A Characterization of $L_2(2^f)$ in Terms of the Number of Character Zeros*

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Abstract. The aim of this paper is to show that $L_2(2^f)$ are the only nonsolvable groups in which every irreducible character of even degree vanishes on just one conjugacy class.

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1. Introduction

For an irreducible character $\chi$ of a finite group $G$, we know that $v(\chi) := \{ g \in G \mid \chi(g) = 0 \}$ is a union of some conjugacy classes of $G$. An old theorem of Burnside asserts that $v(\chi)$ is not empty for any nonlinear $\chi \in \text{Irr}(G)$. It is natural to consider the structure of a finite group provided that the number of character zeros in its character table is very small (see [1], [11], [12] for a few examples). In Berkovich and Kazarin's paper [1], they posed the following question.

Question. Is it true that $L_2(2^f), f \geq 2$ are the only nonabelian simple groups in which every irreducible character of even degree vanishes on just one conjugacy class?

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Our answer to the question is affirmative.

**Theorem A.** Let $G$ be a finite group. If every $\chi \in \text{Irr}(G)$ of even degree vanishes on just one conjugacy class, then $G$ is just one of the following groups:

1. $G$ possesses a normal and abelian Sylow 2-subgroup.
2. $G$ is a Frobenius group with a complement of order 2.
3. $G \cong SL(2,3)$.
4. $G \cong L_2(2^f), f \geq 2$.

In particular, $L_2(2^f) (f \geq 2)$ are the only nonsolvable groups, and therefore the only nonabelian simple groups satisfying the hypothesis.

Instead of proving Theorem A directly, we will study the finite nonsolvable groups $G$ satisfying the following property

\((*)\) every nonlinear $\chi \in \text{Irr}(G)$ of even degree vanishes on at most two conjugacy classes of $G$.

**Theorem B.** If $G$ is a finite nonsolvable group with no nontrivial solvable normal subgroup, then $G$ has the property \((*)\) if and only if $G \cong L_2(7)$ or $L_2(2^f)$ where $f \geq 2$.

In this paper, $G$ always denotes a finite group, a class always means a conjugacy class. We denote by $x^G$ the conjugacy class of $G$ in which $x$ lies. For a subset $A$ of $G$, let $k_G(A)$ be the minimal integer $l$ such that $A$ is a subset of a union of $l$ conjugacy classes of $G$. For $N \triangleleft G$, we put $\text{Irr}(G|N) = \text{Irr}(G) - \text{Irr}(G/N)$; and for $\lambda \in \text{Irr}(N)$, the inertia subgroup of $\lambda$ in $G$ is denoted by $I_G(\lambda)$.

Let $\text{Irr}_2(G)$ be the set of irreducible characters of $G$ with even degree. Our proof depends on the classification theorem of finite simple groups.

### 2. Theorem B

We begin to list some easy results which will be used later.

**Lemma 2.1.** Let $N \triangleleft G$ and set $\overline{G} = G/N$. Then the following results are true.

1. For any $x \in G$, $x^\overline{G}$, viewed as a subset of $G$, is a union of some classes of $G$; furthermore, $k_G(x^\overline{G}) = 1$ if and only if $\chi(x) = 0$ for any $\chi \in \text{Irr}(G|N)$.
2. If $G$ has the property \((*)\), then so has $G/N$.

**Proof.** (1) See [11, Lemma 3(1)].

(2) The result follows directly from (1). \qed

**Lemma 2.2.** For any nonlinear $\chi \in \text{Irr}(G)$, we have:

1. If $G$ is nonsolvable and $k_G(v(\chi)) \leq 2$, then $\chi_{e^G}$ is irreducible.
2. If $v(\chi) \subseteq N$ for some $N \triangleleft G$, then $\gcd(\chi(1), |G/N|) = 1$. In particular, $\chi_N$ is irreducible.
Proof. (1) Suppose that $\chi_{G'}$ is reducible. By [7, Theorem 6.28], we can find a normal subgroup $M$ of $G$ with $G' \leq M < G$ and an irreducible character $\psi$ of $M$ such that $\chi = \psi^G$. It follows that $\chi$ vanishes on $G - M$, and thus $k_{G'}(G - M) \leq 2$. By [13, Theorem 2.2] $G$ is solvable, a contradiction.

(2) See [12, Lemma 2.2].

Next, we need the following Lemma 2.3. An irreducible character $\chi$ of $G$ is called $p$-defect zero for some prime $p$ if $\chi(1)_p = |G|_p$, that is, the $p$-part of the degree $\chi(1)$ equals the $p$-part of the order of $G$. It is well-known that if $\chi \in \text{Irr}(G)$ is $p$-defect zero then $\chi(x) = 0$ whenever $x \in G$ is of order a multiple of $p$.

**Lemma 2.3.** Let $G$ be a nonabelian simple group. Then there exists $\chi \in \text{Irr}_2(G)$ such that $\chi$ is of $p$-defect zero for some prime divisor $p$ of $|G|$.

**Proof.** By [7, Theorem 6.28], we can find a normal subgroup $M$ of $G$ with $G' \leq M < G$ and an irreducible character $\psi$ of $M$. Suppose that $\chi(1) = |G|_p$, that is, the $p$-part of the degree $\chi(1)$ equals the $p$-part of the order of $G$. It is well-known that if $\chi \in \text{Irr}(G)$ is $p$-defect zero then $\chi(x) = 0$ whenever $x \in G$ is of order a multiple of $p$.

Now we are ready to prove Theorem B.

**Proof of Theorem B.** Let $N$ be a minimal normal subgroup of $G$. Since $G$ has no nontrivial solvable normal subgroup, $N$ is nonsolvable.

Step 1. $G$ is almost simple, that is, $N$ is a nonabelian simple group with $N \leq G \leq \text{Aut}(N)$.

Clearly $N = N_1 \times \cdots \times N_s$ is a direct product of isomorphic simple groups $N_i$, $1 \leq i \leq s$. Suppose that $s \geq 2$. Let $\theta_i \in \text{Irr}_2(N_i)$ be of $p$-defect zero (Lemma 2.3), and set $\theta = \theta_1 \times \cdots \times \theta_s$. Then $\theta$ is an irreducible character of $N$, also $\theta^g \in \text{Irr}(N)$ is of $p$-defect zero for any $g \in G$. Let $\chi_0$ be an irreducible constituent of $\theta^G$, let $x_1 \in N_1, x_2 \in N_2$ be of order $p$, and $y_2 \in N_2$ be of a prime order $q$ ($q \neq p$). Now for any $g \in G$, we have

$$\theta^g(x_1) = \theta^g(x_1x_2) = \theta^g(x_1y_2) = 0,$$

and this implies that $\chi_0(x_1) = \chi_0(x_1x_2) = \chi_0(x_1y_2) = 0$. Since $x_1, x_1x_2, x_1y_2$ lie in distinct conjugacy classes, we obtain a contradiction. Thus $N$ is simple.

Suppose that $C_G(N) > 1$. Then $C_G(N)$ contains a minimal normal subgroup $M$ of $G$. Set $T = M \times N$. Arguing on $M \times N$ as in the above paragraph, we conclude that $M, N$ are nonabelian simple groups, and we can find $\psi \in \text{Irr}_2(M)$, $\theta \in \text{Irr}_2(N)$ so that $\psi$ is of $q$-defect zero, and $\theta$ is of $p$-defect zero, where $q, p$ are prime divisors of $|M|$ and $|N|$ respectively. Let $x \in M, y \in N$ be of order $q, p$ respectively. Then for any irreducible constituent $\chi$ of $(\psi \times \theta)^G$, we see
that $\chi(x) = \chi(y) = \chi(xy) = 0$. Clearly, $x, y, xy$ lie in distinct classes of $G$, a contradiction. Thus $C_G(N) = 1$, so $N \leq G \leq \text{Aut}(N)$, and then $G$ is an almost simple group.

Step 2. $N$ is a simple group of Lie type.

Suppose that $N \cong A_n$ for some $n \geq 8$. Let $\pi$ be the permutation character of $N$, and $\delta$ be the mapping of $N$ into $\{0, 1, 2, \cdots \}$ such that $\delta(g)$ is the number of 2-cycles in the standard composition of $g$. Set

$$\lambda = \frac{(\pi - 1)(\pi - 2)}{2} - \delta, \quad \rho = \frac{\pi(\pi - 3)}{2} + \delta.$$  

By [5, V, Theorem 20.6], both $\lambda$ and $\rho$ are irreducible characters of $N$. Observe that either $\lambda(1) = (n - 1)(n - 2)/2$ or $\rho(1) = n(n - 3)/2$ is even. Let $\chi_0$ be an irreducible constituent of $\tau^G$, where $\tau \in \{\lambda, \rho\}$ is of even degree. Since $G/N \leq \text{Out}(N) = \text{Out}(A_n) = Z_2$ ($n \geq 8$, see [2]), it follows that $N = G'$. Now Lemma 2.2(1) implies that $(\chi_0)_x = \tau$.

For even $n$, set

$$a_1 = (1, \ldots, n - 1), \quad a_2 = (1, \ldots, n - 2)(n - 1, n),$$

$$a_3 = (1, \ldots, n - 5)(n - 4, n - 3, n - 2);$$

$$b_1 = (1, \ldots, n - 3), \quad b_2 = (1, 2, \ldots, n - 3)(n - 2, n - 1, n),$$

$$b_3 = (1, \ldots, n - 4)(n - 3, n - 2).$$

For odd $n$, set

$$a_1 = (1, \ldots, n - 2), \quad a_2 = (1, \ldots, n - 4)(n - 3, n - 2, n - 1),$$

$$a_3 = (1, \ldots, n - 5)(n - 4, n - 3);$$

$$b_1 = (1, \ldots, n), \quad b_2 = (1, \ldots, n - 3)(n - 2, n - 1),$$

$$b_3 = (1, \ldots, n - 6)(n - 5, n - 4, n - 3).$$

We see that $\lambda(a_i) = 0 = \rho(b_i)$ for any $i = 1, 2, 3$. Therefore, either $\chi_0(a_1) = \chi_0(a_2) = \chi_0(a_3) = 0$ or $\chi_0(b_1) = \chi_0(b_2) = \chi_0(b_3) = 0$. Clearly $a_1, a_2, a_3$ (or $b_1, b_2, b_3$) lie in distinct classes of $G$. We obtain a contradiction.

Suppose that $N$ is isomorphic to $A_7$ or one of the sporadic simple groups. Assume $G = N$. We obtain a contradiction by [2]. Assume $G > N$. Since $G$ has no nontrivial solvable normal subgroup, $G \leq \text{Aut}(N)$. It follows by [2] that $|\text{Out}(N)| \leq 2$, and so $|G/N| = 2$ and $N = G'$. By Lemma 2.2, every $\theta \in \text{Irr}_2(N)$ is extendable to $\chi \in \text{Irr}(G)$, and that $k_G(v(\theta)) = k_G(v(\chi) \cap N) \leq 1$. By [2], we also get a contradiction.

Note that $A_5 \cong L_2(4) \cong L_2(5), A_6 \cong L_2(9)$. By the classification theorem of finite simple groups, $N$ must be a simple group of Lie type.

Remarks and notation: Since $N$ is one of the simple groups of Lie type, by [15] $N$ has an irreducible character $\chi_0$ of 2-defect zero. Let $\sigma_0$ be an irreducible constituent of $\chi_0^G$. Observe that $\chi_0^g(x) = 0$ for any $g \in G$ and any $x \in N$ of even order. It follows that $\sigma_0(x) = 0$ whenever $x \in N$ is of even order.

Let $P \in \text{Syl}_2(N)$, and $\Delta = \cup_{g \in G} (P^g \setminus \{1\})$. We have

$$\Delta \leq v(\sigma_0),$$

and so $k_G(\Delta) \leq 2$. 

Step 3. If $G = N \cong L_2(q)$ for some odd $q = p^f > 5$, then $G \cong L_2(7)$.

Note that all irreducible characters of $L_2(q)$ are listed in [6, XI, Theorem 5.5, 5.6, 5.7]. Let $\eta \in \text{Irr}_2(G)$ be of degree $p^f + 1$, and $C$ be a Singer cycle of $G$, and $\Xi = \cup_{\eta \in G}(C^n - 1)$. For any $v \in \Xi$, if $p$ is a prime divisor of element order $o(v)$, then either $\sigma_0$ or $\eta$ is $p$-defect zero, and so either $\sigma_0(v) = 0$ or $\eta(v) = 0$. This implies that $k_G(\Xi) \leq 4$. Since $k_G(\Xi) = (q - 1)/4$ (see [5, II, Theorem 8.5]), we have that $q = 7, 9, 11, 13, 17$. By [2], we conclude that $q = 7$ and $G \cong L_2(7)$.

Step 4. If a Sylow 2-subgroup $P$ of $N$ is nonabelian, then $G \cong L_2(7)$.

In this case, since $P$ has an element of order 4, we conclude that $v(\sigma_0) = \Delta \subset N$ and $k_G(v(\sigma_0)) = 2$. By Lemma 2.2 (2), $|G/N|$ is odd and $(\sigma_0)_N = \chi_0$. Therefore $\sigma_0$ is of 2-defect zero, and $\sigma_0(x) = 0$ for any $x \in G$ of even order. This implies that all elements of even order are contained in $\Delta$, thus $C_G(t)$ is a 2-group for any involution $t$ of $G$. Since $P$ is nonabelian, by [14, III, Theorem 5] we conclude that $G$ is isomorphic to one of the following groups: $Sz(q), q = 2^{m+1}, L_2(q)$ where $q$ is a Fermat prime or Mersenne prime, $L_3(4), L_2(9), M_{10}$.

By [2], neither $M_{10}$ nor $L_3(4)$ nor $L_2(9)$ has the property $(*)$. Note that all elements of order 4 in $Sz(2^{m+1})$ constitute two conjugacy classes, which can be easily verified by [6, XI, Theorem 3.10]. Therefore $G \cong Sz(2^{m+1})$ is not the case. Now by step 3, we conclude that $G \cong L_2(7)$.

Step 5. If a Sylow 2-subgroup $P$ of $N$ is abelian, then $G \cong L_2(2^f), f \geq 2$.

Since $P$ is abelian, by [6, XI, Theorem 13.7], $N$ is one of the following groups: $L_2(2^f); L_2(q)$ where $q = 3, 5 \text{ (mod 8)}; 2G_2(q), q = 3^{2m+1}$. Recall that $\sigma_0(x) = 0$ whenever $x \in N$ is of even order.

Suppose that $N \cong 2G_2(q)$. Then all elements of even order in $N$ lie in at least three classes of $G$ (see [6, XI, Theorem 13.4]), a contradiction.

Therefore $N \cong L_2(q)$, where $q = 2^f$ or $q = 3, 5 \text{ (mod 8)}$. Then $\text{Aut}(N) = N\langle \phi, \delta \rangle$, where $\langle \phi \rangle$ is the group of field automorphisms of $N$, $\langle \delta \rangle$ is the group of diagonal automorphisms of $N$.

Case 1. Suppose that $N = L_2(q)$ where $q > 5, q = 3, 5 \text{ (mod 8)}$.

In this case, we have $|N|^2 = 4$. Since $\phi$ and $\delta$ commute modulo $\text{Inn}(N)$, we have $N = G'$. Let $\theta \in \text{Irr}_2(N)$ be such that $q = 3 \text{ (mod 8)}$ then $\theta(1) = q + 1$, and if $q = 5 \text{ (mod 8)}$ then $\theta(1) = q - 1$. By Lemma 2.2 (1), $\theta$ is extendable to an irreducible character $\mu$ of $G$.

Observe that $\theta(1) = 4k$ for some odd $k > 1$, and that $\theta$ is of $r$-defect zero for any prime divisor $r$ of $\theta(1)$. Let $x = x_1x_2 \in N$ be of order $2k$, where $o(x_1) = 2, o(x_2) = k$. We have that $\theta(x_1) = \theta(x_2) = \theta(x) = 0$, and so $\mu(x_1) = \mu(x_2) = \mu(x) = 0$, a contradiction.

Case 2. Suppose that $N \cong L_2(2^f), f \geq 2$.

In this case, $\text{Out}(N) = \langle \phi \rangle$. We need to prove that $G = N$. Observe that if $N \cong L_2(4)(\cong L_2(5))$, then $G = N$ by [2]. Thus we may assume that $f \geq 3$. Suppose that $G > N$. Then $G = G \cap N\langle \phi \rangle$. Following [3, §38] and using the notation of that table for the characters of $L_2(q)$, we may take $\theta_1 \in \text{Irr}(N)$ of degree $2^f - 1$ such that the stabilizer of $\theta$ in $\text{Aut}(N)$ is $N$. Thus $\theta_1$ induces to an irreducible character of $G$. This implies by Lemma 2.2 that $\theta_1^G$ is of odd degree.
In particular, \( G/N \) is a cyclic group of odd order.

Recall that \( \chi_0 \in \text{Irr}(N) \) is of degree \( 2^f \), and \( \sigma_0 \) is an extension of \( \chi_0 \) to \( G \). Since \( \Delta = \bigcup_{g \in G} (P^g - 1) \) is a class of \( N \), it forces \( \Delta \) to be also a class of \( G \). This implies that \( |C_G(t)| = |G/N||P| \) for any \( t \in P - \{1\} \), and so that \( C_G(t) = PA \), where \( A \cap N = 1 \), \( A \cong G/N \). Observe that \( \sigma_0(g) = 0 \) whenever \( g \in G \) is of even order and that \( G/N \) is a cyclic group of odd order. Suppose that there are primes \( r_1, r_2 \) such that \( r_1 r_2 \) divides \( |A| \). We can find \( x_1, x_2 \in C_G(t) = PA \) of order \( 2r_1, 2r_1 r_2 \) respectively, and then \( \sigma_0(t) = \sigma(x_1) = \sigma(x_2) = 0 \). However, \( t, x_1, x_2 \) lie in distinct classes of \( G \), a contradiction. Hence \( |G/N| \) is an odd prime \( q \). Also, we see that \( \Theta \), the set of elements of order \( 2q \), forms a class of \( G \). Let \( w \in A \) be of order \( q \) and \( y = wt \). Since \( \Theta \) is a class of \( G \), all cyclic subgroups of order \( 2q \) are conjugate to \( \langle y \rangle \). Note that distinct subgroups of order \( 2q \) have no common element of order \( 2q \), it follows that

\[
|\Theta| = |G : N_G(\langle y \rangle)|(q - 1).
\]

As \( N_G(\langle w \rangle) = \langle w \rangle N_N(\langle w \rangle) = \langle w \rangle \times (N_G(\langle w \rangle) \cap N) = C_G(w) \), we have

\[
N_G(\langle y \rangle) = N_G(\langle w \rangle) \cap N_G(\langle t \rangle) = C_G(w) \cap C_G(t) = C_G(y).
\]

Then

\[
|G : C_G(y)|(q - 1) = |G : N_G(\langle y \rangle)|(q - 1) = |\Theta| = |y^G| = |G : C_G(y)|,
\]

a contradiction. Thus \( G = N = L_2(2^f) \) as desired. \( \square \)

3. Theorem A

Lemma 3.1. Let \( N \trianglelefteq G \) and \( H/N \) be a Hall \( \pi \)-subgroup of \( G/N \). If \( \eta \in \text{Irr}(H) \) induces to an irreducible character \( \chi \) of \( G \), then \( \chi(x) = 0 \) for any \( \pi' \)-element \( x \in G - N \).

Proof. It follows directly from the definition of induced character. \( \square \)

Lemma 3.2. Let \( G \cong L_2(2^f) \), \( f \geq 3 \) and \( P \in \text{Syl}_2(G) \). If \( H \) is a proper subgroup of \( G \) with \( P \leq H \), then \( H \leq N_G(P) \).

Proof. It is enough to investigate the maximal subgroups of \( L_2(2^f) \) (see [5, II, Theorem 8.27]). \( \square \)

Proof of Theorem A. We need only to prove that if every member of \( \text{Irr}_2(G) \) has just one class of zeros, then \( G \) is one of the types listed in the theorem. Suppose that \( \text{Irr}_2(G) \) is empty. By a well-known theorem of Ito and Michler, \( G \) possesses a normal abelian Sylow 2-subgroup. In what follows, we assume that \( \text{Irr}_2(G) \) is not empty.

Case 1. Suppose that \( G \) is nonsolvable.

By Theorem B, there exists a normal solvable subgroup \( N \) of \( G \) such that \( G/N \cong L_2(7) \) or \( L_2(2^f) \). Clearly \( G/N \cong L_2(7) \) is not the case, and so \( G/N \cong \)
$L_2(2^f)$. Suppose that $N > 1$. To reach a contradiction, we may assume that $N$ is a minimal normal subgroup of $G$, and thus $N$ is an elementary abelian $q$-group for some prime $q$. Let $\chi_0 \in \text{Irr}(G/N)$ be of degree $2^f$. Let $P \leq G$ be such that $P/N \in Syl_2(G/N)$, and $\Delta = \cup_{g \in G} (P^g - N)$. Then

$$\Delta = v(\chi_0), \quad k_G(\Delta) = 1.$$  

For any $\chi \in \text{Irr}(G|N)$, by Lemma 2.1 we conclude that $\chi$ vanishes on $\Delta$, and then by [13, Lemma 1.1] we see that $\chi(1)$ is even. Let us consider the subgroup $P$. For any $t \in P - N$, we have

$$|C_G(t)| = |C_{G/N}(tN)| = |P/N|.$$  

If $q$ is odd, then the above equation yields that $P$ is a Frobenius group with $N$ as its kernel, and this leads to the contradiction that $P/N$, as an abelian Frobenius complement is cyclic. Thus $N$ is a 2-group. Since $N$ is a nontrivial normal subgroup of the 2-group $P$, we can take a non-principal $\lambda_0 \in \text{Irr}(N/N_1) \subseteq \text{Irr}(N)$, where $N/N_1$ is a chief factor of $P$. Clearly $\lambda_0$ is $P$-invariant. Note that if $\chi$ is an irreducible constituent of $\lambda_0^G$, then $\chi \in \text{Irr}(G|N)$ and then $\chi(1)$ is even.

Assume that $I_G(\lambda_0) = G$. Observe that $N$ can be viewed as an irreducible $G$-module over a field $F_2$ of 2 elements. Then $\text{Irr}(N)$ has a natural $G$-module structure induced by the conjugate action of $G$ on $N$, and since $N$ is irreducible, $\text{Irr}(N)$ is also an irreducible $G$-module (see Section 1.6 of [8]). Let $W$ be the set of all $G$-invariant linear character of $N$. Then $W$ is a nontrivial $G$-submodule of $\text{Irr}(N)$, and this implies that $W = \text{Irr}(N)$. Now applying [7, Theorem 6.32] we conclude that $N \leq Z(G)$. By [2, Page xvi, Table 5], we have either $G \cong L_2(2^f) \times N$ or $G \cong SL(2,5)$. If $G \cong L_2(2^f) \times N$, then $|C_G(t)| > |P/N|$ for any $t \in P - N$, a contradiction. If $G \cong SL(2,5)$, then we also obtain a contradiction by [2].

Assume that $I_G(\lambda_0) < G$. By Lemma 3.2, we have that $P \leq I_G(\lambda_0) \leq H$, where $H/N = N_{G/N}(P/N)$. Let $\psi_0$ be an irreducible constituent of $\lambda_0^H$. By Clifford theorem, $\psi_0$ induces to an irreducible character $\eta_0$ of $G$. Observe that $\eta_0 \in \text{Irr}(G|N)$ vanishes on $\Delta$ and is of even degree. Since $H/N$ is a Hall subgroup of $G/N$, it follows by Lemma 3.1 that $\eta_0 = \psi_0^G$ vanishes on some element outside $\Delta$. Thus $\eta_0$ vanishes on at least two classes of $G$, a contradiction.

Case 2. Suppose that $G$ is solvable.

Assume first that there is some $\chi \in \text{Irr}_2(G)$ such that $\chi_{\lambda^G}$ is reducible. By [7, Theorem 6.22], there exist a subgroup $H$ with $G' \leq H < G$ and an irreducible character $\lambda$ of $H$ so that $\chi = \lambda^G$. This implies that $\chi$ vanishes on $G - H$, and so $k_G(G - H) = 1$. Now it is easy to verify in this case that $G$ is a Frobenius group with a complement of order 2 (see [11, Lemma 2(2)]).

In what follows, we assume that $\chi_{\lambda^G}$ is irreducible for any $\chi \in \text{Irr}_2(G)$, and we will show in this case that $G \cong SL(2,3)$. Since $\chi_{\lambda^G}$ is irreducible, $\chi$ vanishes at some element of $G'$, and consequently $v(\chi) \subseteq G'$ because $k_G(v(\chi)) = 1$. It follows by Lemma 2.2 (2) that $gcd(\chi(1), |G/G'|) = 1$ for any $\chi \in \text{Irr}_2(G)$. In particular, $|G/G'|$ is odd.

Let $E < G$ maximal be such that $G/E$ is nonabelian. By [7, Lemma 12.3] $G/E$ is a $p$-group or a Frobenius group. Suppose that $G/E$ is a $p$-group and let $\psi$ be
a nonlinear irreducible character of $G/E$. Being a prime divisor of $|G/G'|$, $p$ is coprime to $\chi(1)$ for any $\chi \in \text{Irr}_2(G)$. Then $\chi_0\psi \in \text{Irr}_2(G)$ for some $\chi_0 \in \text{Irr}_2(G)$, and $p$ is a common divisor of $|G/G'|$ and $(\chi_0\psi)(1)$, a contradiction. Therefore $G/E$ is a Frobenius group with a kernel $N/E$ and a cyclic complement.

For any $\tau_0 \in \text{Irr}_2(N)$, by Frobenius reciprocity $\tau_0$ is extendible to some $\chi_0 \in \text{Irr}_2(G)$, and thus [7, Theorem 12.4] implies that both $\chi_0$ and $\tau_0$ vanish on $N - E$, then

$$k_G(N - E) = 1,$$

and so

$$k_{G/E}(N/E - E/E) = 1.$$

This implies that

$$|N/E| = 1 + |G/N|.$$  

Since $|G/N|$ is odd, $N/E$ is an elementary abelian 2-group. Set $|N/E| = 2^r$ and let $t \in N - E$. We have

$$2^r = |C_G(t)| = |C_N(t)| = |N/E| + \sum_{\eta \in \text{Irr}(N/E)} |\eta(t)|^2.$$  

This implies that $N' = E$, and that for any $\eta \in \text{Irr}(N/E)$ (that is, for any nonlinear $\eta \in \text{Irr}(N)$), $\eta$ must vanish on $N - E$, so $\eta(1)$ is even (see [13, Lemma 1.1]), and hence $\eta$ is extendible to $G$.

Clearly $E > 1$. Let $E/F$ be a chief factor of $G$. If $E/F$ is of odd order, then the above fact implies that $N/F$ is a Frobenius group with the kernel $E/F$, and then being a Frobenius complement, the elementary abelian 2-group $N/E$ is of order 2, which is impossible. Thus $E/F$ is a 2-group. Let us investigate the quotient group $G/F$ and let $K \cong G/N$ be a Hall $2'$-subgroup of $G/F$. Since every nonlinear irreducible character of $N/F$ is of even degree, $K$ acts nontrivially on $N/F$ and fixes every nonlinear irreducible character of $N/F$. By [10, Lemma 19.2], we conclude that $N'/F = E/F \leq Z(G/F)$. Since $N/E, E/F$ are chief factors of $G$, it is easy to see that

$$E/F = N'/F = Z(N/F) = Z(G/F) = \Phi(N/F).$$

Thus $|E/F| = 2$ and $N/F$ is an extraspecial 2-group. Now [4, Ch.5, Theorem 6.5] implies that $2^r - 1 = |G/N|$ divides $2^e + 1$ for some integer $e \leq r/2$. This yields that $2^r = 4$, and so $G/F \cong SL(2,3)$.

To finish the proof of Theorem A, it suffices to show that $F = 1$. Suppose that $F > 1$. Towards a contradiction we may assume that $F$ is a minimal normal subgroup of $G$. Assume that $F$ is a 2-group. Since $|C_G(t)| = 4$ for any $t \in N - E$, there is $x \in N - E$ of order $|N|/2 \geq 8$ ([13, Lemma 1.3]), which leads to the contradiction that $|C_G(x)| \geq 8 > 4$. Assume that $F$ is a $q$-group for some odd prime $q$ and set $P \in Syl_2(N)$. Since $|C_G(t)| = |N/E| = 4$ for any $t \in N - E$, we see that $C_P(x) \leq P'$ for any $1 \neq x \in F$. It follows by [10, Lemma 19.1] that $N = PF$ is a Frobenius group with a complement $P$ and that $P$ is either cyclic or isomorphic to $Q_8$. Then we can find some $\theta_0 \in \text{Irr}(N)$ of degree 8. Let $\chi_0$ be an extension of $\theta_0$ to $G$. We have $N - F \subseteq v(\chi_0)$, a contradiction. Thus $F = 1$, and the proof of Theorem A is complete.
References


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