Point-reflections in Metric Plane

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Abstract. We axiomatize the class of groups generated by the point-reflections of a metric plane with a non-Euclidean metric, the structure of which turns out to be very rich compared to the Euclidean metric case, and state an open problem.

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1. Introduction

There is a very large literature on characterizations of groups of motions in terms of line-reflections or hyperplane-reflections (see [1]), but relatively little about groups generated by point-reflections. This subject has received some attention much later, in [4], [5], [6] (and, in a different setting, with an added differential structure, in e. g. [2] or [3]). The purpose of this paper is to determine the theories of point-reflections that one obtains from the groups of isometries of Bachmann’s metric planes. If the metric plane is elliptic, i. e. if there are three line-reflections whose product is the identity, then the point-reflections coincide with the line-reflections, so that the axiom system of the group generated by point-reflections is identical to the one expressed in terms of line-reflections. The interesting case is thus that of non-elliptic metric planes.

2. Non-elliptic metric planes

2.1. Axiom system in terms of line-reflections

We shall first present non-elliptic metric planes as they appear in [1]. Our language will be a one-sorted one, with variables to be interpreted as ‘rigid motions’,
containing a unary predicate symbol $G$, with $G(x)$ to be interpreted as ‘$x$ is a line-reflection’, a constant symbol 1, to be interpreted as ‘the identity’, and a binary operation $\circ$, with $\circ(a,b)$, which we shall write as $a \circ b$, to be interpreted as ‘the composition of $a$ with $b$’. To improve the readability of the axioms, we introduce the following abbreviations:

$$a^2 :\iff a \circ a,$$
$$\iota(g) :\iff g \neq 1 \land g^2 = 1,$$
$$a|b :\iff G(a) \land G(b) \land \iota(a \circ b),$$
$$J(abc) :\iff \iota((a \circ b) \circ c),$$
$$pq|a :\iff p|q \land G(a) \land J(pqa).$$

The axioms are (we omit universal quantifiers whenever the axioms are universal sentences):

**B 1.** $(a \circ b) \circ c = a \circ (b \circ c)$
**B 2.** $(\forall a)(\exists b) b \circ a = 1$
**B 3.** $1 \circ a = a$
**B 4.** $G(a) \to \iota(a)$
**B 5.** $G(a) \land G(b) \to G(a \circ (b \circ a))$
**B 6.** $(\forall abcd)(\exists g) a|b \land c|d \to G(g) \land J(abg) \land J(cdg)$
**B 7.** $ab|g \land cd|g \land ab|h \land cd|h \to (g = h \lor a \circ b = c \circ d)$
**B 8.** $\bigwedge_{i=1}^{3} pq|a_i \to G(a_1 \circ (a_2 \circ a_3))$
**B 9.** $\bigwedge_{i=1}^{3} g|a_i \to G(a_1 \circ (a_2 \circ a_3))$
**B 10.** $(\exists ghj) g|h \land G(j) \land \neg j|g \land \neg j|h \land \neg J(jgh)$
**B 11.** $(\forall x)(\exists ghj) G(g) \land G(h) \land G(j) \land (x = g \circ h \lor x = g \circ (h \circ j))$
**B 12.** $G(a) \land G(b) \land G(c) \to a \circ (b \circ c) \neq 1$

Since $a \circ b$ with $a|b$ represents a point-reflection, we may think of an unordered pair $(a,b)$ with $a|b$ as a point, an element $a$ with $G(a)$ as a line, two lines $a$ and $b$ for which $a|b$ as a pair of perpendicular lines, and say that a point $(p,q)$ is incident with the line $a$ if $pq|a$. With these figures of speech in mind, the above axioms make the following statements: B1, B2, and B3 are the group axioms for the operation $\circ$; B4 states that line-reflections are involutions; B5 states the invariance of the set of line-reflections, B6 states that any two points can be joined by a line, which is unique according to B7 (we shall denote the line joining the points $(a, b)$ and $(c, d)$ by $\langle (a, b), (c, d) \rangle$); B8 and B9 state that the composition of three reflections in lines that have a common point or a common perpendicular is a line-reflection; B10 states that there are three lines $g, h, j$ such that $g$ are $h$ are
perpendicular, but \( j \) is perpendicular to neither \( g \) nor \( h \), nor does it go through the intersection point of \( g \) and \( h \); B11 states that every motion is the composition of two or three line-reflections, and B12 states that the composition of three line-reflections is never the identity. The function of the last axiom, B12, is to exclude elliptic geometries, and thus to ensure that the perpendicular from a point not on a line to that line is unique. The theory of non-elliptic metric planes, axiomatized by \{B1-B12\} will be denoted by \( \mathcal{B} \).

2.2. Axiom system in terms of ternary geometric operations

The same class of models can also be axiomatized in the following manner: the language \( \mathcal{L} \) contains only one sort of individual variables, to be interpreted as ‘points’, three individual constants \( a_0, a_1, a_2 \), to be interpreted as three non-collinear points, and two operation symbols, \( F \) and \( \pi \). \( F(abc) \) is the foot of the perpendicular from \( c \) to the line \( ab \), if \( a \neq b \), and \( a \) itself if \( a = b \), and \( \pi(abc) \) is the fourth reflection point whenever \( a, b, c \) are collinear points with \( a \neq b \) and \( b \neq c \), and arbitrary otherwise. By ‘fourth reflection point’ we mean the following: if we designate by \( \sigma_x \) the mapping defined by \( \sigma_x(y) = \sigma(xy) \), i. e. the reflection of \( y \) in the point \( x \), then, if \( a, b, c \) are three collinear points, by [1, §3.9, Satz 24b], the composition (product) \( \sigma, \sigma_b, \sigma_a \), is the reflection in a point, which lies on the same line as \( a, b, c \). That point is designated by \( \pi(abc) \).

In order to formulate the axioms in a more readable way, we shall use the following abbreviations:

\[
\begin{align*}
\sigma(ab) & := \pi(aba), \\
R(abc) & := \sigma(F(abc)c), \\
L(abc) & := F(abc) = c \lor a = b,
\end{align*}
\]

where \( \sigma \) has the same meaning as above, \( R(abc) \) stands for the reflection of \( c \) in \( ab \) (a line if \( a \neq b \), the point \( a \) if \( a = b \)), and \( L(abc) \) stands for ‘the points \( a, b, c \) are collinear (but not necessarily distinct)’. The axiom system consists of the following axioms:

C 1. \( F(aab) = a \)

C 2. \( \sigma(aa) = a \)

C 3. \( \sigma(a\sigma(ab)) = b \)

C 4. \( L(aba) \)

C 5. \( L(abc) \rightarrow L(cba) \land L(bac) \)

C 6. \( L(ab\sigma(ab)) \)

C 7. \( L(abF(abc)) \)

C 8. \( \sigma(ax) = \sigma(bx) \rightarrow a = b \)

C 9. \( a \neq b \land F(abx) = F(aby) \rightarrow L(xyF(abx)) \)
The axioms make the following statements: C1 defines the value of \( F(abc) \) when \( a = b \) — it is an axiom with no geometric function (we could have opted to leave it undefined, but that would have lengthened the statements of the axioms C16 and C18); C2: the point \( a \) is a fixed point of the reflection \( \sigma_a \), C3: reflections in points are involutory transformations (or the identity); C8: reflections of a point in two different points do not coincide; C4: \( a \) lies on the line determined by \( a \) and \( b \); C5: collinearity of three points is a symmetric relation; C6: the reflection of \( b \) in \( a \) is collinear with \( a \) and \( b \); C7: for \( a \neq b \), the foot of the perpendicular from \( c \) to the line \( ab \) lies on that line; C9 states the uniqueness of the perpendicular to the line \( ab \) in the point \( F(abx) \); C10: the foot of the perpendicular from \( x \) to the line \( ab \) does not depend on the particular choice of points \( a \) and \( b \) that determine the line \( ab \); C11: if \( x \) is a point outside of the line \( ab \), and \( y \) is a point on the perpendicular from \( x \) to \( ab \), then the feet of the perpendiculars of \( x \) and \( y \) to the line \( ab \) coincide; C12 states that perpendicularity is a symmetric relation (if \( ca \) is perpendicular to \( ab \), then \( ba \) is perpendicular to \( ac \)); C13: if \( xy \) is perpendicular to \( xa \), then so are \( \sigma_a(y)\sigma_a(x) \) and \( \sigma_a(x)\sigma_a(y) \); C14: reflections in points preserve midpoints; C15: reflections in lines preserve the orthogonality relation; C16 and C17 together state the three reflections theorem for lines having a point in common; C18 is the three reflections theorem for lines having a common perpendicular; C19: \( a_0, a_1, a_2 \) are three non-collinear points. With \( \Sigma = \{C1-C19\} \), we proved in [8] the following.

**Theorem 1.** \( \Sigma \) is an axiom system for non-elliptic metric planes. In every model of \( \Sigma \), the operations \( F \) and \( \pi \) have the intended interpretations.
3. Axiom system for metric planes with non-Euclidean metric in terms of point-reflections

We now turn to yet another axiomatization of non-elliptic metric planes with non-Euclidean metric (i.e. in which there exists no rectangle), in terms of motions which are products of point-reflections, the individual constant 1, and the binary operation $\circ$, with $a \circ b$ standing for the composition of the motions $a$ and $b$. In case $a$ is a point-reflection, we will refer to $a$ as a ‘point’ as well. To improve the readability of the axioms we introduce the following abbreviations:

$$P(a) \iff a \neq 1 \land a \circ a = 1$$

$$P(a_1, \ldots, a_n) \iff \bigwedge_{i=1}^n P(a_i)$$

$$L(abc) \iff (a \circ b) \circ c = (c \circ b) \circ a$$

$$\sigma(ab) := (a \circ b) \circ a$$

$$\varphi(eabcd) \iff (\neg L(abc) \land L(abd) \land L(cde) \land \sigma(e\sigma(bc)) = \sigma(\sigma(dbc)))$$

$$\pi(abc) := c \circ (b \circ a)$$

$$\varrho(eabcd) \iff \varphi(eabcu) \land d = \sigma(uc).$$

Here $P(a)$ stands for ‘$a$ is a point-reflection’, given that, in the group generated by point-reflections the only involutory elements are the point-reflections themselves. The subsequent abbreviations will be used only when all the variables that appear in them are point-reflections. $L(abc)$ stands for ‘$a, b, c$ are collinear’; $\sigma(ab)$ is the point obtained by reflecting $b$ in $a$; $\varphi(eabcd)$ holds, in case $a \neq b$, if $d$ is the foot of the perpendicular from $c$ to $ab$ (as shown, for $a, b, c$ not collinear in [9, Prop. 1] ($e$ is a point needed in this construction, see Figure 1)) and, in case $a = b$, if $d = a$; and $\varrho(eabcd)$ stands for ‘$d$ is the reflection of $c$ in the line $ab$ if $a \neq b$ or in point $a$ if $a = b$.

![Figure 1. The definition of perpendicularity in terms of $\sigma$](image-url)
The axioms for this axiom system are: B1, B2, B3, axiom P1, which ensures that, whenever \(a, b, c\) are collinear points, \(\pi(abc)\) is a point as well, the axioms P2 and P3 stating the existence and uniqueness of the foot of the perpendicular from point \(c\) to line \(ab\), whenever \(c\) does not lie on the line \(ab\), as well as P4–P14, which are slightly changed variants of C8–C19.

**P 1.** \(P(a, b, c) \rightarrow a \circ b \neq c\)

**P 2.** \((\forall abc)(\exists de) \ P(a, b, c) \land \neg L(abc) \rightarrow P(e, d) \land \varphi(eabcd)\)

**P 3.** \(P(a, b, c, d, e, d', e') \land \neg L(abc) \land \varphi(eabcd) \land \varphi(e'abcd') \rightarrow d = d'\)

**P 4.** \(P(a, b, x) \land \sigma(ax) = \sigma(bx) \rightarrow a = b\)

**P 5.** \(P(a, b, d, e, f, x, y) \land a \neq b \land \varphi(eabcd) \land \varphi(fabyd) \rightarrow L(xyd)\)

**P 6.** \(P(a, b, c, d, e, f, u, v, x) \land a \neq b \land c \neq d \land L(abc) \land L(abd) \land \varphi(eabxu) \land \varphi(fcdxv) \rightarrow u = v\)

**P 7.** \(P(a, b, d, e, f, u, v, x) \land \neg L(abx) \land \varphi(eabxu) \land L(xyu) \land \varphi(fabyv) \rightarrow u = v\)

**P 8.** \(P(a, b, c, x, y, u) \land \neg L(ab) \land \varphi(xabca) \land \varphi(yacbu) \rightarrow u = a\)

**P 9.** \(P(a, e, f, u, x, y) \land \neg L(axy) \land \varphi(eaxyx) \land \varphi(fa\sigma(ax)\sigma(ay)u) \rightarrow u = \sigma(ax)\)

**P 10.** \(P(a, b, c, d, e, f, m, n, p, q, u, v, x) \land \neg L(abc) \land \varphi(eabca) \land u \neq v \land \varphi(nuva\sigma) \land \varphi(pwibbl) \land \varphi(qwvcc) \land \varphi(ma'bc'x) \rightarrow x = a'\)

**P 11.** \(P(o, a, b, c, m, n, p, q, x, y, z, u, v) \land \neg L(oab) \land \neg L(abc) \land \varphi(moax) \land \varphi(nobyz) \land \varphi(poczu) \land \varphi(qxuvn) \rightarrow \sigma(vx) = u\)

**P 12.** \(P(o, a, b, c, m, n, p, m', n', p', x, y, z, u, x', y', z', u', t, t') \land \neg L(oab) \land \neg L(abc) \land \varphi(moax) \land \varphi(nobyz) \land \varphi(poczu) \land \varphi(qm'obax'y') \land \varphi(qn'oby'z') \land \varphi(qp'ocz'u') \land \sigma(txz) = u \land \sigma(t'x') = u' \rightarrow L(ott')\)

**P 13.** \(P(a, b, c, d, e, f, m, n, p, t, u, v, w, x, y, z, w, o, g) \land L(abc) \land \neg L(abc) \land \neg L(bab) \land \neg L(bc') \land \varphi(mab'a) \land \varphi(nbab'b) \land \varphi(pbc'c) \land \varphi(toa'xy) \land \varphi(ub'b'yz) \land \varphi(vcc'zw) \land \varphi(qxwp(abc)o) \land \varphi(r\pi(abc)g) \rightarrow \sigma(ox) = w \land g = \pi(abc)\)

**P 14.** \((\exists abc) P(a, b, c) \land \neg L(ab)\)

Finally, we need an axiom that lets us know that every rigid motion is a product of point-reflections. We do not know whether every element of the subgroup generated by point-reflections of the motion group \(G\) of a metric plane can be written as a product of at most a certain fixed number of point-reflections (whereas we do know that every element of \(G\) can be written as a product of at most three line-reflections). Unless we establish that there is an upper bound on the number of point-reflections needed (or that such an upper bound does not exist), we cannot determine the first-order theory of the group generated by point-reflections (as it may be either (1) the theory axiomatized by the axioms \{B1–B3,P1–P14\} in case there are, for every natural number \(k\), products of point-reflections that cannot be...
written as a product of at most $k$ point-reflections, or (2) the theory axiomatized by those axioms and an axiom stating that every rigid motion is a product of at most $k$ point-reflections, should $k$ be the least number with this property). \(^1\) What we can do is to determine the $L_{\omega_1\omega}$-theory of point-reflections, i.e. to state that every rigid motion is a product of an unspecified number of point-reflections as an infinite disjunction of first-order formulas. The axiom thus is

\[ P_{15}. \forall a \bigvee_{n=1}^{\infty} (\exists p_1 \ldots p_n) \, P(p_1, \ldots, p_n) \land a = p_1 \circ (\ldots \circ p_n) \ldots \]

Let \( \Pi = \{B1–B3,P1–P15\} \). Given that we can define \( F \) in terms of \( \varphi \), given that \( P2 \) and \( P3 \) ensure the existence and uniqueness of the value of \( F(abc) \) for \( c \) not on \( ab \), that \( C1–C7, C14 \) follow from our definitions of \( L, F \) and \( \sigma \), and that the axioms \( C8–C19 \) follow from their translations \( P4–P14 \) into our language, we deduce that in every model of \( \Pi \) the individual variables \( x \) for which \( P(x) \) holds can be interpreted as points, the defined notions \( F \) and \( \pi \) have the desired geometric interpretation, and thus, that the resulting structure is a non-elliptic metric plane with non-Euclidean metric (the metric cannot be Euclidean, given that we know that \( \sigma \) has the desired interpretation). Given \( P15 \), the map \( \tilde{\alpha} \) defines an isomorphism of \( \mathcal{M} \) onto \( \mathcal{G} \). We have shown that:

**Theorem 2.** \( \Pi \) is an $L_{\omega_1\omega}$-axiom system for the group generated by the point-reflections of non-elliptic metric planes with non-Euclidean metric.

We now turn to metric planes whose metric is Euclidean, also called ‘metric-Euclidean planes’.

4. Point-reflections in metric-Euclidean planes

If axiom \( R \) (“There exists a rectangle” (see [1, \( \S 6,7 \)]) holds in a metric plane, i.e.

\[ (\exists abcd) \, a \neq b \land c \neq d \land a|c \land a|d \land b|c \land b|d, \]

is added to \( \mathcal{B} \), then the group generated by point-reflections can be described very simply by means of \( B1–B3 \) and

\(^1\)It was shown in [11] that, under additional assumptions, it is possible to write every product of point-reflections as a product of at most 4 point-reflections, but no such reduction is known in the general non-Euclidean metric case.
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E 1. \((\exists ab)\ P(a, b) \land a \neq b\)

E 2. \(P(a, b) \land a \circ b = b \circ a \rightarrow a = b\)

E 3. \((\forall x)(\exists ab)\ P(a, b) \land (x = a \lor x = a \circ b)\)

E 4. \(P(a, b, c) \rightarrow P(a \circ (b \circ c))\)

Let \(\mathfrak{M}\) be a model of B1–B3, E1–E4, let \(M\) be its universe, and let \(P = \{m \in M : P(m)\}\) be the set of points in \(M\). We define on \(P \times P\) an equivalence relation \(\sim\) by \((a, b) \sim (c, d)\) if and only if \(a \circ c = b \circ d\), and denote by \([a, b]\) the equivalence class of \((a, b)\). Let \(G := P \times P/\sim\). We define on \(G\) an addition operation + by \([a, b] + [c, d] := [a, b \circ c \circ d]\), which turns \(G\) into an Abelian group, as can be easily checked. We fix a point \(o\) in \(P\), and consider all elements in \(G\) written as \([o, x]\) (notice that \([a, b] = [o, b \circ (a \circ o)]\)). Writing \(x\) for \([o, x]\), we check that \(\sigma(ab) = 2a - b\) (i. e. that \([o, \sigma(ab)] + [o, b] = [o, a] + [o, a]\)). By E2 we know that \(G\) must satisfy \(2x = 0 \rightarrow x = 0\). Any metric-Euclidean plane can be embedded in a Gaussian plane associated with the pair of fields \((K, L)\), where \(K \subset L\), \([L : K] = 2\) (a generalization of the Gauss plane over \((\mathbb{C}, \mathbb{R})\), see [7]), the points being elements of \(L\), and thus the algebraic representation of point-reflections \(\tilde{x}\) is given by \(\tilde{x}(y) = 2x - y\) and \((\tilde{x} \cdot \tilde{y})(z) = 2(x - y) + z\). Given that the only operations involved in the description of point-reflections and their composition are + and −, every first-order sentence that is true in all metric-Euclidean planes must hold over arbitrary Abelian groups which satisfy \(2x = 0 \rightarrow x = 0\) as well.

**Theorem 3.** \{B1–B3, E1–E4\} is an axiom system for the group generated by the point-reflections of metric-Euclidean planes.

Related results have been proved in [10].

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**References**


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