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A Note on Rees Algebras and the MFMC Property

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Abstract. We study irreducible representations of Rees cones and characterize the max-flow min-cut property of clutters in terms of the normality of Rees algebras and the integrality of certain polyhedra. Then we present some applications to combinatorial optimization and commutative algebra. As a byproduct we obtain an effective method, based on the program *Normaliz* [4], to determine whether a given clutter satisfies the max-flow min-cut property. Let $\mathcal{C}$ be a clutter and let I be its edge ideal. We prove that $\mathcal{C}$ has the max-flow min-cut property if and only if I is normally torsion free, that is, $I^i = I^{(i)}$ for all $i \geq 1$, where $I^{(i)}$ is the i-th symbolic power of I.

1. Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let $I \subset R$ be a monomial ideal minimally generated by $x^{v_1}, \ldots, x^{v_q}$. As usual we will use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Consider the $n \times q$ matrix A with column vectors $v_1, \ldots, v_q$.

A *clutter* with vertex set X is a family of subsets of X, called edges, none of which is included in another. A basic example of clutter is a graph. If A has entries in $\{0, 1\}$, then A defines in a natural way a clutter $\mathcal{C}$ by taking $X = \{x_1, \ldots, x_n\}$ as vertex set and $E = \{S_1, \ldots, S_q\}$ as edge set, where S_i is the support of x^{v_i},

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i.e., the set of variables that occur in x^{v_i}. In this case we call I the *edge ideal* of the clutter $\mathcal{C}$ and write $I = I(\mathcal{C})$. Edge ideals are also called *facet ideals* [9]. This notion has been studied by Faridi [10] and Zheng [18]. The matrix A is often referred to as the incidence matrix of $\mathcal{C}$.

The *Rees algebra* of I is the R-subalgebra:

$$R[It] := R[\{x^{v_1}t, \ldots, x^{v_q}t\}] \subset R[t],$$

where t is a new variable. In our situation $R[It]$ is also a K-subalgebra of $K[x_1, \ldots, x_n, t]$.

The *Rees cone* of I is the rational polyhedral cone in $\mathbb{R}^{n+1}$, denoted by $\mathbb{R}_+\mathcal{A}'$, consisting of the non-negative linear combinations of the set

$$\mathcal{A}' := \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the i-th unit vector. Thus $\mathcal{A}'$ is the set of exponent vectors of the set of monomials $\{x_1, \ldots, x_n, x^{v_1}t, \ldots, x^{v_q}t\}$, that generate $R[It]$ as a K-algebra.

The first main result of this note (Theorem 3.2) shows that the irreducible representation of the Rees cone, as a finite intersection of closed half-spaces, can be expressed essentially in terms of the vertices of the *set covering polyhedron*:

$$Q(A) := \{x \in \mathbb{R}^n \mid x \geq 0,\ xA \geq \mathbf{1}\}.$$

Here $\mathbf{1} = (1, \ldots, 1)$. The second main result (Theorem 3.4) is an algebro-combinatorial description of the max-flow min-cut property of the clutter $\mathcal{C}$ in terms of a purely algebraic property (the normality of $R[It]$) and an integer programming property (the integrality of the rational polyhedron $Q(A)$). Some applications will be shown. For instance we give an effective method, based on the program *Normaliz* [4], to determine whether a given clutter satisfy the max-flow min-cut property (Remark 3.5). We prove that $\mathcal{C}$ has the max-flow min-cut property if and only if $I^i = I^{(i)}$ for $i \geq 1$, where $I^{(i)}$ is the i-th symbolic power of I (Corollary 3.14). There are other interesting links between algebraic properties of Rees algebras and combinatorial optimization problems of clutters [11].

Our main references for Rees algebras and combinatorial optimization are [3], [14] and [12] respectively.

2. Preliminaries

For convenience we quickly recall some basic results, terminology, and notation from polyhedral geometry.

A set $C \subset \mathbb{R}^n$ is a *polyhedral set* (resp. *cone*) if $C = \{x \mid Bx \leq b\}$ for some matrix B and some vector b (resp. $b = 0$). By the finite basis theorem [17, Theorem 4.1.1] a polyhedral cone $C \subsetneq \mathbb{R}^n$ has two representations:

Minkowski representation: $C = \mathbb{R}_+\mathcal{B}$ with $\mathcal{B} = \{\beta_1, \ldots, \beta_r\}$ a finite set, and

Implicit representation: $C = H^+_{c_1} \cap \cdots \cap H^+_{c_s}$ for some $c_1, \ldots, c_s \in \mathbb{R}^n \setminus \{0\}$, where $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{R}_+\mathcal{B}$ is the cone generated by $\mathcal{B}$ consisting of the set of linear combinations of $\mathcal{B}$ with coefficients in $\mathbb{R}_+$, H_{c_i} is the hyperplane of $\mathbb{R}^n$ through the origin with normal vector c_i, and $H_{c_i}^+ = \{x|\, \langle x, c_i\rangle \geq 0\}$ is the positive *closed half-space* bounded by H_{c_i}. Here $\langle\ ,\ \rangle$ denotes the usual inner product. These two representations satisfy the *duality theorem for cones*:

$$H_{\beta_1}^+ \cap \cdots \cap H_{\beta_r}^+ = \mathbb{R}_+c_1 + \cdots + \mathbb{R}_+c_s, \tag{1}$$

see [13, Corollary 7.1a] and its proof. The *dual cone* of C is defined as

$$C^* := \bigcap_{c\in C} H_c^+ = \bigcap_{a\in\mathcal{B}} H_a^+.$$

By the duality theorem $C^{**} = C$. An implicit representation of C is called *irreducible* if none of the closed half-spaces $H_{c_1}^+, \ldots, H_{c_s}^+$ can be omitted from the intersection. Note that the left hand side of equation (1) is an irreducible representation of C^* if and only if no proper subset of $\mathcal{B}$ generates C.

3. Rees cones, normality and the MFMC property

To avoid repetitions, throughout the rest of this note we keep the notation and assumptions of Section 1.

Notice that the Rees cone $\mathbb{R}_+\mathcal{A}'$ has dimension $n+1$. A subset $F \subset \mathbb{R}^{n+1}$ is called a *facet* of $\mathbb{R}_+\mathcal{A}'$ if $F = \mathbb{R}_+\mathcal{A}' \cap H_a$ for some hyperplane H_a such that $\mathbb{R}_+\mathcal{A}' \subset H_a^+$ and $\dim(F) = n$. It is not hard to see that the set

$$F = \mathbb{R}_+\mathcal{A}' \cap H_{e_i} \quad (1 \leq i \leq n+1)$$

defines a facet of $\mathbb{R}_+\mathcal{A}'$ if and only if either $i = n+1$ or $1 \leq i \leq n$ and $\langle e_i, v_j\rangle = 0$ for some column v_j of A. Consider the index set

$$\mathcal{J} = \{1 \leq i \leq n|\ \langle e_i, v_j\rangle = 0 \text{ for some } j\} \cup \{n+1\}.$$

Using [17, Theorem 3.2.1] it is seen that the Rees cone has a unique irreducible representation

$$\mathbb{R}_+\mathcal{A}' = \left(\bigcap_{i\in\mathcal{J}} H_{e_i}^+\right) \bigcap \left(\bigcap_{i=1}^{r} H_{a_i}^+\right) \tag{2}$$

such that $0 \neq a_i \in \mathbb{Q}^{n+1}$ and $\langle a_i, e_{n+1}\rangle = -1$ for all i. A point x_0 is called a *vertex* or an *extreme point* of $Q(A)$ if $\{x_0\}$ is a proper face of $Q(A)$.

Lemma 3.1. *Let $a = (a_{i1}, \ldots, a_{iq})$ be the i-th row of the matrix A and define $k = \min\{a_{ij}|\, 1 \leq j \leq q\}$. If $a_{ij} > 0$ for all j, then e_i/k is a vertex of $Q(A)$.*

Proof. Set $x_0 = e_i/k$. Clearly $x_0 \in Q(A)$ and $\langle x_0, v_j\rangle = 1$ for some j. Since $\langle x_0, e_\ell\rangle = 0$ for $\ell \neq i$, the point x_0 is a basic feasible solution of $Q(A)$. Then by [1, Theorem 2.3] x_0 is a vertex of $Q(A)$. □

Theorem 3.2. *Let V be the vertex set of $Q(A)$. Then*

$$\mathbb{R}_+\mathcal{A}' = \left(\bigcap_{i\in\mathcal{J}} H^+_{e_i}\right)\bigcap\left(\bigcap_{\alpha\in V} H^+_{(\alpha,-1)}\right)$$

is the irreducible representation of the Rees cone of I.

Proof. Let $V = \{\alpha_1,\dots,\alpha_p\}$ be the set of vertices of $Q(A)$ and let

$$\mathcal{B} = \{e_i|\, i\in\mathcal{J}\}\cup\{(\alpha,-1)|\,\alpha\in V\}.$$

First we dualize equation (2) and use the duality theorem for cones to obtain

$$\begin{aligned}(\mathbb{R}_+\mathcal{A}')^* &= \{y\in\mathbb{R}^{n+1}|\,\langle y,x\rangle\geq 0,\,\forall\, x\in\mathbb{R}_+\mathcal{A}'\}\\ &= H^+_{e_1}\cap\cdots\cap H^+_{e_n}\cap H^+_{(v_1,1)}\cap\cdots\cap H^+_{(v_q,1)}\\ &= \sum_{i\in\mathcal{J}}\mathbb{R}_+e_i+\mathbb{R}_+a_1+\cdots+\mathbb{R}_+a_r. \qquad (3)\end{aligned}$$

Next we show the equality

$$(\mathbb{R}_+\mathcal{A}')^* = \mathbb{R}_+\mathcal{B}. \qquad (4)$$

The right hand side is clearly contained in the left hand side because a vector α belongs to $Q(A)$ if and only if $(\alpha,-1)$ is in $(\mathbb{R}_+\mathcal{A}')^*$. To prove the reverse containment observe that by equation (3) it suffices to show that $a_k\in\mathbb{R}_+\mathcal{B}$ for all k. Writing $a_k=(c_k,-1)$ and using $a_k\in(\mathbb{R}_+\mathcal{A}')^*$ gives $c_k\in Q(A)$. The set covering polyhedron can be written as

$$Q(A)=\mathbb{R}_+e_1+\cdots+\mathbb{R}_+e_n+\mathrm{conv}(V),$$

where conv(V) denotes the convex hull of V, this follows from the structure of polyhedra by noticing that the characteristic cone of $Q(A)$ is precisely $\mathbb{R}^n_+$ (see [13, Chapter 8]). Thus we can write

$$c_k=\lambda_1e_1+\cdots+\lambda_ne_n+\mu_1\alpha_1+\cdots+\mu_p\alpha_p,$$

where $\lambda_i\geq 0$, $\mu_j\geq 0$ for all i,j and $\mu_1+\cdots+\mu_p=1$. If $1\leq i\leq n$ and $i\notin\mathcal{J}$, then the i-th row of A has all its entries positive. Thus by Lemma 3.1 we get that e_i/k_i is a vertex of $Q(A)$ for some $k_i>0$. To avoid cumbersome notation we denote e_i and $(e_i,0)$ simply by e_i, from the context the meaning of e_i should be clear. Therefore from the equalities

$$\sum_{i\notin\mathcal{J}}\lambda_ie_i=\sum_{i\notin\mathcal{J}}\lambda_ik_i\left(\frac{e_i}{k_i}\right)=\sum_{i\notin\mathcal{J}}\lambda_ik_i\left(\frac{e_i}{k_i},-1\right)+\left(\sum_{i\notin\mathcal{J}}\lambda_ik_i\right)e_{n+1}$$

we conclude that $\sum_{i\notin\mathcal{J}}\lambda_ie_i$ is in $\mathbb{R}_+\mathcal{B}$. From the identities

$$\begin{aligned}a_k &= (c_k,-1)=\lambda_1e_1+\cdots+\lambda_ne_n+\mu_1(\alpha_1,-1)+\cdots+\mu_p(\alpha_p,-1)\\ &= \sum_{i\notin\mathcal{J}}\lambda_ie_i+\sum_{i\in\mathcal{J}\setminus\{n+1\}}\lambda_ie_i+\sum_{i=1}^{p}\mu_i(\alpha_i,-1)\end{aligned}$$

we obtain that $a_k \in \mathbb{R}_+\mathcal{B}$, as required. Taking duals in equation (4) we get

$$\mathbb{R}_+\mathcal{A}' = \bigcap_{a\in\mathcal{B}} H_a^+. \tag{5}$$

Thus, by the comments at the end of Section 2, the proof reduces to showing that $\beta \notin \mathbb{R}_+(\mathcal{B}\setminus\{\beta\})$ for all $\beta \in \mathcal{B}$. To prove this we will assume that $\beta \in \mathbb{R}_+(\mathcal{B}\setminus\{\beta\})$ for some $\beta \in \mathcal{B}$ and derive a contradiction.

Case (I): $\beta = (\alpha_j, -1)$. For simplicity assume $\beta = (\alpha_p, -1)$. We can write

$$(\alpha_p, -1) = \sum_{i\in\mathcal{J}} \lambda_i e_i + \sum_{j=1}^{p-1} \mu_j(\alpha_j, -1), \qquad (\lambda_i \geq 0; \mu_j \geq 0).$$

Consequently

$$\begin{aligned} \alpha_p &= \sum_{i\in\mathcal{J}\setminus\{n+1\}} \lambda_i e_i + \sum_{j=1}^{p-1} \mu_j \alpha_j & (6)\\ -1 &= \lambda_{n+1} - (\mu_1 + \cdots + \mu_{p-1}). & (7) \end{aligned}$$

To derive a contradiction we claim that $Q(A) = \mathbb{R}^n_+ + \mathrm{conv}(\alpha_1, \ldots, \alpha_{p-1})$, which is impossible because by [2, Theorem 7.2] the vertices of $Q(A)$ would be contained in $\{\alpha_1, \ldots, \alpha_{p-1}\}$. To prove the claim note that the right hand side is clearly contained in the left hand side. For the other inclusion take $\gamma \in Q(A)$ and write

$$\begin{aligned} \gamma &= \sum_{i=1}^{n} b_i e_i + \sum_{i=1}^{p} c_i \alpha_i \qquad (b_i, c_i \geq 0; \sum_{i=1}^{p} c_i = 1)\\ &\overset{(6)}{=} \delta + \sum_{i=1}^{p-1} (c_i + c_p\mu_i)\alpha_i \qquad (\delta \in \mathbb{R}^n_+). \end{aligned}$$

Therefore using the inequality

$$\sum_{i=1}^{p-1}(c_i + c_p\mu_i) = \sum_{i=1}^{p-1} c_i + c_p\left(\sum_{i=1}^{p-1}\mu_i\right) \overset{(7)}{=} (1-c_p) + c_p(1+\lambda_{n+1}) \geq 1$$

we get $\gamma \in \mathbb{R}^n_+ + \mathrm{conv}(\alpha_1, \ldots, \alpha_{p-1})$. This proves the claim.

Case (II): $\beta = e_k$ for some $k \in \mathcal{J}$. First we consider the subcase $k \leq n$. The subcase $k = n+1$ can be treated similarly. We can write

$$e_k = \sum_{i\in\mathcal{J}\setminus\{k\}} \lambda_i e_i + \sum_{i=1}^{p} \mu_i(\alpha_i, -1), \qquad (\lambda_i \geq 0; \mu_i \geq 0).$$

From this equality we get $e_k = \sum_{i=1}^{p} \mu_i\alpha_i$. Hence $e_k A \geq (\sum_{i=1}^{p}\mu_i)\mathbf{1} > 0$, a contradiction because $k \in \mathcal{J}$ and $\langle e_k, v_j\rangle = 0$ for some j. □

Clutters with the max-flow min-cut property. For the rest of this section we assume that A is a $\{0,1\}$-matrix, i.e., I is a square-free monomial ideal.

Definition 3.3. *The clutter* $\mathcal{C}$ *has the max-flow min-cut* (MFMC) *property if both sides of the LP-duality equation*

$$\min\{\langle \alpha, x\rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \mid y \geq 0; Ay \leq \alpha\} \tag{8}$$

have integral optimum solutions x *and* y *for each non-negative integral vector* α.

It follows from [13, pp. 311–312] that $\mathcal{C}$ has the MFMC property if and only if the maximum in equation (8) has an optimal integral solution y for each non-negative integral vector α. In optimization terms [12] this means that the clutter $\mathcal{C}$ has the MFMC property if and only if the system of linear inequalities $x \geq 0;\ xA \geq \mathbf{1}$ that define $Q(A)$ is *totally dual integral* (TDI). The polyhedron $Q(A)$ is said to be *integral* if $Q(A)$ has only integral vertices.

Next we recall two descriptions of the integral closure of $R[It]$ that yield some formulations of the normality property of $R[It]$. Let $\mathbb{N}\mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integer coefficients. The Rees algebra of the ideal I can be written as

$$\begin{aligned} R[It] &= K[\{x^a t^b \mid (a,b) \in \mathbb{N}\mathcal{A}'\}] && (9)\\ &= R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t]. && (10)\end{aligned}$$

According to [16, Theorem 7.2.28] and [15, p. 168] the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\begin{aligned} \overline{R[It]} &= K[\{x^a t^b \mid (a,b) \in \mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+\mathcal{A}'\}] && (11)\\ &= R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots, && (12)\end{aligned}$$

where $\overline{I^i} = (\{x^a \in R \mid \exists\, p \geq 1; (x^a)^p \in I^{pi}\})$ is the integral closure of I^i and $\mathbb{Z}\mathcal{A}'$ is the subgroup of $\mathbb{Z}^{n+1}$ generated by $\mathcal{A}'$. Notice that in our situation we have the equality $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{n+1}$. Hence, by equations (9) to (12), we get that $R[It]$ is a normal domain if and only if any of the following two conditions hold: (a) $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$, (b) $I^i = \overline{I^i}$ for $i \geq 1$.

Theorem 3.4. *The clutter* $\mathcal{C}$ *has the* MFMC *property if and only if* $Q(A)$ *is an integral polyhedron and* $R[It]$ *is a normal domain.*

Proof. $\Rightarrow$) By [13, Corollary 22.1c] the polyhedron $Q(A)$ is integral. Next we show that $R[It]$ is normal. Take $x^\alpha t^{\alpha_{n+1}} \in \overline{R[It]}$. Then $(\alpha, \alpha_{n+1}) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. Hence $Ay \leq \alpha$ and $\langle y, \mathbf{1}\rangle = \alpha_{n+1}$ for some vector $y \geq 0$. Therefore one concludes that the optimal value of the linear program

$$\max\{\langle y, \mathbf{1}\rangle \mid y \geq \mathbf{0};\ Ay \leq \alpha\}$$

is greater or equal than α_{n+1}. Since A has the MFMC property, this linear program has an optimal integral solution y_0. Thus there exists an integral vector y_0' such that

$$\mathbf{0} \leq y_0' \leq y_0 \quad \text{and} \quad |y_0'| = \alpha_{n+1}.$$

Therefore

$$\begin{pmatrix}\alpha\\ \alpha_{n+1}\end{pmatrix} = \begin{pmatrix}A\\ \mathbf{1}\end{pmatrix} y_0' + \begin{pmatrix}A\\ \mathbf{0}\end{pmatrix}(y_0 - y_0') + \begin{pmatrix}\alpha\\ 0\end{pmatrix} - \begin{pmatrix}A\\ \mathbf{0}\end{pmatrix} y_0$$

and $(\alpha, \alpha_{n+1}) \in \mathbb{N}\mathcal{A}'$. This proves that $x^\alpha t^{\alpha_{n+1}} \in R[It]$, as required.

$\Leftarrow$) Assume that A does not satisfy the MFMC property. There exists an $\alpha_0 \in \mathbb{N}^n$ such that if y_0 is an optimal solution of the linear program:

$$\max\{\langle y, \mathbf{1}\rangle |\ y \geq \mathbf{0};\ Ay \leq \alpha_0\}, \qquad (*)$$

then y_0 is not integral. We claim that also the optimal value $|y_0| = \langle y_0, \mathbf{1}\rangle$ of this linear program is not integral. If $|y_0|$ is integral, then $(\alpha_0, |y_0|)$ is in $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. As $R[It]$ is normal, we get that $(\alpha_0, |y_0|)$ is in $\mathbb{N}\mathcal{A}'$, but this readily yields that the linear program $(*)$ has an integral optimal solution, a contradiction. This completes the proof of the claim.

Now, consider the dual linear program:

$$\min\{\langle x, \alpha_0\rangle |\ x \geq \mathbf{0},\ xA \geq \mathbf{1}\}.$$

By [17, Theorem 4.1.6]) the optimal value of this linear program is attained at a vertex x_0 of $Q(A)$. Then by the LP duality theorem [12, Theorem 3.16] we get $\langle x_0, \alpha_0\rangle = |y_0| \notin \mathbb{Z}$. Hence x_0 is not integral, a contradiction to the integrality of the set covering polyhedron $Q(A)$. □

Remark 3.5. The program *Normaliz* [4, 5] computes the irreducible representation of a Rees cone and the integral closure of $R[It]$. Thus one can effectively use Theorems 3.2 and 3.4 to determine whether a given clutter $\mathcal{C}$ as the max-flow min-cut property. See example below for a simple illustration.

Example 3.6. Let $I = (x_1x_5, x_2x_4, x_3x_4x_5, x_1x_2x_3)$. Using *Normaliz* [4] with the input file:

```
4
5
1 0 0 0 1
0 1 0 1 0
0 0 1 1 1
1 1 1 0 0
3
```

we get the output file:

```
9 generators of integral closure of Rees algebra:
  1  0  0  0  0  0
  0  1  0  0  0  0
  0  0  1  0  0  0
  0  0  0  1  0  0
  0  0  0  0  1  0
  1  0  0  0  1  1
```

```
 0  1  0  1  0  1
 0  0  1  1  1  1
 1  1  1  0  0  1

10 support hyperplanes:
   0   0   1   1   1  -1
   1   0   0   0   0   0
   0   1   0   0   0   0
   0   0   0   0   0   1
   0   0   1   0   0   0
   1   0   0   1   0  -1
   0   0   0   1   0   0
   0   0   0   0   1   0
   0   1   0   0   1  -1
   1   1   1   0   0  -1
```

The first block shows the exponent vectors of the generators of the integral closure of $R[It]$, thus $R[It]$ is normal. The second block shows the irreducible representation of the Rees cone of I, thus using Theorem 3.2 we obtain that $Q(A)$ is integral. Altogether Theorem 3.4 proves that the clutter $\mathcal{C}$ associated to I has the max-flow min-cut property.

Definition 3.7. *A set $C \subset X$ is a minimal vertex cover of a clutter $\mathcal{C}$ if every edge of $\mathcal{C}$ contains at least one vertex in C and C is minimal w.r.t. this property. A set of edges of $\mathcal{C}$ is independent if no two of them have a common vertex. We denote by $\alpha_0(\mathcal{C})$ the smallest number of vertices in any minimal vertex cover of $\mathcal{C}$, and by $\beta_1(\mathcal{C})$ the maximum number of independent edges of $\mathcal{C}$.*

Definition 3.8. *Let $X = \{x_1, \ldots, x_n\}$ and let $X' = \{x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s}\}$ be a subset of X. A minor of I is a proper ideal I' of $R' = K[X \setminus X']$ obtained from I by making $x_{i_k} = 0$ and $x_{j_\ell} = 1$ for all k, ℓ. The ideal I is considered itself a minor. A minor of $\mathcal{C}$ is a clutter $\mathcal{C}'$ that corresponds to a minor I'.*

Recall that a ring is called *reduced* if 0 is its only nilpotent element. The *associated graded ring* of I is the quotient ring $\mathrm{gr}_I(R) := R[It]/IR[It]$.

Corollary 3.9. *If the associated graded ring $\mathrm{gr}_I(R)$ is reduced, then $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for any minor $\mathcal{C}'$ of $\mathcal{C}$.*

Proof. As the reducedness of $\mathrm{gr}_I(R)$ is preserved if we make a variable x_i equal to 0 or 1, we may assume that $\mathcal{C}' = \mathcal{C}$. From [8, Proposition 3.4] and Theorem 3.2 it follows that the ring $\mathrm{gr}_I(R)$ is reduced if and only if $R[It]$ is normal and $Q(A)$ is integral. Hence by Theorem 3.4 we obtain that the LP-duality equation

$$\min\{\langle \mathbf{1}, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq \mathbf{1}\}$$

has optimum integral solutions x, y. To complete the proof notice that the left hand side of this equality is $\alpha_0(\mathcal{C})$ and the right hand side is $\beta_1(\mathcal{C})$. □

Next we state an algebraic version of a conjecture [6, Conjecture 1.6] which to our best knowledge is still open:

Conjecture 3.10. If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then the associated graded ring $\mathrm{gr}_I(R)$ is reduced.

Proposition 3.11. *Let B be the matrix with column vectors $(v_1, 1), \ldots, (v_q, 1)$. If $x^{v_1}, \ldots, x^{v_q}$ are monomials of the same degree $d \geq 2$ and $\mathrm{gr}_I(R)$ is reduced, then B diagonalizes over $\mathbb{Z}$ to an identity matrix.*

Proof. As $R[It]$ is normal, the result follows from [7, Theorem 3.9]. □

This result suggests the following weaker conjecture:

Conjecture 3.12. (Villareal) Let A be a $\{0, 1\}$-matrix such that the number of 1's in every column of A has a constant value $d \geq 2$. If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then the quotient group $\mathbb{Z}^{n+1}/((v_1, 1), \ldots, (v_q, 1))$ is torsion-free.

Symbolic Rees algebras. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of the edge ideal $I = I(\mathcal{C})$ and let $C_k = \{x_i | \, x_i \in \mathfrak{p}_k\}$, for $k = 1, \ldots, s$, be the corresponding minimal vertex covers of the clutter $\mathcal{C}$. We set

$$\ell_k = (\textstyle\sum_{x_i \in C_k} e_i, -1) \quad (k = 1, \ldots, s).$$

The *symbolic Rees algebra* of I is the K-subalgebra:

$$R_s(I) = R + I^{(1)}t + I^{(2)}t^2 + \cdots + I^{(i)}t^i + \cdots \subset R[t],$$

where $I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i$ is the i-th symbolic power of I.

Corollary 3.13. *The following conditions are equivalent*

(a) $Q(A)$ *is integral.*

(b) $\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+$.

(c) $\overline{R[It]} = R_s(I)$, *i.e.,* $\overline{I^i} = I^{(i)}$ *for all* $i \geq 1$.

Proof. The integral vertices of $Q(A)$ are precisely the vectors $a_1, \ldots, a_s$, where $a_k = \sum_{x_i \in C_k} e_i$ for $k = 1, \ldots, s$. Hence by Theorem 3.2 we obtain that (a) is equivalent to (b). By [8, Corollary 3.8] we get that (b) is equivalent to (c). □

Corollary 3.14. *Let $\mathcal{C}$ be a clutter and let I be its edge ideal. Then $\mathcal{C}$ has the max-flow min-cut property if and only if $I^i = I^{(i)}$ for all $i \geq 1$.*

Proof. It follows at once from Corollary 3.13 and Theorem 3.4. □

References

[1] Bertsimas, D.; Tsitsiklis, J. N.: *Introduction to linear optimization*. Athena Scientific, Massachusetts 1997.

[2] Brøndsted, A.: *Introduction to Convex Polytopes*. Graduate Texts in Mathematics **90**, Springer-Verlag, 1983. Zbl 0509.52001

[3] Brumatti, P.; Simis, A.; Vasconcelos, W. V.: *Normal Rees algebras*. J. Algebra **112** (1988), 26–48. Zbl 0641.13009

[4] Bruns, W.; Koch, R.: *Normaliz – a program for computing normalizations of affine semigroups*. 1998. Available via anonymous ftp from ftp.mathematik.Uni-Osnabrueck.DE/pub/osm/kommalg/software

[5] Bruns, W.; Koch, R.: *Computing the integral closure of an affine semigroup*. Effective methods in algebraic and analytic geometry, 2000 (Kraków). Univ. Iagel. Acta Math. **39** (2001), 59–70. Zbl 1006.20045

[6] Cornuéjols, G.: *Combinatorial optimization: Packing and covering*. CBMS-NSF Regional Conference Series in Applied Mathematics **74**, SIAM (2001). Zbl 0972.90059

[7] Escobar, C.; Martínez-Bernal, J.; Villarreal, R. H.: *Relative volumes and minors in monomial subrings*. Linear Algebra Appl. **374** (2003), 275–290. Zbl 1051.52008

[8] Escobar, C.; Villarreal, R. H.; Yoshino, Y.: *Torsion freeness and normality of blowup rings of monomial ideals. Commutative Algebra*, Lect. Notes Pure Appl. Math. **244**, 69–84. Chapman & Hall/CRC, Boca Raton, FL, 2006. Zbl 1097.13002

[9] Faridi, S.: *The facet ideal of a simplicial complex*. Manuscr. Math. **109** (2002), 159–174. Zbl 1005.13006

[10] Faridi, S.: *Cohen-Macaulay properties of square-free monomial ideals*. J. Comb. Theory, Ser. A **109**(2) (2005), 299–329. Zbl 1101.13015

[11] Gitler, I.; Reyes, E.; Villarreal, R. H.: *Blowup algebras of square–free monomial ideals and some links to combinatorial optimization problems*. Rocky Mountain J. Math., to appear.

[12] Korte, B.; Vygen, J.: *Combinatorial Optimization. Theory and Algorithms*. Algorithms and Combinatorics **21**, Springer-Verlag, Berlin 2000. Zbl 0953.90052

[13] Schrijver, A.: *Theory of Linear and Integer Programming*. John Wiley & Sons, New York 1986. Zbl 0665.90063

[14] Vasconcelos, W. V.: *Arithmetic of Blowup Algebras*. London Math. Soc., Lecture Note Series **195**, Cambridge University Press, Cambridge 1994. Zbl 0813.13008

[15] Vasconcelos, W. V.: *Computational Methods of Commutative Algebra and Algebraic Geometry*. Springer-Verlag, Berlin 1998. Zbl 0896.13021

[16] Villarreal, R. H.: *Monomial Algebras*. Pure and Applied Mathematics **238**, Marcel Dekker, New York 2001. Zbl 1002.13010

[17] Webster, R.: *Convexity*. Oxford University Press, Oxford 1994. Zbl 0835.52001

[18] Zheng, X.: *Resolutions of facet ideals*. Commun. Algebra **32**(6) (2004), 2301–2324. Zbl 1089.13014

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