);


double find_hijk(double xi, double xj, double xk) {
    double ak = asin(xk);
    double a, b, h;
    double alpha, beta;
    double upper = cos(ak * 2.0); /* trapezoid becomes triangle */
    double lower = -xk; /* trapezoid becomes rectangle */
    while (upper - lower > 0.001) { /* binary search */
        h = (upper + lower) / 2.0;
        alpha = acos(h);
        beta = alpha - ak * 2.0;
        a = sin(beta);
        b = (cos(beta) - h) / 2.0;
        if (a * a + b * b <= xi * xi)
            upper = h;
        else
            lower = h;
    }
    return upper;
}

References


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