Notes on the Algebra and Geometry of Polynomial Representations

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Abstract. The paper deals with a semi-algebraic set $A$ in $\mathbb{R}^d$ constructed by the inequalities $p_i(x) > 0$, $p_i(x) \geq 0$, and $p_i(x) = 0$ for a given list of polynomials $p_1, \ldots, p_m$, and presents several statements that fit into the following template. Assume that in a neighborhood of a boundary point the semi-algebraic set $A$ can be described by an irreducible polynomial $f$. Then $f$ is a factor of a certain multiplicity of some of the polynomials $p_1, \ldots, p_m$. Special attention is paid to the case when $A$ is a polytope.

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1. Introduction

In what follows $x := (x_1, \ldots, x_d)$ is a variable vector in $\mathbb{R}^d$ ($d \in \mathbb{N}$). As usual, $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_d]$ denotes the ring of polynomials in variables $x_1, \ldots, x_d$ and coefficients in $\mathbb{R}$. A subset $A$ of $\mathbb{R}^d$ is said to be semi-algebraic if

$$A = \{ x \in \mathbb{R}^d : \Phi((\text{sign } p_1(x) \in E_1), \ldots, (\text{sign } p_m(x) \in E_m)) \},$$

(1.1)
where $\Phi$ is a boolean formula, $p_1, \ldots, p_m \in \mathbb{R}[x]$, and $E_1, \ldots, E_m$ are non-empty subsets of $\{0, 1\}$; see also [1], [8], and [6]. We call (1.1) a representation of $A$ by polynomials $p_1, \ldots, p_m$. We distinguish the following particular types of semi-algebraic sets:

\[
(p_1, \ldots, p_m)_{\geq 0} := \{ x \in \mathbb{R}^d : p_1(x) \geq 0, \ldots, p_m(x) \geq 0 \}, \quad (1.2)
\]

\[
(p_1, \ldots, p_s)_{> 0} := \{ x \in \mathbb{R}^d : p_1(x) > 0, \ldots, p_m(x) > 0 \}, \quad (1.3)
\]

\[
Z(p_1, \ldots, p_m) := \{ x \in \mathbb{R}^d : p_1(x) = 0, \ldots, p_m(x) = 0 \}. \quad (1.4)
\]

Sets representable by (1.2), (1.3), and (1.4), respectively, are called elementary open semi-algebraic, elementary closed semi-algebraic, and algebraic, respectively.

A subset $P$ of $\mathbb{R}^d$ is said to be a polytope if $P$ is the convex hull of a non-empty and finite set of points; see [14]. It is known that a set $P$ in $\mathbb{R}^d$ is a polytope if and only if $P$ is non-empty, bounded, and can be represented by $P = (p_1, \ldots, p_m)_{\geq 0}$, where $p_1, \ldots, p_m \in \mathbb{R}[x]$ ($m \in \mathbb{N}$) are of degree one (the so-called $H$-representation). Thus, polytopes are just special elementary closed semi-algebraic sets. The study of polynomial representations of polygons and polytopes was initiated in [7] and [12]; see also the survey [13]. In [12] it was noticed that, if a $d$-dimensional polytope $P$ is represented by

\[
P = (q_1, \ldots, q_m)_{\geq 0} \quad (1.5)
\]

with $q_1, \ldots, q_m \in \mathbb{R}[x]$, then $m \geq d$. In [9] it was conjectured that every $d$-dimensional polytope in $\mathbb{R}^d$ can be represented by (1.5) with $m = d$. This conjecture has recently been confirmed by L. Bröcker [10]; see also [4], [3], and [5] for further related results. We refer to [1, Chapter 5] and [8, §6.5 and §10.4] for results on minimal representations of general elementary semi-algebraic sets. In this paper we derive necessary conditions on representations of polytopes consisting of $d$ polynomials.

**Theorem 1.1.** Let $P$ be a $d$-dimensional polytope in $\mathbb{R}^d$ with $m$ facets such that

\[
P = (p_1, \ldots, p_m)_{\geq 0} = (q_1, \ldots, q_d)_{\geq 0},
\]

where $p_1, \ldots, p_m, q_1, \ldots, q_d \in \mathbb{R}[x]$ and $p_1, \ldots, p_m$ are of degree one. Then every $p_i, i \in \{1, \ldots, m\}$, is a factor of precisely one polynomial $q_j$ with $j \in \{1, \ldots, d\}$. Furthermore, for $i$ and $j$ as above, the factor $p_i$ of $q_j$ is of odd multiplicity. □

Theorem 1.1 improves Proposition 2.1(i) from [12]. In [7] it was shown that every convex polygon $P$ in $\mathbb{R}^2$ can be represented by two polynomials. We are able to determine the precise structure of such minimal representations.

**Theorem 1.2.** Let $P$ be a convex polygon in $\mathbb{R}^2$ with $m \geq 7$ edges and let

\[
P = (p_1, \ldots, p_m)_{\geq 0} = (q_1, q_2)_{\geq 0},
\]

where $p_1, \ldots, p_m, q_1, q_2 \in \mathbb{R}[x]$ and $p_1, \ldots, p_m$ are of degree one. Then there exist $k_1, \ldots, k_m \in \mathbb{N}$ and $g_1, g_2 \in \mathbb{R}[x]$ such that $\{q_1, q_2\} = \{p_1^{k_1} \cdots p_m^{k_m} 1, g_2\}$ and the following conditions are fulfilled:
1. $k_1, \ldots, k_m$ are odd;
2. $g_1, g_2$ are not divisible by $p_i$ for every $i \in \{1, \ldots, m\}$;
3. $g_2(y) = 0$ for every vertex $y$ of $P$.

It is not hard to see that the set $(q_1, q_2) \geq 0$ in Theorem 1.2 does not depend on the concrete choice of odd numbers $k_1, \ldots, k_m$. More precisely, for $g_1, g_2$ as in Theorem 1.2 we have $P = (p_1 \cdots p_m g_1, g_2) \geq 0$. In [7] the polynomials $q_1, q_2$ representing $P$ were defined in such a way that $g_1 = 1$ and $k_1 = \cdots = k_m = 1$ see Figure 1 for an illustration of this result and Theorem 1.2. We also remark that the assumption $m \geq 7$ cannot be relaxed in general, since Theorem 1.2 would not hold if $P$ is a centrally symmetric hexagon. In fact, assume that $P$ is a centrally symmetric hexagon and $p_1, \ldots, p_6$ are polynomials of degree one such that $Z(p_1) \cap P, \ldots, Z(p_6) \cap P$ are consecutive edges of $P$. Then $P = (q_1, q_2) \geq 0$ for $q_1 := p_1 p_3 p_5$ and $q_2 := p_2 p_4 p_6$; see Figure 2. It will be seen from the proof of Theorem 1.2 that the assumption $m \geq 7$ can be relaxed to $m \geq 5$ for the case when $P$ does not have parallel edges.

Theorems 1.1 and 1.2 are obtained as corollaries of the more general Theorem 2.2 given in Section 2. Theorem 2.2 and Corollaries 2.3–2.5 from Section 2 are results
analogous to Theorems 1.1 and 1.2 for more general classes of semi-algebraic sets. Even though the mentioned general results are somewhat technical, they can be of independent interest (see also Section 2.2 providing related examples).

2. The proofs

The origin in $\mathbb{R}^d$ is denoted by $o$. Given $c \in \mathbb{R}^d$ and $\rho > 0$ by $B^d(c, \rho)$ we denote the open Euclidean ball with center $c$ and radius $\rho$. The abbreviations int and bd stands for the interior and boundary, respectively. By dim we denote the dimension.

We refer to [2] and [11] for standard notions and results from commutative algebra and algebraic geometry. The notion of dimension of a semi-algebraic set can be defined in several equivalent ways; for details see [8, §2.8]. Given a polynomial $p \in \mathbb{R}[x]$, by $\nabla p$ we denote the gradient of $p$. The statement of the following lemma is known (see [8, Theorem 4.5.1]).

**Lemma 2.1.** Let $f$ be a polynomial irreducible over $\mathbb{R}[x]$. Then $\dim Z(f) = d - 1$ if and only if for some $y \in \mathbb{R}^d$ one has $f(y) = 0$ and $\nabla f(y) \neq o$. Furthermore, if $\dim Z(f) = d - 1$ and $p \in \mathbb{R}[x]$, then the following conditions are equivalent:

(i) $\dim(Z(f) \cap Z(p)) = d - 1$.

(ii) $Z(f) \subseteq Z(p)$.

(iii) $f$ is a factor of $p$. 

\[ \Box \]

In the proofs below we shall deal with polynomials $p_1, \ldots, p_m$. Throughout the proofs $f_1, \ldots, f_n$ will denote the polynomials irreducible over $\mathbb{R}[x]$ which are involved in the prime factorization of the product $p_1 \cdots p_m$ (see [11, p. 149]), i.e.

$$p_1 \cdots p_m = f_1^{s_1} \cdots f_n^{s_n}$$

for some $s_1, \ldots, s_n \in \mathbb{N}$ and for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$ the polynomials $f_i$ and $f_j$ do not coincide up to a constant multiple.

**Theorem 2.2.** Let $A$ be a semi-algebraic set in $\mathbb{R}^d$ given by (1.1) and let $f$ be a polynomial irreducible over $\mathbb{R}[x]$. Then the following statements hold true.

I. One has $\text{bd } A \subseteq \bigcup_{i=1}^m Z(p_i)$.

II. If

$$\dim(\text{bd } A \cap Z(f)) = d - 1, \quad (2.1)$$

then $f$ is a factor of $p_i$ for some $i \in \{1, \ldots, m\}$.

III. If there exist $a \in Z(f)$ and $\varepsilon > 0$ such that

$$\dim(Z(f) \cap B^d(a, \varepsilon)) = d - 1, \quad (2.2)$$
$$\dim((f)_{\geq 0} \cap B^d(a, \varepsilon)) = A \cap B^d(a, \varepsilon), \quad (2.3)$$

then (2.1) is fulfilled and, moreover, $f$ is an odd-multiplicity factor of $p_i$ for some $i \in \{1, \ldots, m\}$. 

IV. If there exist \( a \in Z(f) \) and \( \varepsilon > 0 \) such that
\[
\dim(Z(f) \cap B^d(a, \varepsilon)) = d - 1,
\]

then (2.1) is fulfilled and, moreover, \( f \) is an odd-multiplicity factor of \( p_i \) for some \( i \in \{1, \ldots, m\} \).

\( \square \)

**Proof.** For \( x \in \mathbb{R}^d \) we define
\[
\Psi(x) := \Phi((\text{sign} \ p_1(x) \in E_1), \ldots, (\text{sign} \ p_m(x) \in E_m)).
\]

Part I: Let \( x_0 \notin \bigcup_{i=1}^m Z(p_i) \), that is \( p_i(x_0) \neq 0 \) for every \( i = 1, \ldots, m \). Then there exists an \( \varepsilon > 0 \) such that the sign of every \( p_i(x) \), \( i \in \{1, \ldots, m\} \), remains constant on \( B^d(x_0, \varepsilon) \). It follows that \( \Psi(x) \) is constant for \( x \in B^d(x_0, \varepsilon) \). Consequently, either \( B^d(x_0, \varepsilon) \subseteq A \) or \( B^d(x_0, \varepsilon) \cap A = \emptyset \). Hence \( x_0 \) is either an interior or an exterior point of \( A \), and we get the conclusion of Part I.

Part II: By Part I we have \( \text{bd} A \subseteq \bigcup_{i=1}^m Z(f_i) \). Consequently
\[
d - 1 \overset{(2.1)}{=} \dim(\text{bd} A \cap Z(f)) \leq \dim \left( \bigcup_{i=1}^m Z(p_i) \cap Z(f) \right)
\]
\[= \max_{1 \leq i \leq m} \dim(Z(p_i) \cap Z(f)) \leq d - 1.
\]

Hence \( \dim Z(f) = d - 1 \) and for some \( i \in \{1, \ldots, m\} \) one has \( \dim(Z(p_i) \cap Z(f)) = d - 1 \). Then Lemma 2.1 yields the assertion of Part II.

Part III: Let \( a \in Z(f) \) and \( \varepsilon > 0 \) satisfy (2.2) and (2.3). From (2.2) it follows that \( \dim Z(f) = d - 1 \). By Lemma 2.1, there exists \( a' \in Z(f) \cap B^d(a, \varepsilon) \) such that \( \nabla f(a') \neq 0 \). We choose \( \varepsilon' > 0 \) such that \( B^d(a', \varepsilon') \subseteq B^d(a, \varepsilon) \) and \( \nabla f(x) \neq 0 \) for every \( x \in B^d(a', \varepsilon') \). Let us show that
\[
Z(f) \cap B^d(a', \varepsilon') \subseteq \text{bd} A.
\]

(2.4)

Consider an arbitrary point \( x \in Z(f) \cap B^d(a', \varepsilon') \). In view of (2.3) we have \( x \in A \). On the other hand, since \( f(x) = 0 \) and \( \nabla f(x) \neq 0 \), there exists a sequence \( (x^k)_{k=1}^{+\infty} \) of points from \( B^d(a', \varepsilon') \) such that \( f(x^k) < 0 \) for every \( k \in \mathbb{N} \) and \( x^k \to x \), as \( k \to +\infty \). Since \( x^k \notin (f)_{k=0}^\infty \) and \( x^k \in B^d(a, \varepsilon) \), in view of (2.3) it follows that \( x^k \notin A \) for every \( k \in \mathbb{N} \). Hence, \( x \) is a point of \( A \) and is a limit of a sequence of points lying outside \( A \). The latter implies (2.4). Since \( f(a') = 0 \) and \( \nabla f(x) \neq 0 \) for every \( x \in Z(f) \cap B^d(a', \varepsilon') \) it follows that \( Z(f) \cap B^d(a', \varepsilon') \) has dimension \( d - 1 \). Consequently, we have
\[
d - 1 = \dim(Z(f) \cap B^d(a', \varepsilon')) \overset{(2.4)}{=} \dim(Z(f) \cap \text{bd} A \cap B^d(a', \varepsilon'))
\]
\[\leq \dim(Z(f) \cap \text{bd} A) \leq \dim(Z(f)) = d - 1.
\]

Hence \( \dim(Z(f) \cap \text{bd} A) = d - 1 \). By Part II, it follows that \( f \) coincides, up to a constant multiple, with \( f_i \) for some \( i \in \{1, \ldots, n\} \). Without loss of generality we
assume that \( f = f_1 \). By Lemma 2.1, we can choose \( a'' \in Z(f) \cap B^d(a', \varepsilon') \) such that \( f_i(a'') \neq 0 \) for \( i \in \{2, \ldots, n\} \). This means the sign of the polynomials \( f_i, i = \{2, \ldots, n\} \), remains constant on \( B^d(a'', \varepsilon'') \). We prove the statement of Part III by contradiction. Assume that whenever \( f \) is factor of \( p_i, i \in \{1, \ldots, m\} \), this factor is of even multiplicity. Since \( \nabla f(a'') \neq 0 \), we can choose \( x_0, y_0 \in B^d(a'', \varepsilon'') \) such that \( f(x_0) > 0 \) and \( f(y_0) < 0 \). Since the signs of \( f_2, \ldots, f_n \) do not change on \( B^d(a'', \varepsilon'') \) and since \( f_1 = f \) appears with an even multiplicity only, we obtain sign \( p_j(x_0) = \text{sign} p_j(y_0) \) for every \( j = 1, \ldots, m \). Hence \( \Psi(x_0) = \Psi(y_0) \). But by (2.3), \( x_0 \in A \) and \( y_0 \notin A \), which implies that \( \Psi(x_0) \neq \Psi(y_0) \), a contradiction.

The proof of Part IV is omitted, since it is analogous to the proof of Part III. 

Let us give an informal interpretation of Theorem 2.2. Let \( f \) be an irreducible polynomial such that \( Z(f) \) is a \((d - 1)\)-dimensional algebraic surface. Consider a semi-algebraic set \( A \) given by (1.1). If the boundary of \( A \) coincides locally with a part of \( Z(f) \), then \( f \) is a factor of some \( p_i \). If \( A \) coincides locally with a part of \( (f)_{\geq 0} \), then \( f \) is an odd-multiplicity factor of some \( p_i \). Furthermore, if in a neighborhood of a boundary point the set \( A \) coincides locally with a part of \( Z(f) \), then \( f \) is a factor of at least two different polynomials \( p_i \) or an even-multiplicity factor of at least one polynomial \( p_i \).

We remark that (2.2) cannot be replaced by the weaker condition \( \dim Z(f) = d - 1 \) and \( Z(f) \cap B^d(a, \varepsilon) \neq \emptyset \), since the algebraic set \( Z(f) \) corresponding to an irreducible polynomial \( f \) can have “parts” of dimensions strictly smaller than \( \dim Z(f) \). In fact, for \( d = 2 \) the irreducible polynomial \( f(x) := x_1^2 + x_2^2 - x_3^2 \) generates the cubic curve \( Z(f) \) with isolated point at the origin. For \( d = 3 \), for the irreducible polynomial \( f(x) = x_3^2 x_1 - x_2^2 \) the set \( Z(f) \) is the well-known Whitney umbrella, which is a two-dimensional algebraic surface with the one-dimensional “handle” \( Z(x_2, x_3) \).

**Corollary 2.3.** Let \( A \) be a semi-algebraic set given by

\[
A = \left\{ x \in \mathbb{R}^d : \Phi((p_1(x) \geq 0), \ldots, (p_m(x) \geq 0)) \right\},
\]

where \( \Phi \) is a boolean formula and \( p_1, \ldots, p_m \in \mathbb{R}[x] \setminus \{0\} \), and let \( f \) be a polynomial irreducible over \( \mathbb{R}[x] \). Then the following statements hold true.

I. If there exist \( b \in Z(f) \) and \( \varepsilon > 0 \) such that

\[
\dim(Z(f) \cap B^d(b, \varepsilon)) = d - 1,
\]

\[
Z(f) \cap B^d(b, \varepsilon) = A \cap B^d(b, \varepsilon),
\]

then (2.1) is fulfilled, and furthermore \( f \) is a factor of \( p_i \) and \( p_j \) for some \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \) or \( f \) is an even-multiplicity factor of \( p_i \) for some \( i \in \{1, \ldots, m\} \).

II. If there exist \( a, b \in \mathbb{R}^d \) and \( \varepsilon > 0 \) such that equalities (2.2), (2.3), (2.5), and (2.6) are fulfilled, then \( f \) is a factor of \( p_i \) and an odd-multiplicity factor of \( p_j \) for some \( i, j \in \{1, \ldots, m\} \) with \( i \neq j \).
Proof. Part I: Let $b \in Z(f)$ and $\varepsilon > 0$ satisfy (2.5) and (2.6). From (2.5) it follows that $\dim Z(f) = d - 1$. By Lemma 2.1, there exists $b' \in Z(f) \cap B^d(b, \varepsilon)$ such that $\nabla f(b') \neq 0$. Choose $\varepsilon' > 0$ such that $B^d(b', \varepsilon') \subseteq B^d(b, \varepsilon)$ and $\nabla f(x) \neq 0$ for every $x \in B^d(b', \varepsilon')$. Using arguments analogous to those from the proof of Theorem 2.2(III) we show the inclusion $Z(f) \cap B^d(b', \varepsilon') \subseteq \text{bd} A$ and (2.1). Hence, by Theorem 2.2(II), $f$ coincides, up to a constant multiple, with $f_i$ for some $i \in \{1, \ldots, n\}$. Without loss of generality we assume that $f = f_i$. If $f$ is a factor of $p_i$ and $p_j$ for some $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we are done. We consider the opposite case, that is, for some $i \in \{1, \ldots, m\}$ the polynomial $f$ is a factor of precisely one polynomial $p_i$ with $i \in \{1, \ldots, m\}$, say $p_1$. We show by contradiction that in this case the factor $f$ of $p_1$ has even multiplicity. Assume the contrary, i.e., the factor $f$ of $p_1$ has odd multiplicity. Analogously to the arguments from the proof of Theorem 2.2, we choose $b'' \in Z(f)$ and $\varepsilon'' > 0$ such that $B^d(b'', \varepsilon'') \subseteq B^d(b', \varepsilon')$ and $f_i(x) \neq 0$ for every $i \in \{2, \ldots, n\}$ and every $x \in B^d(b', \varepsilon'')$. By the choice of $b''$ and $\varepsilon''$ we have sign $p_i(x) = \text{sign } p_i(b'')$ for all $i \in \{2, \ldots, m\}$ and $x \in B^d(b'', \varepsilon'')$. Since $\nabla f(b'') \neq 0$, there exist points $x_0, y_0 \in B^d(b', \varepsilon')$ such that $f(x_0) f(y_0) < 0$. Then $p_1(x_0) p_1(y_0) < 0$. Consequently, either $p_1(x_0) > 0$ or $p_1(y_0) > 0$. Without loss of generality we assume that $p_1(x_0) > 0$. It follows that $(p_i(x_0) \geq 0) \equiv (p_i(b'') \geq 0)$ for $i = 1, \ldots, m$. Hence $x_0 \in A$. But since $f(x_0) \neq 0$, in view of (2.6), we get $x_0 \notin A$, a contradiction.

Part II: By Theorem 2.2(III) $f$ is a factor of odd multiplicity of some $p_i$ with $i \in \{1, \ldots, m\}$. Furthermore, for some $j \in \{1, \ldots, m\}$ with $i \neq j$ the polynomial $f$ is a factor of $p_j$, since otherwise we would get a contradiction to Part I. \hfill \Box

Corollary 2.4. Let $p_1, \ldots, p_m \in \mathbb{R}[x] \setminus \{0\}$ and $A := (p_1, \ldots, p_m)_{\geq 0}$. Let $f$ be a polynomial irreducible over $\mathbb{R}[x]$. Assume that there exist $b \in Z(f)$ and $\varepsilon > 0$ such that equalities (2.5) and (2.6) are fulfilled and additionally

$$
\dim(\text{int } A \cap Z(f)) = d - 1.
$$

Then $f$ is a factor of $p_i$ for some $i \in \{1, \ldots, m\}$ and, for every $i \in \{1, \ldots, m\}$ such that $p_i$ is divisible by $f$, the factor $f$ of $p_i$ has even multiplicity.

Proof. By Corollary 2.3(1), $f$ is a factor of some $p_i$, say $p_1$. Without loss of generality we assume that $f_1 = f$. Let us show that the factor $f$ of $p_1$ is of even multiplicity. Assume the contrary. In view of Lemma 2.1, we can choose $a' \in \text{int } A \cap Z(f)$ such that $\nabla f(a') \neq 0$. We fix $\varepsilon' > 0$ such that $\nabla f(x) \neq 0$ for every $x \in B^d(a', \varepsilon')$. By Lemma 2.1 we can choose $a'' \in B^d(a', \varepsilon')$ such that $f_i(a'') \neq 0$ for every $i \in \{2, \ldots, n\}$. Fix $\varepsilon'' > 0$ such that for every $i \in \{2, \ldots, n\}$ the sign of $f_i$ remains constant on $B^d(a'', \varepsilon'')$. Since $\nabla f(a'') \neq 0$, there exist $x_0$ and $y_0$ in $B^d(a'', \varepsilon'')$ with $f(x_0) f(y_0) < 0$. Hence $p_1(x_0) p_1(y_0) < 0$, and we get that either $x_0$ or $y_0$ does not belong to $A$, a contradiction. \hfill \Box

Corollary 2.5. Let $p_1, \ldots, p_m \in \mathbb{R}[x] \setminus \{0\}$ and $A := (p_1, \ldots, p_m)_{\geq 0}$. Let $f$ be a polynomial irreducible over $\mathbb{R}[x]$. Assume that there exist $b \in Z(f)$ and $\varepsilon > 0$ such that equalities (2.5) and (2.6) are fulfilled and additionally

$$
\dim(\text{int } A \cap Z(f)) = d - 1.
$$

Then $f$ is a factor of $p_i$ for some $i \in \{1, \ldots, m\}$ and, for every $i \in \{1, \ldots, m\}$ such that $p_i$ is divisible by $f$, the factor $f$ of $p_i$ has even multiplicity.
such that
\[ \dim(\text{bd} A \cap Z(f) \cap B^d(b, \varepsilon)) = d - 1, \quad (2.8) \]
\[ B^d(b, \varepsilon) \setminus Z(f) = A \cap B^d(b, \varepsilon). \quad (2.9) \]

Then \( f \) is a factor of \( p_i \) for some \( i \in \{1, \ldots, m\} \) and, for every \( i \in \{1, \ldots, m\} \) such that \( p_i \) is divisible by \( f \), the factor \( f \) of \( p_i \) has even multiplicity.

Proof. Equality (2.8) implies (2.1), and hence, by Theorem 2.2 (II), \( f \) is a factor of \( p_i \) for some \( i \in \{1, \ldots, m\} \). The rest of the proof is analogous to the proof of Corollary 2.4.

Now we are able to prove Theorems 1.1 and 1.2 from the introduction.

Proof of Theorem 1.1. Let us prove the first part of the assertion. Assume the contrary, say \( p_1 \) is a factor of both \( q_1 \) and \( q_2 \). Then within the \((d - 1)\)-dimensional affine space \( Z(p_1) \) the facet \( P \cap Z(p_1) \) of \( P \) is represented by \( d - 2 \) polynomials \( q_3, \ldots, q_d \) in the following way
\[ P \cap Z(p_1) = \{ x \in Z(p_1) : q_3(x) \geq 0, \ldots, q_d(x) \geq 0 \}. \]

This yields a contradiction to the fact that a \( k \)-dimensional convex polytope cannot be represented (in the above form) by less than \( k \) polynomials; see [12, Corollary 2.2]. The second part of the assertion follows directly from Theorem 2.2 (III).

Proof of Theorem 1.2. For \( j \in \{1, 2\} \) denote by \( I_j \) the set of indices \( i \in \{1, \ldots, m\} \) for which \( p_i \) is a factor of \( q_j \). By Corollary 2.4 it follows that \( I_1 \cup I_2 = \{1, \ldots, m\} \). Furthermore, \( I_1 \cap I_2 = \emptyset \), by Theorem 1.1. Let us show that either \( I_1 \) or \( I_2 \) is empty. Assume the contrary. We show that then there exist \( i \in I_1 \) and \( j \in I_2 \) such that the edges \( Z(p_i) \cap P \) and \( Z(p_j) \cap P \) of \( P \) are not adjacent and not parallel. Since \( m \geq 7 \), after possibly exchanging the roles of \( q_1 \) and \( q_2 \), we may assume that the cardinality of \( I_2 \) is at least four. Let us take an arbitrary \( i \in I_1 \). Then there exist at least two sides of the form \( Z(p_j) \cap P, \ j \in I_2 \), which are not adjacent to \( Z(p_i) \cap P \). One of these sides is not parallel to \( Z(p_i) \cap P \). The intersection point \( y \) of \( Z(p_i) \) and \( Z(p_j) \) lies outside \( P \) and fulfills the equalities \( q_1(y) = q_2(y) = 0 \), a contradiction to the inclusion \( (q_1, q_2)_{\geq 0} \subseteq P \). Hence \( I_1 \) or \( I_2 \) is empty. Without loss of generality we assume that \( I_2 = \emptyset \).

For \( i \in \{1, \ldots, m\} \) let \( k_i \) be the multiplicity of the factor \( p_i \) of \( p_1 \). Then \( q_1 = p_1^{k_1} \cdots p_m^{k_m} g_1 \) for some polynomial \( g_1 \), and statements 1 and 2 follow directly from Theorem 2.2 (III).

It remains to verify condition 2 (which involves \( g_2 = g_2 \)). This condition can be deduced from Proposition 2.1 (ii) in [12], but below we also give a short proof. We argue by contradiction. Let \( y \) be a vertex of \( P \) with \( g_2(v) > 0 \). Up to reordering the sequence \( p_1, \ldots, p_m \) we may assume that \( p_1(v) = 0 \). Clearly, any point \( y' \) lying in \( Z(p_1) \setminus P \) and sufficiently close to \( y \) fulfills the conditions \( q_1(y') = 0 \) and \( q_2(y') > 0 \). Hence \( y' \in P \), a contradiction to the inclusion \( (q_1, q_2)_{\geq 0} \subseteq P \).
3. Examples to Theorem 2.2 and its corollaries

Each of the examples below is supplied with a figure referring to the case $d = 2$. Let

$$A := \{ x \in \mathbb{R}^d : x_d > 0 \text{ and } ((x_1 - 1)^2 + x_2^2 + \cdots + x_d^2 \leq 1 \text{ or } x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1) \},$$

see Figure 3. By Theorem 2.2, if $A$ is given by (1.1), then the polynomials $x_d$, $(x_1 - 1)^2 + x_2^2 + \cdots + x_d^2 - 1$, $(x_1 + 1)^2 + x_2^2 + \cdots + x_d^2 - 1$ are factors of odd multiplicity of some of the polynomials $p_1, \ldots, p_m$.

![Figure 3. Illustration to Theorem 2.2](image)

The set

$$A := \{ x \in \mathbb{R}^d : (1 - x_1^2 - \cdots - x_d^2)(x_d + 2)^2 \geq 0 \},$$

$$= \{ x \in \mathbb{R}^d : (1 - x_1^2 - \cdots - x_d^2)(x_d + 2) \geq 0, \ x_d + 2 \geq 0 \},$$

depicted in Figure 4 is the disjoint unit of a closed unit ball centered at $o$ and a hyperplane given by the equation $x_d + 2 = 0$. By Corollary 2.3 (I), if $A$ is given by (1.1), then $x_d + 2$ is a factor of at least two polynomials $p_i$ or a factor of even multiplicity of at least one polynomial $p_1, \ldots, p_m$. From (3.1) and (3.2) we see that both of these possibilities are indeed realizable. Figure 5 depicts the semi-algebraic set

$$A := \{ x \in \mathbb{R}^2 : x_d \geq 0, \ (1 - x_1^2 - \cdots - x_d^2)x_d \geq 0 \},$$

$$= \{ x \in \mathbb{R}^2 : x_d \geq 0, \ (1 - x_1^2 - \cdots - x_d^2)x_d^2 \geq 0 \}. \tag{3.3}$$

By Corollary 2.3 (II), if $A$ is given by (1.1) with $E_1 = \ldots = E_m = \{0, 1\}$, the polynomial $x_d$ is a factor of at least two polynomials $p_i$ and an odd-multiplicity factor of at least one polynomial $p_i$. By (3.3) we see that the above conclusion cannot be strengthened. In fact, (3.3) provides a representation $A = (p_1, p_2)_{\geq 0}$ such that $x_d$ is an odd-multiplicity factor of precisely one polynomial $p_i$.

![Figure 4. Illustration to Corollary 2.3(I)](image)  
![Figure 5. Illustration to Corollary 2.3(II)](image)
Figure 6 presents the semi-algebraic set

\[ A := \{ x \in \mathbb{R}^d : (1 - x_1^2 - \cdots - x_d^2) x_d^2 \geq 0 \}, \]

which serves as an illustration of Corollary 2.4. By Corollary 2.4, if \( A = (p_1, \ldots, p_m)_{\geq 0} \) for \( p_1, \ldots, p_m \in \mathbb{R}[x] \), some of these polynomials are divisible by \( x_d \), and furthermore, if \( p_i \) is divisible by \( x_d \), the multiplicity of the factor \( x_d \) of \( p_i \) is even.

Figure 7 depicts the semi-algebraic set

\[ A := \{ x \in \mathbb{R}^2 : (1 - x_1^2 - \cdots - x_d^2) x_d^2 > 0 \} \]

illustrating Corollary 2.5. By Corollary 2.5, if \( A = (p_1, \ldots, p_m)_{> 0} \) for some polynomials \( p_1, \ldots, p_m \in \mathbb{R}[x] \), then \( x_d \) is a factor of at least one \( p_i \) and \( f \) cannot be a factor of \( p_i \) of odd multiplicity. We notice that Corollary 2.5 is in a certain sense an analogue of Corollary 2.4 for elementary open semi-algebraic sets (since the conclusions of both corollaries are the same).

Finally, we present examples of semi-algebraic sets for which we can verify that they are not elementary semi-algebraic (see also similar examples given in [1, p. 24]). We define the closed semi-algebraic set

\[ A := \{ x \in \mathbb{R}^d : x_d = 0 \text{ or } (x_1 - 3)^2 + x_2^2 + \cdots + x_d^2 \leq 1 \text{ or } (x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \text{ and } x_d \geq 0) \}, \]

see Figure 8. We can show that \( A \) is not elementary closed. In fact, let us assume the contrary, that is \( A = (p_1, \ldots, p_m)_{\geq 0} \) for some polynomials \( p_1, \ldots, p_m \in \mathbb{R}[x] \). Then, by Theorem 2.2 (III) applied for \( a = o \) and \( 0 < \varepsilon < 1 \), we get that \( x_d \) is a factor of odd multiplicity of \( p_i \) for some \( i \in \{1, \ldots, m\} \). Since (2.7) is fulfilled for \( f = x_d \), we can apply Corollary 2.4 obtaining that \( x_d \) is a factor of even multiplicity of \( p_i \), a contradiction. Now we introduce the open semi-algebraic set

\[ A := \{ x \in \mathbb{R}^d : x_1^2 + x_2^2 + \cdots + x_d^2 < 1 \text{ and } x_d > 0 \text{ or } (x_1 - 3)^2 + x_2^2 + \cdots + x_d^2 < 1 \text{ and } x_d \neq 0 \}, \]
see Figure 9. By Theorem 2.2 (IV) and Corollary 2.5 (applied for $f(x) = x_d$) $A$ is not elementary open.

Figure 8. A closed semi-algebraic set which is not elementary closed

Figure 9. An open semi-algebraic set which is not elementary open

References


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