Cardinality Estimates for Piecewise Congruences of Convex Polygons

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Abstract. Two convex polygons $P, P' \subseteq \mathbb{R}^2$ are congruent by dissection with respect to a given group $G$ of transformations of $\mathbb{R}^2$ if both can be dissected into the same finite number $k$ of polygonal pieces $Q_1, \ldots, Q_k$ and $Q'_1, \ldots, Q'_k$ such that corresponding pieces $Q_i, Q'_i$ are congruent with respect to $G$, $1 \leq i \leq k$. In that case $\text{deg}_G(P, P')$ denotes the smallest $k$ with the above property.

For the group Isom$^+$ of proper Euclidean isometries we prove two general upper estimates for $\text{deg}_{\text{Isom}^+}(P, P')$, the first one in terms of the numbers of vertices and the diameters of $P, P'$, the second one depending moreover on the radii of inscribed circles. A particular result concerns regular polygons $P, P'$.

For the groups Sim$^+$ and Sim of proper and general similarities we give upper bounds for $\text{deg}_{\text{Sim}^+}(P, P')$ and $\text{deg}_{\text{Sim}}(P, P')$ in terms of the numbers of vertices.

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1. Introduction

Given a group $G$ of affine transformations of the Euclidean plane $\mathbb{R}^2$, two polygons $P, P' \subseteq \mathbb{R}^2$ are called congruent by dissection (or equidissectable) with respect to $G$ if there exist a number $k \in \{1, 2, \ldots\}$ and dissections of $P$ into polygons $Q_1, \ldots, Q_k$ and of $P'$ into polygons $Q'_1, \ldots, Q'_k$ such that, for every $i \in \{1, \ldots, k\}$, $Q_i$ and $Q'_i$ are congruent with respect to $G$. Here a polygon is meant to be a union of finitely many triangles. We say that $P$ is dissected into $Q_1, \ldots, Q_k$ if $P = Q_1 \cup \cdots \cup Q_k$ and the interiors of distinct pieces $Q_i, Q_j$, $i \neq j$, are disjoint. If $P$ and $P'$ are equidissectable, the minimal number $k$ admitting dissections with the above property is called the degree of the congruence by dissection of $P$ and $P'$. This optimal number of pieces is denoted by $\deg_G(P, P')$.

Of course, if $P$ and $P'$ are congruent by dissection with respect to a subgroup $H$ of $G$, then $P$ and $P'$ are equidissectable with respect to $G$, too, and

$$\deg_G(P, P') \leq \deg_H(P, P').$$

The present paper is mainly devoted to degree estimates for congruences by dissection of convex polygons with respect to the group Isom of Euclidean isometries and the subgroup $\text{Isom}^+$ of proper isometries.

The classical Wallace-Bolyai-Gerwien theorem says that any two polygons of the same area are equidissectable with respect to Isom (see [4] and [10, Chapter 3] for historical remarks). The group containing all translations and all central reflections is known to be the smallest subgroup of Isom satisfying the above property (see [5, 1]). However, the question for the degree of congruences by dissection is rather open. The following theorem by Hertel seems to give the first upper estimate for $\deg_{\text{Isom}}(P, P')$ concerning general polygons of equal area.

**Theorem 1.** ([6], Satz 2) Let $P_m$ and $P'_n$ be an $m$-gon and an $n$-gon of the same area whose diameters are $d$ and $d'$, respectively. Suppose that there exist dissections of $P_m$ into $m-2$ triangles $T_1, \ldots, T_{m-2}$ and of $P'_n$ into $n-2$ triangles $T'_1, \ldots, T'_{n-2}$ and define

$$c = \min\{\text{diam}(T_1), \ldots, \text{diam}(T_{m-2}), \text{diam}(T'_1), \ldots, \text{diam}(T'_{n-2})\}.$$

Then

$$\deg_{\text{Isom}}(P_m, P'_n) \leq 4(m - 2)(n - 2) \left(\frac{\max\{d,d'\}}{c} + 2\right)^2.$$

Our first goal in Section 2 is an upper estimate for $\deg_{\text{Isom}}(P_m, P'_n)$ for arbitrary convex $P_m$ and $P'_n$ only depending on $m, n, d$, and $d'$. We shall see that Theorem 1 gives a bound of that kind depending on $m$ and $n$ like a polynomial of degree 4 and of quadratic behaviour in $\max\{\frac{d}{d'}, \frac{d'}{d}\}$. Then we establish a stronger estimate for $\deg_{\text{Isom}^+}(P_m, P'_n)$ quadratic in $m, n$ and linear in $\max\{\frac{d}{d'}, \frac{d'}{d}\}$ (see Theorem 2). An important technical step to this main result concerns the piecewise congruence of triangles (see Lemma 3).

The estimate of Theorem 2 can be improved if $P_m$ and $P'_n$ are known to contain inscribed circles of sufficiently large radii (see Theorem 3). In particular,
deg_isom^+(P_m^r, P_n^r) \leq 7(m + n - 1) for regular polygons of the same area with m and n vertices, respectively (see Theorem 4). This improves the bound

\[ \deg_{\text{Isom}}(P_m^r, P_n^r) \leq (2m + 4)(n + 1) \] for \( 3 \leq m < n \)

(1)
given by Doyen and Landuyt without proof (see [2]).

Nevertheless, our estimates do not use to be sharp in particular situations. For example, isometric congruences by dissection of regular \( P_m^r \) and \( P_n^r \) with very small \( m, n \), say \( m, n \leq 12 \), are known to require much less than \( 7(m + n - 1) \) pieces. We refer to Theobald’s frequently updated web page [9].

Let us point out that the estimates for \( \deg_{\text{Isom}}^+(P_m, P_n) \) from Theorems 2, 3, and 4 can be realized by dissections into convex pieces.

In the last section we shall summarize new degree estimates for congruences by dissection with respect to similarities. The paper is closed with a remark about the group of translations.

We use the following notations: Given three points \( x_1, x_2, x_3 \in \mathbb{R}^2 \), the symbols \( l(x_1, x_2), x_1x_2, |x_1x_2| = d(x_1, x_2), \angle x_1x_2x_3, \) and \( \|\angle x_1x_2x_3\| \) stand for the straight line passing through \( x_1, x_2 \), the line segment between \( x_1, x_2 \), the length of that segment (which is the Euclidean distance of \( x_1, x_2 \)), the angle determined by \( x_1, x_2, x_3 \), and the size of that angle, respectively. \( \text{int}(A), \text{bd}(A), \text{conv}(A), \text{diam}(A) = \sup\{d(x_1, x_2) : x_1, x_2 \in A\}, \) and \( \lambda(A) \) denote the interior, the boundary, the convex hull, the diameter, and the area measure, respectively, of a (measurable) subset \( A \subseteq \mathbb{R}^2 \). \( \lfloor \xi \rfloor \) and \( \lceil \xi \rceil \) are the largest lower and the smallest upper integer bound of \( \xi \in \mathbb{R} \).

2. Congruence by dissection with respect to isometries

2.1. An estimate in terms of vertex numbers and diameters based on Theorem 1

Corollary. Let \( P_m \) and \( P_n^r \) be convex polygons of the same area whose numbers of vertices are \( m \) and \( n \) and whose diameters are \( d \) and \( d^r \), respectively. Then

\[ \deg_{\text{Isom}}(P_m, P_n^r) \leq 4(m - 2)(n - 2) \left( \max\{d, d^r\} \max\left\{ \frac{\lfloor m/2 \rfloor}{d}, \frac{\lfloor n/2 \rfloor}{d^r} \right\} + 2 \right)^2. \]

In particular

\[ \deg_{\text{Isom}}(P_m, P_n^r) < mn(m + n)^2 \left( \max\{d, d^r\} \right)^2. \]

For proving this corollary, we need a lower bound for the value \( c \) from Theorem 1.

Lemma 1. Every convex \( m \)-gon \( P_m \) admits a dissection into \( m - 2 \) triangles \( T_1, \ldots, T_{m-2} \) such that

\[ \min\{\text{diam}(T_1), \ldots, \text{diam}(T_{m-2})\} \geq \frac{1}{\lfloor m/2 \rfloor} \text{diam}(P_m). \]

One can obtain a strict inequality if \( m \neq 3 \).
Case 2. diam(P) = max_{1 \leq i < j \leq 4} d(x_i, x_j) < d_1 + d_2.

Cutting P_4 along x_1x_3 gives a dissection into two triangles each having a diameter of at least

\[ d(x_1, x_3) = d_1 \geq \frac{d_1 + d_2}{2} > \frac{\text{diam}(P_4)}{2} = \frac{1}{\left\lceil \frac{1}{2} \right\rceil} \text{diam}(P_4). \]

Now let \( m \geq 5 \). Let \( x_1, \ldots, x_m \) be the vertices of \( P_m \) in their order along the boundary of \( P_m \).

Case 1. diam(P_m) is the length of a diagonal, say of \( x_1x_k, k \in \{3, \ldots, m - 1\} \).

Then \( x_1x_k \) dissects \( P_m \) into a \( k \)-gon \( P_{k,1} = \text{conv}\{x_1, \ldots, x_k\} \) and an \((m - k - 2)\)-gon \( P_{m-k+2,2} = \text{conv}\{x_k, x_{k+1}, \ldots, x_m, x_1\} \), both with diameter \( d(x_1, x_k) = \text{diam}(P_m) \). Application of the induction hypothesis to \( P_{k,1} \) and \( P_{m-k+2,2} \) yields a dissection of \( P_m \) with the required properties.

Case 2. diam(P_m) is the length of an edge of \( P_m \), say of \( x_1x_m \).

Case 2.1. \( d(x_1, x_m) > \frac{1}{\left\lceil \frac{1}{2} \right\rceil} \text{diam}(P_m) \).

We cut \( P_m \) along \( x_1x_3 \) into the triangle \( T = \triangle x_1x_2x_3 \) and the \((m - 1)\)-gon \( P_{m-1} = \text{conv}\{x_3, x_4, \ldots, x_m, x_1\} \) of diameter \( d(x_1, x_m) = \text{diam}(P_m) \). Then \( \text{diam}(T) \geq d(x_1, x_3) > \frac{1}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m) \). Dissection of \( P_{m-1} \) according to the induction hypothesis gives \( m - 3 \) additional triangles of sufficiently large diameters.

Case 2.2. \( d(x_1, x_3) \leq \frac{1}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m) \).

Then

\[ d(x_3, x_m) > d(x_1, x_m) - d(x_1, x_3) \geq \left(1 - \frac{1}{\left\lceil \frac{1}{4} \right\rceil}\right) \text{diam}(P_m) = \frac{m - 3}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m). \]

We split \( P_m \) along \( x_3x_m \) into the quadrilateral \( P_{4,1} = \text{conv}\{x_1, x_2, x_3, x_m\} \) with \( \text{diam}(P_{4,1}) = d(x_1, x_m) = \text{diam}(P_m) \) and into \( P_{m-2,2} = \text{conv}\{x_3, \ldots, x_m\} \) with \( \text{diam}(P_{m-2,2}) \geq d(x_3, x_m) > \frac{m - 3}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m) \). The induction hypothesis gives dissections of \( P_{4,1} \) and of \( P_{m-2,2} \) into two and into \( m - 4 \) triangles, respectively. The two pieces \( T \) of \( P_{4,1} \) satisfy

\[ \text{diam}(T) > \frac{1}{2} \text{diam}(P_{4,1}) = \frac{1}{2} \text{diam}(P_m) \geq \frac{1}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m). \]

The \( m - 2 \) pieces \( U \) of \( P_{m-2,2} \) have diameters

\[ \text{diam}(U) \geq \frac{1}{\left\lceil \frac{1}{2} \right\rceil} \text{diam}(P_{m-2,2}) > \frac{1}{\left\lceil \frac{1}{2} \right\rceil} \frac{m - 3}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m) = \frac{1}{\left\lceil \frac{1}{4} \right\rceil} \text{diam}(P_m). \]

This completes the proof. \( \square \)

**Proof of the Corollary.** We apply Theorem 1 to \( P_m \) and \( P'_n \). According to Lemma 1 there exist dissections of \( P_n \) and \( P'_n \) such that the value \( c \) from the theorem satisfies

\[ c \geq \min \left\{ \frac{d}{\left\lceil \frac{1}{4} \right\rceil}, \frac{d}{\left\lceil \frac{1}{2} \right\rceil} \right\}, \quad \text{that is} \quad \frac{1}{c} \leq \max \left\{ \frac{\left\lceil \frac{1}{4} \right\rceil}{d}, \frac{\left\lceil \frac{1}{2} \right\rceil}{d} \right\}. \]
Combining this with the statement of the theorem we obtain the first estimate of the corollary. The second one is proved by

\[
\deg_{\text{isom}}(P_m, P'_n) \leq 4(m - 2)(n - 2)\left(\max\{d, d'\} \max\left\{\frac{m}{2}, \frac{n}{2}\right\} + 2\right)^2
\]
\[
\leq 4(m - 2)(n - 2)\left(\max\left\{\frac{d}{\delta}, \frac{d'}{\delta}\right\} \max\left\{\frac{m}{2}, \frac{n}{2}\right\} + 2\right)^2
\]
\[
= (m - 2)(n - 2)\left(\max\{d, d'\} \max\{m, n\} + 4\right)^2
\]
\[
\leq (m - 2)(n - 2)\left((m + n - 3) + 4\right)^2\left(\max\left\{\frac{d}{\delta}, \frac{d'}{\delta}\right\}\right)^2
\]
\[
= (m - 2)(m + n + 1)(n - 2)(m + n + 1)\left(\max\left\{\frac{d}{\delta}, \frac{d'}{\delta}\right\}\right)^2
\]
\[
< m(m + n)(m + n)\left(\max\left\{\frac{d}{\delta}, \frac{d'}{\delta}\right\}\right)^2. \quad \square
\]

We close this subsection by a claim showing that Lemma 1 is sharp. In this sense the estimate (2) is best possible. This justifies the formulation of the corollary.

**Lemma 2.** For every \( m \in \{3, 4, 5, \ldots\} \) and every \( \varepsilon > 0 \), there exists a convex \( m \)-gon \( P_m \) such that every dissection of \( P_m \) into \( m - 2 \) triangles \( T_1, \ldots, T_{m-2} \) satisfies

\[
\min\{\text{diam}(T_1), \ldots, \text{diam}(T_{m-2})\} < \left(\frac{1}{\Delta} + \varepsilon\right) \text{diam}(P_m).
\]

**Sketch of the proof.** We assume \( m \geq 4 \), because the case \( m = 3 \) is trivial. Let \( \delta > 0 \) be small. We define \( P_m = \text{conv}\{x_1, \ldots, x_m\} \) where

\[
x_i = \begin{cases} 
(\cos((i - 1)\delta + \delta^2), \sin((i - 1)\delta + \delta^2)) & \text{if } i \text{ is odd,} \\
(\cos(i\delta), \sin(i\delta)) & \text{if } i \text{ is even.}
\end{cases}
\]

Then \( \text{diam}(P_m) = d(x_1, x_m) \) (if \( \delta \) is sufficiently small).

Suppose that \( P_m \) is dissected into \( m - 2 \) triangles \( T_1, \ldots, T_{m-2} \). It can be shown by Euler’s formula that all vertices of the triangles are vertices of \( P_m \), too. Any such triangulation contains two different triangles each of them sharing two edges with the boundary of \( P_m \). Only one of them can contain the long edge \( x_1 x_m \). Hence at least one of them is of the form \( T = \Delta x_i x_{i+1} x_{i+2} \) with \( 1 \leq i \leq m - 2 \).

Consequently, \( \text{diam}(T) = d(x_i, x_{i+2}) = d(x_1, x_3) = d(x_2, x_4) \). We obtain

\[
\frac{\text{diam}(T)}{\text{diam}(P_m)} = \frac{d(x_1, x_3)}{d(x_1, x_m)} \quad \text{and} \quad \lim\delta_{\varepsilon 0} \frac{d(x_1, x_3)}{d(x_1, x_m)} = \lim\delta_{\varepsilon 0} \frac{2\delta}{\Delta} \frac{1}{\Delta} = \frac{1}{\Delta}. 
\]

Thus \( P_m \) satisfies the claim of Lemma 2 if \( \delta \) is sufficiently small. \( \square \)

### 2.2. Congruence by dissection of triangles

**Lemma 3.** Let \( T \) and \( T' \) be triangles of the same area having the diameters \( d \) and \( d' \), respectively. Then

\[
\deg_{\text{isom}^+}(T, T') \leq 4\left[\frac{1}{2} \max\left\{\frac{d}{\delta}, \frac{d'}{\delta}\right\}\right] + 3.
\]
Proof. We use the method of crossposing triangle strips (see [3, Chapter 12]).

Let \( T = \triangle x_0 x_1 \hat{x}_0 \) with \( d = d(x_0, x_1) \), \( c \) denotes the centre of \( x_0 \hat{x}_0 \). Let \( \tau \) be the translation mapping \( x_0 \) onto \( x_1 \) and let \( \sigma \) be the central reflection with respect to \( c \). We define \( x_i = \tau^i(x_0) \), \( T_i = \tau^i(T) \), \( \hat{x}_i = \sigma(x_i) \), and \( \hat{T}_i = \sigma(T_i) \) for \( i \in \mathbb{Z} \). Then the triangles \( T_i, \hat{T}_i, i \in \mathbb{Z} \), form an infinite dissection of a strip \( \Sigma \) bounded by \( l = l(x_0, x_1) \) and \( \hat{l} = l(\hat{x}_0, \hat{x}_1) \) (see Figure 1). The sizes of the inner angles of \( T \) at \( x_0, x_1, \hat{x}_0 \) are denoted by \( \alpha, \beta, \gamma \), respectively. Since \( d(x_0, x_1) = d \) is the diameter of \( T \), the width of \( \Sigma \) is at most \( \sqrt{2} \cdot d \). Based on \( T' \), we introduce a strip \( \Sigma' \) and respective terms \( c', \tau', \sigma', x'_i, T'_i, \hat{x}'_i, \hat{T}'_i, l', \hat{l}, \alpha', \beta', \gamma' \) analogously. We assume \( T = \triangle x_0 x_1 \hat{x}_0 \) and \( T' = \triangle x'_0 x'_1 \hat{x}'_0 \) to be oriented in the same way.

Without loss of generality, \( d \leq d' \). Then \( \alpha' \leq \alpha \) or \( \beta' \leq \beta \), because \( T \) and \( T' \) have the same area. Again without loss of generality, \( \alpha' \leq \alpha \).

We suppose that \( c' = c \) (which can be obtained by translating \( T' \)). Finally, we assume that the intersection \( \Sigma \cap \Sigma' \) is a parallelogram \( P \), whose vertices \( p_1, p_2, p_3, p_4 \) represent the intersections \( l \cap l', \hat{l} \cap l', \hat{l} \cap l', \hat{l} \cap l' \), respectively, such that \( d(p_2, p_3) = d' \) and \( \delta = |\angle p_1 p_2 p_3| \leq \frac{\pi}{3} \) (see Figure 2). In fact, this situation can be obtained by suitably rotating \( \Sigma' \) around \( c \), because the width of \( \Sigma \) does not exceed \( \sqrt{2}d \leq \sqrt{3}d' \).

The midpoints \( c_1 \) and \( c_3 \) of \( p_1p_2 \) and \( p_3p_4 \) satisfy \( d(c, c_1) = d(c, c_3) = \frac{d'}{2} \) and hence agree with the centres of \( x'_0 \hat{x}'_0 \) and \( x'_0 \hat{x}'_1 \), respectively. The area of \( P \) is twice that of...
of $T'$ and so twice that of $T$, too. This shows that $d(p_1, p_2) = d$ and the midpoints $c_2$ and $c_4$ of $p_2p_3$ and $p_1p_4$ coincide with those of $x_1\hat{x}_0$ and $x_0\hat{x}_1$, respectively.

The edges of the triangles $T_1, T_i, T_i', T_i''$ dissect $P$ into finitely many polygons that appear in $k$ pairs symmetric with respect to $c$. Each pair consists of one member contained in $\bigcup_{i \in Z} T_i$ and one member covered by $\bigcup_{i \in Z} T_i'$. Images of the $k$ first members under suitable integer powers of $\tau$ form a dissection of $T$ (see Figure 3). Similarly, we find one element in every pair of symmetric pieces of $P$ such that images of these $k$ elements under suitable integer powers of $\tau'$ constitute a dissection of $T'$. This shows that $\deg_{\text{bom}^+}(T, T') \leq k$.

It remains to establish an upper bound for $k$. We prepare this by proving

$$\alpha + \beta' + \delta \leq \pi. \tag{3}$$

Among all triangles $\triangle x_0x_1y$ with $y \in \hat{l}$ and of diameter $d(x_0, x_1) = d$ the isosceles one with $d(x_1, y) = d(x_0, x_1) = d$ and $|\angle x_1x_0y| = |\angle x_0yx_1| = \alpha_0 \geq \frac{\pi}{2}$ maximizes the size of the inner angle at $x_0$, in particular $\alpha \leq \alpha_0$. Its area coincides with that of $T$ and can be computed by $\frac{1}{2}d^2 \sin(|\angle x_0x_1y|) = \frac{1}{2}d^2 \sin(\pi - 2\alpha_0) = \frac{1}{2}d^2 \sin 2\alpha_0$. So

$$\alpha \leq \alpha_0, \text{ where } \frac{\pi}{3} \leq \alpha_0 < \frac{\pi}{2} \text{ and } d^2 \sin 2\alpha_0 = 2\lambda(T) = \lambda(P) = dd' \sin \delta.$$ 

Similarly,

$$\beta' \leq \beta'_0, \text{ where } \frac{\pi}{3} \leq \beta'_0 < \frac{\pi}{2} \text{ and } d^2 \sin 2\beta'_0 = dd' \sin \delta.$$ 

These admit the estimate

$$dd' \sin(\pi - \delta) = \frac{1}{2}d^2 \sin \delta$$

$$\leq \frac{1}{2} \left( \frac{d \cos \alpha_0}{d' \cos \beta'_0} + \frac{d' \cos \beta'_0}{d \cos \alpha_0} \right) dd' \sin \delta$$

$$= \frac{1}{2} \left( \frac{d \cos \alpha_0}{d' \cos \beta'_0} d^2 \sin 2\beta'_0 + \frac{d' \cos \beta'_0}{d \cos \alpha_0} d^2 \sin 2\alpha_0 \right)$$

$$= dd' \left( \cos \alpha_0 \sin \beta'_0 + \cos \beta'_0 \sin \alpha_0 \right)$$

$$= dd' \sin(\alpha_0 + \beta'_0).$$

Hence $\sin(\alpha_0 + \beta'_0) \geq \sin(\pi - \delta)$ and therefore $\alpha_0 + \beta'_0 \leq \pi - \delta$, because $\alpha_0 + \beta'_0, \pi - \delta \in \left[\frac{\pi}{2}, \pi\right]$. This implies (3), namely $\alpha + \beta' + \delta \leq \alpha_0 + \beta'_0 + \delta \leq \pi$.

Since $\delta < \frac{\pi}{2}$ and $d(x_0, c_4) = \frac{d(x_0, \hat{x}_1)}{2} \leq \frac{d}{2} \leq \frac{d'}{2} = d(p_1, c_4)$ in the triangle $\triangle x_0p_1c_4$, we have $p_1 \in \bigcup_{i = 0}^{\infty} x_i x_{i+1}$. Thus $x_0\hat{x}_0$ meets $p_1p_4$ as well as $p_2p_3$ and
Theorem 3. Let \( P \) be a polygon, \( c \) a point \( \not\in \partial P \) outside \( P \), \( \delta > 0 \) a positive real number, and \( T \) be a convex polygonal arc with \( |\partial T| < \delta \). Then for any \( \alpha \in \mathbb{R}_+ \), we have the following bounds on the number of \( \delta \)-isometric congruences of \( P \) into \( T \):

\[
\deg_{\text{isom}}^+(T, T') \leq \left\lfloor \frac{\max\{d(c,c_1),d(c,c_2)\}}{\delta/2} \right\rfloor + 3 + \left\lfloor \frac{d(x_0,c_1)}{\delta} \right\rfloor + 3.
\]

Proof. Let \( x_0 \in \partial P \backslash \partial T \). By the definition of \( \deg_{\text{isom}}^+ \), it is enough to show that for each \( \alpha \), there exist \( \alpha' \in \mathbb{R}_+ \) and \( \beta' \in \mathbb{R}_+ \) such that

\[
\beta' \leq \alpha' \leq \alpha + \pi
\]

and

\[
\beta' - \alpha' \leq \beta + \alpha - \pi.
\]

The latter is equivalent to

\[
\beta' - \beta \leq \alpha' - \alpha + \pi > 0.
\]

This is clear from (3). Thus \( x_0 \in \partial T \backslash \partial T_\alpha \), but misses \( \text{int}(P \cap T') \) and \( \text{int}(P \cap T_\alpha) \). The segment \( x_{i_0} \partial T_\alpha \) may intersect all three open polygons \( \text{int}(P \cap T') \), \( \text{int}(P \cap T_\alpha) \), and \( \text{int}(P \cap T_{\alpha+\pi/2}) \). However, dissecting by both \( x_{i_0} \partial T_\alpha \) and \( x_0 \partial T_\alpha \) enlarges the number of \( \delta \)-pieces of \( P \) by at most 4.

The last observation shows that the number \( k \) of pieces in \( P \) is bounded by \( k \leq (4i_0 - 1) + 4 = 4i_0 + 3 \). Now the proof of Lemma 3 is completed by

\[
\deg_{\text{isom}}^+(T, T') \leq 4\left\lfloor \frac{d(x_0,c_1)}{\delta/2} \right\rfloor + 3 + \left\lfloor \frac{d(x_0,c_2)}{\delta/2} \right\rfloor + 3 + 4\left\lfloor \frac{d(c_1,c_2)}{\delta/2} \right\rfloor + 3 = 4\left\lfloor \frac{d}{\delta/2} \right\rfloor + 3 + 4\left\lfloor \frac{d}{\delta/2} \right\rfloor + 3.
\]

\( \square \)
2.3. An improved estimate in terms of vertex numbers and diameters

**Theorem 2.** Let $P_m$ and $P'_n$ be convex polygons of the same area whose numbers of vertices are $m$ and $n$ and whose diameters are $d$ and $d'$, respectively. Then

$$\deg_{braid}(P_m, P'_n) \leq (m + n - 5) \left(4 \left[ \max \left\{ \frac{\pi}{d}, \frac{\pi}{d'} \right\} \right] + 3 \right).$$

In particular

$$\deg_{braid}(P_m, P'_n) < 2(m + n)^2 \max \left\{ \frac{d}{\pi}, \frac{d'}{\pi} \right\}.$$ 

The proof is prepared by two more lemmas.

**Lemma 4.** Let the area $\lambda(T)$ of a triangle $T$ be represented as a sum $\lambda(T) = \lambda_1 + \cdots + \lambda_k$ of $k$ positive real numbers. Then one can dissect $T$ into $k$ triangles $T_1, \ldots, T_k$ such that

$$\lambda(T_i) = \lambda_i \quad \text{and} \quad \diam(T_i) > \frac{\diam(T)}{2} \quad \text{for} \quad 1 \leq i \leq k.$$ 

**Proof.** Let $T = \triangle x_1x_2x_3$ and suppose that $d(x_1, x_2) \geq d(x_2, x_3) \geq d(x_1, x_3)$. We fix points $y_1, \ldots, y_{k-1}$ of the edge $x_1x_3$ such that $d(x_1, y_1) = \frac{\lambda_1}{\lambda(T)} d(x_1, x_3)$, $d(y_{i-1}, y_i) = \frac{\lambda_i}{\lambda(T)} d(x_1, x_3)$ for $2 \leq i \leq k-1$, and $d(y_{k-1}, x_3) = \frac{\lambda_k}{\lambda(T)} d(x_1, x_3)$. Then $T$ splits into $T_1 = \triangle x_1x_2y_1$, $T_i = \triangle y_{i-1}x_2y_i$ for $2 \leq i \leq k-1$, and $T_k = \triangle y_{k-1}x_2x_3$. The areas of these triangles are proportional to the lengths of their edges contained in $x_1x_3$. Hence $\lambda(T_i) = \lambda_i$, $1 \leq i \leq k$. An estimate of their diameters can be obtained by the aid of the orthogonal projection $\pi$ onto the long edge $x_1x_2$, namely

$$\diam(T_i) \geq d(y_{i-1}, x_2) > d(\pi(y_{i-1}), x_2) > d(\pi(x_3), x_2) \geq \frac{d(x_1, x_2)}{2} = \frac{\diam(T)}{2}$$

for $2 \leq i \leq k$. For $T_1$ we even have $\diam(T_1) = d(x_1, x_2) = \diam(T)$. \hfill $\square$

**Lemma 5.** Let $P_m$ and $P'_n$ be as in Theorem 2. Then there exist dissections of $P_m$ into $m + n - 5$ triangles $T_1, \ldots, T_{m+n-5}$ and of $P'_n$ into $m + n - 5$ triangles $T'_1, \ldots, T'_{m+n-5}$ such that, for $1 \leq i \leq m + n - 5$,

$$\lambda(T_i) = \lambda(T'_i), \quad \frac{d}{2(\pi)} < \diam(T_i) \leq d, \quad \text{and} \quad \frac{d'}{2(\pi)} < \diam(T'_i) \leq d'.$$

**Proof.** By Lemma 1, there exist dissections of $P_m$ into triangles $S_1, \ldots, S_{m-2}$ with $\diam(S_i) \geq \frac{d}{(\pi)}$ and of $P'_n$ into triangles $S'_1, \ldots, S'_{n-2}$ with $\diam(S'_j) \geq \frac{d'}{(\pi)}$. Let

$$\mu_i = \sum_{i=1}^{m} \lambda(S_i), \quad 0 \leq i \leq m - 2.$$ 

Then the intervals $I_i = [\mu_{i-1}, \mu_i], \quad 1 \leq i \leq m - 2$, have the lengths $\lambda(S_i)$ and together constitute a dissection of $[0, \lambda(P_m)]$. Similarly, let

$$\nu_j = \sum_{i=1}^{j} \lambda(S'_i), \quad 0 \leq j \leq n - 2.$$ 

The intervals $J_j = [\nu_{j-1}, \nu_j], \quad 1 \leq j \leq n - 2$, have the lengths $\lambda(S'_j)$ and form a dissection of $[0, \lambda(P'_n)] = [0, \lambda(P_m)]$. The numbers $\mu_1, \ldots, \mu_m-3, \nu_1, \ldots, \nu_{n-3}$ cut $[0, \lambda(P_m)]$ into at most $m + n - 5$ subintervals each being completely covered by some $I_i$ and by some $J_j$. Hence there exists a dissection of $[0, \lambda(P_m)]$ into closed intervals $K_1, \ldots, K_{m+n-5}$ of positive lengths which refines both subdivisions $\{I_1, \ldots, I_{m-2}\}$ and $\{J_1, \ldots, J_{n-2}\}$ simultaneously.
The interval $I_i$ of length $\lambda(S_i)$ splits into suitable $K_{i_1}, \ldots, K_{i_l}$ of the lengths $\lambda_{i_1}, \ldots, \lambda_{i_l}$. By Lemma 4, $S_i$ can be decomposed into triangles $T_{i_1}, \ldots, T_{i_l}$ such that

$$\lambda(T_{i_r}) = \lambda_i$$

and

$$\text{diam}(T_{i_r}) > \frac{\text{diam}(S_i)}{2} \geq \frac{d}{2l(\frac{l}{2})}$$

for $1 \leq r \leq l$.

This gives the dissection of $P_m$ into $T_1, \ldots, T_{m+n-5}$. In the same way we can dissect $P_n'$ into triangles $T'_1, \ldots, T'_{m+n-5}$ with

$$\lambda(T'_i) = \lambda_i$$

and

$$\text{diam}(T'_i) > \frac{d}{2l(\frac{l}{2})}$$

for $1 \leq i \leq m + n - 5$.

by partitioning $S'_1, \ldots, S'_{n-2}$. In particular $\lambda(T_i) = \lambda_i = \lambda(T'_i)$. This completes the proof.

Proof of Theorem 2. We dissect $P_m$ and $P_n'$ by Lemma 5. Then, for every $i \in \{1, \ldots, m + n - 5\}$, we apply Lemma 3 to $T_i$ and $T'_i$. This gives dissections of $P_m$ and $P_n'$ proving the first estimate. The second one can be shown as follows.

$$\deg_{\text{Isom}^+}(P_m, P'_n) < (m + n - 5) \left( 4 \left( \max \left\{ \frac{\lfloor d \rfloor}{d}, \frac{\lfloor d' \rfloor}{d'} \right\} + 1 \right) + 3 \right)$$

$$\leq (m + n - 5) \left( 4 \max \left\{ \frac{\lfloor d \rfloor}{d}, \frac{\lfloor d' \rfloor}{d'} \right\} + 7 \right)$$

$$= (m + n - 5) \left( 2 \max \left\{ \frac{\lfloor d \rfloor}{d}, \frac{\lfloor d' \rfloor}{d'} \right\} + 7 \right)$$

$$\leq (m + n - 5) \left( 2(m + n - 3) \max \left\{ \frac{d}{d'}, \frac{d'}{d} \right\} + 7 \right)$$

$$= 2(m + n - 5) \left( m + n + \frac{1}{2} \right) \max \left\{ \frac{d}{d'}, \frac{d'}{d} \right\}$$

$$< 2(m + n)^2 \max \left\{ \frac{d}{d'}, \frac{d'}{d} \right\}. \quad \square$$

2.4. An estimate in terms of vertex numbers, diameters, and radii of inscribed circles

Theorem 3. Let $P_m$ be a convex $m$-gon of diameter $d$ containing a circle of radius $r$ and let $P_n'$ be a convex $n$-gon of the same area having the diameter $d'$ and containing a circle of radius $r'$. Then

$$\deg_{\text{Isom}^+}(P_m, P'_n) \leq (m + n - 1) \left( 4 \left[ \frac{1}{2} \max \left\{ \frac{d}{d'}, \frac{d'}{d} \right\} \right] + 3 \right).$$

In particular

$$\deg_{\text{Isom}^+}(P_m, P'_n) < (m + n) \left( 2 \max \left\{ \frac{d}{d'}, \frac{d'}{d} \right\} + 7 \right).$$

Again the problem is reduced to piecewise congruences of triangles.

Lemma 6. Let $P_m$ and $P_n'$ be as in Theorem 3. Then there exist dissections of $P_m$ into $k \leq m + n - 1$ triangles $T_1, \ldots, T_k$ and of $P_n'$ into $k$ triangles $T'_1, \ldots, T'_k$ such that, for $1 \leq i \leq k$,

$$\lambda(T_i) = \lambda(T'_i), \quad r < \text{diam}(T_i) \leq d, \quad \text{and} \quad r' < \text{diam}(T'_i) \leq d'.$$
Proof. Let \( x_0, \ldots, x_{m-1} \) and \( x'_0, \ldots, x'_{n-1} \) be the vertices of \( P_m \) and \( P'_n \), respectively, ordered counterclockwise along the boundaries. Let \( c \) and \( c' \) be the midpoints of the inscribed circles of the respective polygons. We define a bijection \( p \) of the half-open interval \([0, \lambda(P_m)]\) onto \( \text{bd}(P_m) \) such that the counterclockwise arc from \( x_0 \) to \( p(\lambda) \) along \( \text{bd}(P_m) \) together with the segments \( cx_0 \) and \( cp(\lambda) \) bounds a polygon of area \( \lambda \). Similarly, we introduce \( p' : [0, \lambda(P'_n)] = [0, \lambda(P'_n)] \rightarrow \text{bd}(P'_n) \).

Now let \( \{0 = p^{-1}(x_0), \ldots, p^{-1}(x_{m-1})\} \cup \{0 = (p')^{-1}(x'_0), \ldots, (p')^{-1}(x'_{n-1})\} = \{\lambda_0, \ldots, \lambda_{k-1}\} \) be ordered such that \( 0 = \lambda_0 < \cdots < \lambda_{k-1} \). Of course, \( k \leq m+n-1 \). We define

\[
T_i = \begin{cases} 
\text{conv}\{c, p(\lambda_{i-1}), p(\lambda_i)\}, & 1 \leq i \leq k-1, \\
\text{conv}\{c, p(\lambda_{k-1}), x_0\}, & i = k,
\end{cases}
\]

and

\[
T'_i = \begin{cases} 
\text{conv}\{c', p'(\lambda_{i-1}), p'(\lambda_i)\}, & 1 \leq i \leq k-1, \\
\text{conv}\{c', p'(\lambda_{k-1}), x'_0\}, & i = k,
\end{cases}
\]

this way obtaining dissections of \( P_m \) into \( T_1, \ldots, T_k \) and of \( P'_n \) into \( T'_1, \ldots, T'_k \). Then \( \lambda(T_i) = \lambda_i - \lambda_{i-1} = \lambda(T'_i) \) for \( 1 \leq i \leq k-1 \) and \( \lambda(T_k) = \lambda(P_n) - \lambda_{k-1} = \lambda(T'_k) \). Since every \( T_i \) contains the centre \( c \) as well as at least one vertex outside the inscribed circle of radius \( r \), the lower estimate is obvious. The upper one is trivial. The triangles \( T'_i \) behave analogously.

Now Theorem 3 can be inferred from Lemma 6 as Theorem 2 has been proved by Lemma 5.

2.5. An estimate for regular polygons

Theorem 4. Let \( P^r_m \) and \( P^r_n \) be regular polygons of the same area having \( m \) and \( n \) vertices, respectively. Then

\[
\deg_{\text{isom}}(P^r_m, P^r_n) \leq 7(m+n-1).
\]

Theorem 4 is an immediate consequence of the following claim and of Lemma 3.

Lemma 7. Let \( P^r_m \) and \( P^r_n \) be regular polygons of area 1 having \( m \) and \( n \) vertices, respectively. Then there exist dissections of \( P^r_m \) into \( k \leq m+n-1 \) triangles \( T_1, \ldots, T_k \) and of \( P^r_n \) into \( k \) triangles \( T'_1, \ldots, T'_k \) such that, for \( 1 \leq i \leq k \),

\[
\lambda(T_i) = \lambda(T'_i) \quad \text{and} \quad \frac{1}{2} < \frac{\text{diam}(T_i)}{\text{diam}(T'_i)} < 2.
\]

Proof. Simple trigonometric calculations show that the radius \( r_m \) of the largest inscribed circle, the radius \( R_m \) of the smallest circumscribed circle, and the edge length \( e_m \) of \( P^r_m \) are

\[
r_m = \frac{1}{\sqrt{m \tan \frac{\pi}{m}}}, \quad R_m = \sqrt{\frac{1+\tan^2\frac{\pi}{m}}{m \tan \frac{\pi}{m}}}, \quad e_m = 2\sqrt{\frac{1}{m} \tan \frac{\pi}{m}}.
\]

We assume \( 3 \leq m < n \) without loss of generality.
Case 1. \( n = 4 \). Then \( m = 3 \). We cut \( P_3^r \) along an axis of symmetry into \( T_1, T_2 \) and \( P_3' \) along a diagonal into \( T_1', T_2' \). Then \( \lambda(T_i) = \lambda(T'_i) = \frac{1}{2} \) and \( \frac{\text{diam}(T_i)}{\text{diam}(T'_i)} = \frac{e_i}{2r_i} = 2^4 \frac{3}{4} = 1.07... \) for \( i = 1, 2 \).

Case 2. \( n \geq 5 \). Now we define dissections of \( P_m^r \) into \( T_1, \ldots, T_k, \) \( k \leq m+n-1 \), and of \( P_n^r \) into \( T_1', \ldots, T'_k \) as we did in the proof of Lemma 6. We obtain in particular
\[
\lambda(T_i) = \lambda(T'_i), \quad r_m < \text{diam}(T_i), \quad \text{and} \quad r_n < \text{diam}(T'_i)
\]
for \( 1 \leq i \leq k \). For the remaining estimate of \( \frac{\text{diam}(T_i)}{\text{diam}(T'_i)} \) let \( i \) be fixed.

Case 2.1. \( m = 3 \). The triangle \( T'_i \) is of the form \( \triangle c'y_1'y_2', c' \) being the centre of \( P_n^r \) and \( y_1', y_2' \) lying on a common edge of \( P_n^r \). We can estimate \( \text{diam}(T'_i) = \max\{|c'y_1'|, |c'y_2'|, |y_1'y_2'|\} \) by
\[
r_5 \leq r_n < \text{diam}(T'_i) \leq \max\{R_n, R_n, e_n\} \leq \max\{R_5, e_5\} = e_5.
\]
Similarly, we obtain \( T_i = \triangle cy_1y_2 \), where \( c \) is the centre of \( P_3^r \) and \( y_1, y_2 \) lie on a common edge of \( P_3^r \). The respective triangle \( T'_i \), which has the same area as \( T_i \), is contained in one of the \( n \) pairwise congruent triangles defined as the convex hulls of \( c' \) and an edge of \( P_n^r \). Hence \( \lambda(T_i) = \lambda(T'_i) \leq \frac{1}{n} \leq \frac{1}{5} \). Since \( r_3 \) is the height of \( T_i \) over the edge \( y_1y_2 \), we obtain \( \frac{|y_1y_2|}{2} = \lambda(T_i) \leq \frac{1}{5} \) and hence \( |y_1y_2| \leq \frac{2}{5r_3} \). Thus \( \text{diam}(T_i) \) satisfies the estimate
\[
r_3 < \text{diam}(T_i) = \max\{|cy_1|, |cy_2|, |y_1y_2|\} \leq \max\{R_3, 3, \frac{2}{5r_3}\} = \frac{2}{5r_3}.
\]

Combining this with (4) we obtain the required inequalities, namely
\[
0.57... = \frac{r_3}{e_5} < \frac{\text{diam}(T_i)}{\text{diam}(T'_i)} < \frac{2}{r_5} = 1.73...
\]

Case 2.2. \( m \geq 4 \). Arguments similar to those for showing (4) give
\[
r_4 \leq r_m < \text{diam}(T_i) \leq \max\{R_m, R_m, e_m\} \leq \max\{R_4, e_4\} = e_4,
\[
r_4 < r_n < \text{diam}(T'_i) \leq \max\{R_n, R_n, e_n\} < \max\{R_4, e_4\} = e_4.
\]

Consequently, \( \frac{1}{2} = \frac{r_4}{e_4} < \frac{\text{diam}(T_i)}{\text{diam}(T'_i)} < \frac{r_4}{e_4} = 2 \). This completes the proof. \( \square \)

3. Remarks concerning similarities and translations

3.1. Congruence by dissection with respect to similarities

Any two polygons \( P, P' \) are congruent by dissection with respect to the group Sim of similarities, since \( P \) and the similar image \( \sqrt{\frac{\lambda(P)}{\lambda(P')}} P' \) of \( P' \) have the same area and hence are congruent by dissection with respect to Isom by the Wallace-Bolyai-Gerwien theorem. It is shown in [7] that any convex \( m \)-gon \( P_m \) and any convex \( n \)-gon \( P_n \) satisfy \( \text{deg}_{\text{Sim}}(P_m, P_n) \leq 3(\max\{m, n\} - 2) \). This motivates the definition
\[
\text{deg}_{\text{Sim}}(m, n) = \max\{\text{deg}_{\text{Sim}}(P_m, P_n') : P_m \text{ a convex } m \text{-gon, } P_n' \text{ a convex } n \text{-gon}\}\]
for \( m, n \geq 3 \). We introduce \( \deg_{\text{Sim}^+}(m, n) \) analogously, \( \text{Sim}^+ \) denoting the subgroup of proper similarities. Of course,

\[
\deg_{\text{Sim}}(m, n) = \deg_{\text{Sim}^+}(n, m) \leq \deg_{\text{Sim}^+}(m, n) = \deg_{\text{Sim}^+}(n, m).
\]

The above mentioned estimate from [7] now reads as

\[
\deg_{\text{Sim}^+}(m, n) \leq 3(n - 2) \quad \text{for all} \quad 3 \leq m \leq n.
\] (5)

We could show the following.

**Theorem 5.** ([8], Theorems 5 and 6) For arbitrary \( 3 \leq m \leq n \),

\[
\deg_{\text{Sim}^+}(m, n) \leq \begin{cases} 
\left\lfloor \frac{5m - 9}{2} \right\rfloor = \left\lfloor \frac{7m + 8n - 27}{6} \right\rfloor & \text{if } n = m, \\
\left\lfloor \frac{7m + 8n - 24}{6} \right\rfloor & \text{if } n > m
\end{cases}
\] (6)

and

\[
\deg_{\text{Sim}}(m, n) \leq \begin{cases} 
n + 3 & \text{if } m = 3, \\
m + n + \left\lfloor \frac{m}{3} \right\rfloor & \text{if } 4 \leq m \leq 11, \\
m + n + 4 & \text{if } m \geq 12.
\end{cases}
\] (7)

The estimate (6) can be realized by dissections into convex pieces, whereas our construction for (7) uses up to two non-convex simple polygons in the corresponding dissections. The bound (7) is stronger than (6) if \( n \) is sufficiently large, but can be improved by (6) for certain small \( m, n \).

The estimate (6) is better than (5) apart from the case \( m = n = 3 \), where both bounds attain the value 3. In the latter situation the estimates are sharp. This is a consequence of the following observation.

**Lemma 8.** ([8], Corollary 2) Two triangles \( T, T' \) satisfy \( \deg_{\text{Sim}}(T, T') \leq 2 \) if and only if \( T \) has an angle of size \( \alpha \) and \( T' \) has an angle of size \( \alpha' \) such that \( \alpha = \alpha' \) or \( \alpha + \alpha' = \pi \).

### 3.2. Congruence by dissection with respect to translations

Two polygons of the same area are not necessarily equidissectable with respect to the group Trans of translations. Since directions are invariant under translations, congruence by dissection with respect to Trans depends on the directional behaviour of the corresponding polygons, too. We recall a necessary and sufficient condition from [5].

Let \( S^1 \) be the set of all vectors of length 1 in \( \mathbb{R}^2 \). Given a (not necessarily convex) polygon \( P \) and a direction \( x \in S^1 \), \( l(P, x) \) is to denote the sum of the lengths of all edges of \( P \) having the outer normal vector \( x \). If no such edges exist, we put \( l(P, x) = 0 \). Now let \( L_x(P) = l(P, x) - l(P, -x) \). It is shown in [5] that two polygons \( P, P' \) are congruent by dissection with respect to Trans if and only if they have the same area and \( L_x(P) = L_x(P') \) for all \( x \in S^1 \).

Suppose from now on that \( P, P' \) are arbitrary convex polygons congruent by dissection under translations. In contrast with Theorem 2, a general upper
estimate for \( \text{deg}_{\text{Trans}}(P, P') \) cannot be given in terms of the vertex numbers and the diameters of \( P \) and \( P' \). For example, the two rectangles \( R(m) = [0, m] \times [0, 1] \) and \( R'(m) = [0, 1] \times [0, m] \) satisfy \( \text{deg}_{\text{Trans}}(R(m), R'(m)) = m \) for every \( m \in \{1, 2, \ldots\} \), even though their diameters agree. The last observation reveals a more general principle.

Given a bounded non-empty set \( A \subseteq \mathbb{R}^2 \) and a vector \( x \in S^1 \), the width of \( A \) in direction \( x \) is given by

\[
w(A, x) = \sup \{ \langle a_1 - a_2, x \rangle : a_1, a_2 \in A \},
\]

where \( \langle \cdot, \cdot \rangle \) stands for the scalar product. Of course, this functional is monotone and invariant under translations, that is, \( w(A, x) \leq w(B, x) \) if \( A \subseteq B \subseteq \mathbb{R}^2 \) and \( w(A, x) = w(A + t, x) \) for all \( t \in \mathbb{R}^2 \).

Since \( P \) can be dissected into \( \text{deg}_{\text{Trans}}(P, P') \) translates \( Q_i \) of pieces of \( P' \), there is a convex set of width \( w(P, x) \) that can be covered by \( \text{deg}_{\text{Trans}}(P, P') \) sets of width \( w(Q_i, x) \leq w(P', x) \). This yields \( \text{deg}_{\text{Trans}}(P, P') \geq \left\lceil \frac{w(P, x)}{w(P', x)} \right\rceil \) and, by symmetry,

\[
\text{deg}_{\text{Trans}}(P, P') \geq \sup \left\{ \left\lceil \max \left\{ \frac{w(P, x)}{w(P', x)}, \frac{w(P', x)}{w(P, x)} \right\} \right\} : x \in S^1 \}
\]
as a general lower estimate for \( \text{deg}_{\text{Trans}}(P, P') \).

We denote the right hand side of the last formula by \( W(P, P') \). Can one find a general upper bound for \( \text{deg}_{\text{Trans}}(P, P') \) only in terms of \( W(P, P') \) and the vertex numbers of \( P \) and \( P' \)?

References


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