Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 49 (2008), No. 1, 269-275.

The Special Cuts of the 600-cell

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Abstract. A polytope is called regular-faced if each of its facets is a regular polytope. The 4-dimensional regular-faced polytopes were determined by G. Blind and R. Blind [2], [5], [6]. Regarding this classification, the class of such polytopes not completely known is the one which consists of polytopes obtained by removing some set of non-adjacent vertices (an independent set) of the 600-cell. These independent sets are enumerated up to isomorphism, and we show that the number of polytopes in this last class is 314 248 344.

MSC 2000: 52B11, 52B15
Keywords: Grand antiprism, regular-faced polytope, regular polytope, semiregular polytope, 600-cell, (snub) 24-cell, symmetry group

1. Introduction

A *d-dimensional polytope* is the convex hull of a finite number of vertices in $\mathbb{R}^d$. A d-dimensional polytope is called *regular* if its isometry group is transitive on its flags. The regular polytopes are (see, for example, [8]) the regular n-gon, d-dimensional simplex α_d, hypercube γ_d, cross-polytope β_d, the 3-dimensional Icosahedron *Ico*, Dodecahedron *Dod*, the 4-dimensional 600-cell, 120-cell and 24-cell.

A polytope is called *regular-faced* if its facets (i.e., $(d-1)$-dimensional faces) are regular polytopes. If, in addition, its symmetry group is vertex-transitive then it is called *semiregular*. Several authors have considered this geometric generalization of the regular polytopes. An overview of this topic has been given by Martini [13], [14]. The 3-dimensional regular-faced polytopes have been determined by Johnson [11] and Zalgaller [19]; see [1], [10], [17], [18] for some beautiful presentations. The three papers [5], [6], [2] give a complete enumeration for the cases with dimension $d \geq 4$. G. Blind and R. Blind [4] characterized all types of semiregular polytopes.

Given a d-dimensional regular polytope P, $Pyr(P)$ denotes, if it exists, the regular faced $(d+1)$-dimensional polytope obtained by taking the convex hull of P and a special vertex v. The *bipyramid* $BPyr(P)$ denotes a $(d+1)$-dimensional polytope defined as the convex hull of P and two vertices, v_1 and v_2, on each side of P. The list of regular-faced d-polytopes for $d \geq 4$ is:

1. the regular d-polytopes,
2. two infinite families of d-polytopes ($Pyr(\beta_{d-1})$ and $BPyr(\alpha_{d-1})$),
3. the semiregular polytopes n_{21} with $n \in \{0,1,2,3,4\}$ of dimension $n+4$ and the semiregular 4-dimensional octicosahedric polytope,
4. three 4-polytopes ($Pyr(Ico)$, $BPyr(Ico)$ and the union of 0_{21}+$Pyr(\beta_3)$, where β_3 is a facet of 0_{21}), and
5. any special cut 4-polytope, arising from the 600-cell by the following procedure: if C is a subset of the 120 vertices of the 600-cell, such that any two vertices in C are not adjacent, then the special cut 600_C is the convex hull of all vertices of the 600-cell, except those in C.

This paper presents the enumeration of all such special cuts (see Table 1 and [9] for the results). The enumeration of special cuts with 2, 23 and 24 vertices is done in [3]. The ones with 3, 4, 5, 6, 21, and 22 vertices are enumerated by Kirrmann [12]. Also, Martini [13] enumerated the number of special cuts with n vertices for $n \leq 6$.

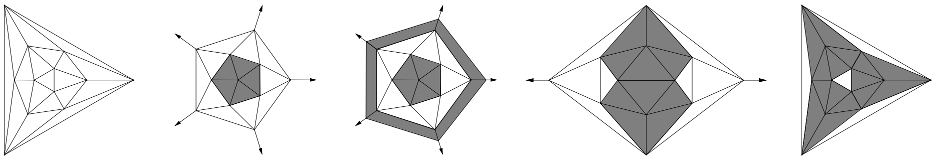
2. Geometry of special cuts

The 600-cell has 120 vertices, and its symmetry group is the Coxeter group H_4 with 14 400 elements. A subset C of the vertex set of the 600-cell is called *independent* if any two vertices in C are not adjacent. Given an independent subset C of the vertex-set of 600-cell, denote by 600_C the polytope obtained by taking the convex hull of the remaining vertices. Two polytopes 600_C and $600_{C'}$ are isomorphic if and only if C and C' are equivalent under H_4.

If C is reduced to a vertex v, then the 20 3-dimensional simplex facets containing v are transformed into an icosahedral facet of $600_{\{v\}}$, which we denote by Ico_v. It is easy to see that if one takes two vertices v and v' of the 600-cell, then the set of simplices containing v and v' are disjoint if and only if v and v' are not adjacent. Therefore, if C is an independent set of the vertices of the 600-cell then 600_C is regular-faced and is called a *special cut*. The name special cut comes from the fact that 600_C can be obtained from the 600-cell by cutting it with the hyperplanes corresponding to the facet defined by the icosahedra Ico_v for $v \in C$.

	1	2	3	4	5	6	8	9	10	12	16	18	20
1													
2							1			2			3
3	1	21		6		3	1		1	2			3
4	187	184	2	40		7	6			3			2
5	3721	938	4	79		21	3		1	7			1
6	41551	3924	17	212		34	18		6	8			
7	321809	12093	53	322		63	4		19	12			4
8	1792727	32714	102	672	1	102	40		28	17	3		
9	7284325	70006	170	815		137	6		14	19		1	2
10	21539704	129924	282	1349	2	190	43		4	16			3
11	45979736	194232	420	1346		251	6		11	15			3
12	69895468	247136	505	1781		236	57	1	37	21	4	1	12
13	74365276	252040	527	1457		266	6		58	20			7
14	54266201	213377	553	1545		255	43		26	31			9
15	26605433	142212	478	1041	2	181	4	1	5	19		1	4
16	8612476	76249	316	837		165	39		5	14	4		
17	1824397	31465	216	461		116	4		16	6			3
18	252764	10001	123	273		45	20		25	10		1	
19	22673	2360	49	120		39	3		12	8			1
20	1202	388	18	40		17	5		1	7			
21	22	37	6	12		5	1					1	
22						5	1						
23													
24													
	24	30	32	36	40	48	72	100	120	144	192	240	576
1									1				
2												1	
3				1									
4	3		1		1								
5								1					
6	2			1		1	1						
7	1												
8	6					2					1		
9	2			1									
10	8				1			1					
11													
12	5				1	2			1	1			
13									2				
14	7				1				1				
15	2	1											
16	4					1					1		
17	1												
18	4			1		1							
19													
20	2		1			1						1	
21	2												
22	2					1							
23	1												
24													1

Table 1. The number of special cuts between 1 and 24 vertices with the orders of their symmetry groups



Case I Case II Case III Case IV Case V

Figure 1. The local possibilities for vertices

	1	2	3	4	5	6	8	10	12	16
10										
11		1								
12	18	9		4			1			1
13	1555	146		23						
14	39597	980		52		4	4			
15	221823	2997	9	64	2	4		3	1	
16	341592	4573	10	113		16	7		11	1
17	192266	4081	9	59		7				
18	49741	2251	19	54		26	8		2	
19	6771	838	7	39		7		6		
20	598	199	6	14		12	2	1	5	
21	17	20	2	11						
22						3				
24										

	18	20	24	30	40	48	100	144	240	576
10							1			
11										
12			1					1		
13										
14		2			1					
15		1		1						
16						1				
17										
18	1		1							
19										
20			1			1			1	
21										
22			2							
24										1

Table 2. The number of maximal special cuts between 10 and 24 vertices (the cut size is in the first column) and their symmetry group orders (the column headers)

$\|C\|$	$\|\operatorname{Aut}(600_C)\|$	maximal	conn.	vertex orbits
24	576	yes	no	$(96, V, C_{3v})$
20	240	yes	no	$(40, V, C_{3v})$, $(60, IV, C_{2v})$
16	192	no	yes	$(96, IV, C_s)$, $(8, I, T_h)$
8	192	no	yes	$(96, II, C_s)$, $(16, I, T)$
12	144	yes	yes	$(36, IV, C_{2v})$, $(72, II, C_s)$
10	100	yes	yes	$(100, II, C_1)$, $(10, III, D_5)$

Table 3. Some highly symmetric special cuts

A vertex v of a special cut 600_C can be contained in at most 3 icosahedra $(Ico_w)_{w \in C}$. The possible ways of having a vertex of 600_C contained in an icosahedron Ico_w are listed in Figure 1. It is easy to see that an independent set C has at most 24 vertices. Table 1 gives the number of special cuts of each size and group order (the sizes are listed in the first column and the group orders are the column headers). A special cut 600_C is called *maximal* if we cannot add any vertices to C and still have a special cut; a list of these is given in Table 2, again with the information referring to symmetry groups.

Table 3 provides some information about highly symmetric special cuts. The column "conn." refers to the connectivity of the graph defined by the simplices of 600_C, with two simplices adjacent if they share a 2-dimensional face. In the column "vertex orbits", the sizes of the vertex orbits, their types according to Figure 1 and the nature of the vertex stabilizer according to its Schoenflies symbol are listed. The 143 cases with at least 20 symmetries are available from [9].

- The snub 24-cell (also called *tetricosahedric polytope*) is the semiregular polytope obtained as 600_C with $|C| = 24$. Its symmetry group has order 576 and its facets are 24 icosahedra and 120 3-dimensional simplices in two orbits O_1, O_2 with $|O_1| = 24$ and $|O_2| = 96$. The simplices in O_1 are adjacent only to simplices in O_2. The 24 vertices of C form a 24-cell, hence the name *snub* 24-*cell*. Coxeter [8] provides further details.
- The vertex set of the 24-cell can be split into three cross-polytopes β_4. Selecting one or two of these cross-polytopes gives two special cuts with 8 and 16 vertices and 192 symmetries.
- It is easy to see that the minimum size of a maximal special cut is at least 10. One of size 10 can be constructed as follows (indicating that the minimum size of a maximal cut is 10). The vertex set of the 600-cell is partitioned into two cycles of 10 vertices each and a set containing the 100 remaining vertices. The convex hull of the 100 remaining vertices is called a *Grand antiprism* (discovered by Conway [7]). Taking a maximum independent set of each of these cycles (a total of 10 vertices since five are selected from each cycle) gives the unique (up to isomorphism) maximal special cut of order 10.

3. Enumeration methods

The *skeleton* of a polytope P is the graph formed by its vertices and edges. Enumerating the special cuts of the 600-cell is the same as enumerating the independent sets of its skeleton.

In order to ensure correctness, the independent sets were enumerated by two entirely different methods and the results were checked to ensure that they agreed. The first method used to enumerate the independent sets was a parent-child search (see [15]). The 120 vertices of the 600-cell were numbered and then the search considered only the independent sets which were lexicographically minimum in their orbit. The method is then the following: given a lexicographically minimum independent set S, we consider all ways to add a vertex v such that $v > \max(S)$

and $S \cup \{v\}$ is still a lexicographically minimum independent set. Given a lexicographically minimum independent set $S = \{v_1, v_2, \dots, v_k\}$ with $v_i < v_{i+1}$, this method provides a canonical path to obtain S; first one obtains $\{v_1\}$, then $\{v_1, v_2\}$, until one gets S.

The second method is explained in [16]. The algorithm uses a novel algorithmic trick combined with appropriate data structures to decrease the running time of the search. One advantageous feature of this algorithm is that the symmetries of the independent sets generated are available with no additional computation required. This method proved much faster by a factor of 1000; the relationship between the performance difference which can be attributed to the algorithm versus the quality of the programming has not been determined.

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Received August 25, 2007