Ideal Structure of Hurwitz Series Rings

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Abstract. We study the ideals, in particular, the maximal spectrum and the set of idempotent elements, in rings of Hurwitz series.

Let $A$ be a commutative ring with identity. The elements of the ring $HA$ of Hurwitz series over $A$ are formal expressions of the type $f = \sum_{i=0}^{\infty} a_i X^i$ where $a_i \in A$ for all $i$. Addition is defined termwise. The product of $f$ by $g = \sum_{i=0}^{\infty} b_i X^i$ is defined by $f \ast g = \sum_{n=0}^{\infty} c_n X^n$ where $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ and $\binom{n}{k}$ is a binomial coefficient. Recently, many authors turned to this ring and discovered interesting applications in it. See for example [1] and [2]. The natural homomorphism $\epsilon : HA \rightarrow A$, is defined by $\epsilon(f) = a_0$.

1. Generalities

1.1. Proposition. $HA$ is an integral domain if and only if $A$ is an integral domain with zero characteristic.

Proof. $\Leftarrow$ See [1, Corollary 2.8].

$\Rightarrow$ Since $A \subset HA$, then $A$ is a domain. Suppose that $A$ has a positive characteristic $m$. Then $X \ast X^{m-1} = (\binom{m-1+1}{1}) X^m = m X^m = 0$.

1.2. Proposition. Let $I$ be an ideal of $A$. Then $HA/\epsilon^{-1}(I) \simeq A/I$ and $HA/HI \simeq H(A/I)$. In particular

a) $\epsilon^{-1}(I)$ is a radical ideal of $HA \iff I$ is a radical ideal of $A$.  

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Proof. The map \( \psi : HA \rightarrow A/I \), defined by \( \psi = \tau \circ \epsilon \) where \( \tau \) is the canonical surjection of \( A \) onto \( A/I \), is a surjective homomorphism with \( \ker \psi = \epsilon^{-1}(I) \), so \( HA/\epsilon^{-1}(I) \approx A/I \).

The map \( \phi : HA \rightarrow H(A/I) \), defined for \( f = \sum_{i=0}^{\infty} a_iX^i \) by \( \phi(f) = \sum_{i=0}^{\infty} \bar{a}_iX^i \), is a surjective homomorphism, with \( \ker \phi = HI \), so \( HA/HI \approx H(A/I) \).

Now (a), (b) and (c) follow from the first isomorphism.

(d) \( HI \in Spec(HA) \iff HA/HI \) an integral domain \( \iff H(A/I) \) an integral domain \( \iff A/I \) an integral domain with zero characteristic \( \iff I \in Spec(A) \) and \( A/I \) has zero characteristic.

The inverse implication in (d) of the proposition was proved in [1, Prop. 2.7].

Example. Let \( A = \mathbb{F}_q \) be the finite field of \( q \) elements. Since \( X^q - 1 = qX^q = 0 \), then \( H0 = 0 \) is not prime in \( HF_q \).

1.3. Corollary. The set of maximal ideals of \( HA \) is \( \text{Max}(HA) = \{ \epsilon^{-1}(M) : M \in \text{Max}(A) \} \). In particular, the Jacobson radical \( \text{Rad}(HA) = \epsilon^{-1}(\text{Rad}(A)) \). The ring \( HA \) is local (resp. quasi local) if and only if \( A \) is local (resp. quasi local).

\[ \begin{align*}
\text{Examples.} & \quad 1) \text{Max}(HZ) = \{ \epsilon^{-1}(p\mathbb{Z}) : p \text{ prime integer} \}.
\end{align*} \]

2) For any field \( K \), \( HK \) is local with maximal ideal \( \epsilon^{-1}(0) \).

3) Contrary to the case of the ring of usual formal power series over a field, the element \( X \) does not generate the maximal ideal \( \epsilon^{-1}(0) \) of \( HF_2 \). Indeed, for any \( f = \sum_{n=0}^{\infty} a_nX^n \in HF_2 \), \( X \cdot f = \sum_{n=0}^{\infty} (n+1)a_nX^{n+1} = \sum_{n=0}^{\infty} (n+1)a_nX^{n+1} = \sum_{k=0}^{\infty} a_{2k}X^{2k+1}. \)

1.4. Proposition. If \( P \subset Q \) are consecutive prime ideals in \( A \), then \( \epsilon^{-1}(P) \subset \epsilon^{-1}(Q) \) are consecutive prime ideals in \( HA \).

Proof. Let \( R \in Spec(HA) \) such that \( \epsilon^{-1}(P) \subset R \subset \epsilon^{-1}(Q) \). There is an \( f = a_0 + a_1X + \cdots \in R \setminus \epsilon^{-1}(P) \). Then \( a_0 \not\in P \) and \( a_0 = f - (a_1X + \cdots) \in R \) since \( a_1X + \cdots \in \epsilon^{-1}(P) \subset R \). Therefore \( a_0 \in R \cap A \) and \( P = \epsilon^{-1}(P) \cap A \subset R \cap A \subset \epsilon^{-1}(Q) \cap A = Q \). Since \( P \subset Q \) are consecutive, then \( R \cap A = Q \). For any element \( g = b_0 + b_1X + \cdots \in \epsilon^{-1}(Q) \), \( b_0 \in Q \subset R \) and \( b_1X + \cdots \in \epsilon^{-1}(P) \subset R \), so \( g \in R \) and \( \epsilon^{-1}(Q) = R \).
2. Idempotent elements in Hurwitz series ring

For \( f \in HA \), the ideal \( c(f) \) generated by the coefficients of \( f \) in \( A \) is called the content of \( f \).

2.1. Proposition. Suppose that for any \( P \in \text{Spec}(A) \), \( A/P \) has zero characteristic. If \( f \) and \( g \) are such that \( f \cdot g = 0 \), then \( c(f)\cdot c(g) \subseteq \text{Nil}(A) \). Moreover, if \( A \) is reduced, then each coefficient of \( f \) annihilates \( g \).

Proof. By Proposition 1.2, for any \( P \in \text{Spec}(A) \), \( HP \in \text{Spec}(HA) \). Since \( f \cdot g = 0 \in HP \), then \( f \) or \( g \in HP \). If \( a \) is a coefficient of \( f \) and \( b \) a coefficient of \( g \), then \( ab \in P \). So \( ab \in \bigcap \{ P : P \in \text{Spec}(A) \} = \text{Nil}(A) \) and \( c(f)\cdot c(g) \subseteq \text{Nil}(A) \).

Example. The result is not true in general. Suppose for example that \( A \) has positive characteristic \( n \). Then \( X \cdot X^{n-1} = \binom{n-1+1}{1}X^n = nX^n = 0 \), with \( c(X) = c(X^{n-1}) = A \), so \( c(X)c(X^{n-1}) = A \not\subseteq \text{Nil}(A) \).

As usual, \( \text{Bool}(A) \) will mean the set of idempotent elements in the ring \( A \).

2.2. Corollary. Suppose \( A \) is reduced and \( A/P \) has zero characteristic, for every \( P \in \text{Spec}(A) \). Then \( \text{Bool}(HA) = \text{Bool}(A) \).

Proof. Let \( f = \sum_{i=0}^{\infty} a_iX^i \in HA \), with \( f \cdot f = f \). Then \( f-1 = (a_0-1)+\sum_{i=1}^{\infty} a_iX^i \) and \( f \cdot (f-1) = 0 \). By Proposition 2.1, for \( i \geq 1 \), \( a_i^2 = 0 \), so \( a_i = 0 \) and \( f = a_0 \in A \).

More generally, we have the following result.

2.3. Proposition. For any ring \( A \), \( \text{Bool}(HA) = \text{Bool}(A) \).

Proof. Let \( f = \sum_{i=0}^{\infty} a_iX^i \in HA \) be such that \( f \cdot f = f \). Then \( a_0^2 = a_0 \) and \( 2a_0a_1 = a_1 \Rightarrow 2a_0^2a_1 = a_0a_1 \Rightarrow 2a_0a_1 = a_0a_1 \Rightarrow a_0a_1 = 0 \). Suppose by induction that \( a_0a_i = 0 \), for \( 1 \leq i < n \). The coefficient of \( X^n \) in \( f \cdot f = f \) is \( \sum_{i=0}^{n}(\binom{n}{i})a_ia_{n-i} = a_n \Rightarrow a_0(\sum_{i=0}^{n}(\binom{n}{i})a_ia_{n-i}) = a_0a_n \Rightarrow a_0(\binom{n}{i})a_ia_n + (\binom{n}{i})a_ia_0) = a_0a_n \Rightarrow 2a_0^2a_n = a_0a_n \Rightarrow 2a_0a_n = a_0a_n \Rightarrow a_0a_n = 0 \). So for each \( i \geq 1 \), \( a_0a_i = 0 \). Suppose that \( f \not\in A \) and let \( k = \min \{ i \in \mathbb{N}^+ : a_i \neq 0 \} \), \( g = \sum_{i=k}^{\infty} a_iX^i \), then \( k \geq 1 \), \( a_k \neq 0 \), \( f = a_0 + g \), \( a_0 \cdot g = \sum_{i=k}^{\infty} a_0a_iX^i = 0 \). Since \( f \cdot f = f \), then \( (a_0 + g) \cdot (a_0 + g) = a_0 + g \Rightarrow a_0^2 + g \cdot g = a_0 + g \Rightarrow g \cdot g = g = (\binom{k}{k})a_k^2X^2k + \cdots = a_kX^k + \cdots \Rightarrow a_k = 0 \), which is impossible. So \( f = a_0 \in A \).

A ring \( A \) is called PS if the socle \( \text{Soc}(A) \) is projective. By [3, Theorem 2.4], a ring \( A \) is PS if and only if for every maximal ideal \( M \) of \( A \) there is an idempotent \( e \) of \( A \) such that \( (0 : M) = eA \). In [2, Theorem 3.2], Zhongkui Liu proved the following result:

“If \( A \) has zero characteristic and if \( A \) is a PS-ring, then \( HA \) is a PS-ring”. His proof is not correct, it uses in many places the wrong fact:

“If \( A \) has zero characteristic, \( n \in \mathbb{N}^+ \) and \( x \in A \), then \( nx = 0 \) implies \( x = 0 \)”.

But this is not true. Take for example: \( A = \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \), \( n \geq 2 \) an integer and \( x = (0,1) \). When I wrote to Liu, he proposed to replace the condition “\( A \) has zero characteristic” by “\( A \) is \( \mathbb{Z} \)-torsion free”. With this change the proof becomes correct.
In the next proposition, I avoid the hypothesis “A is a PS-ring” in the theorem of Liu and I give a short and simple proof.

2.4. Proposition. If A is torsion free as a \( \mathbb{Z} \)-module, then \( HA \) is a PS-ring.

**Proof.** If \( M \in \text{Max}(HA) \), there is \( M \in \text{Max}(A) \) such that \( M = e^{-1}(M) \) by Corollary 1.3, so \( X \in M \). Let \( f = \sum_{i=0}^{\infty} a_i X^i \in (0 : M) \), then \( 0 = X \ast f = \sum_{i=0}^{\infty} (i+1) a_i X^{i+1} = \sum_{i=0}^{\infty} i a_i X^{i} \). For each \( i \in \mathbb{N} \), \( (i+1) a_i = 0 \), but \( A \) is \( \mathbb{Z} \)-torsion free, then \( a_i = 0 \) and \( f = 0 \).

2.5. Lemma. Suppose that A is reduced and \( A/P \) has zero characteristic for any \( P \in \text{Spec}(A) \). For \( f \in HA \), let \( I_f = (0 : c(f)) \). Then:

a) For every \( f \in HA \), \( (0 : f) = HI_f \).

b) If \( J \) is an ideal of \( HA \) and \( L = \sum_{f \in J} c(f) \), then \( (0 : J) = H(0 : L) \).

**Proof.**

a) Put \( f = \sum_{i=0}^{\infty} a_i X^i \). By Proposition 2.1, \( g = \sum_{i=0}^{\infty} b_i X^i \in (0 : f) \iff f \ast g = 0 \iff \forall i, j \in \mathbb{N}, a_i b_j = 0 \iff \forall j \in \mathbb{N}, b_j \in (0 : c(f)) = I_f \iff g \in HI_f \).

b) By part a), \( (0 : J) = \bigcap_{f \in J} (0 : f) = \bigcap_{f \in J} HI_f = H\big(\bigcap_{f \in J} I_f\big) \). But \( \bigcap_{f \in J} I_f = \bigcap_{f \in J} (0 : c(f)) = (0 : \sum_{f \in J} c(f)) = (0 : L) \). So \( (0 : J) = H(0 : L) \).

2.6. Proposition. If A is reduced with \( A/P \) has zero characteristic for every \( P \in \text{Spec}(A) \), then \( HA \) is a PS-ring.

**Proof.** Let \( M \in \text{Max}(HA) \). By Corollary 1.3, \( X \in M \), then \( \sum_{f \in M} c(f) = A \). By the preceding lemma, \( (0 : M) = H(0 : A) = H0 = (0) \).

**Conjecture.** In [4, Proposition 4], Xue showed that the ring \( A[[X]] \) is always PS, for any ring \( A \). In the light of this theorem and the preceding results I conjecture that the ring \( HA \) is also PS.

2.7. Definition. A quasi-Baer ring is a ring A such that for any ideal \( I \) of A there is an idempotent \( e \) of A with \( (0 : I) = eA \).

The following lemma is well known. We include its proof for the sake of the reader.

2.8. Lemma. Any quasi-Baer ring is reduced.

**Proof.** Let \( a \) be a nilpotent element of the quasi-Baer ring \( A \) and \( n \geq 1 \) the smallest integer such that \( a^n = 0 \). Let \( (0 : aA) = eA \), with \( e \in A \) and \( e^2 = e \). If \( n \geq 2 \), then \( a^{n-1} \in eA \), put \( a^{n-1} = eb \), with \( b \in A \). Since \( ae = 0 \), then \( 0 = a^{n-1} e = be^2 = be = a^{n-1} \), which is impossible.

2.9. Proposition. If A is a quasi-Baer ring with \( A/P \) has zero characteristic for every \( P \in \text{Spec}(A) \), then \( HA \) is a quasi-Baer ring.
Proof. Let \( J \) be an ideal of \( HA \) and \( L = \sum_{f \in J} c(f) \). There is \( e \in \text{Bool}(A) \) such that 
\[
(0 : L) = eA.
\]
By Lemma 2.5, 
\[
(0 : J) = H(0 : L) = H(eA) = e \ast HA.
\]

3. Hurwitz series over a noetherian ring

3.1. Lemma. Let \( I \) be an ideal of \( A \). Then \( HI = I \ast HA \) if and only if for any countable subset \( S \) of \( I \) there is a finitely generated ideal \( F \) of \( A \) such that \( S \subseteq F \subseteq I \).

Proof. \( \Rightarrow \) A countable subset of \( I \) is a sequence \((a_i)_{i \in \mathbb{N}}\) of elements of \( I \). Let 
\[
f = \sum_{i=0}^{\infty} a_i X^i \in HI = I \ast HA.
\]
There are \( b_1, \ldots, b_n \in I \) and \( g_1, \ldots, g_n \in HA \) such that 
\[
f = b_1 \ast g_1 + \cdots + b_n \ast g_n.
\]
If \( F = b_1A + \cdots + b_nA \), then \( \{a_i : i \in \mathbb{N}\} \subseteq F \).

\( \Leftarrow \) Since \( I \subset HI \), then \( I \ast HA \subseteq HI \). Now, let 
\[
f = \sum_{i=0}^{\infty} a_i X^i \in HI.
\]
There is a finitely generated ideal \( F = b_1A + \cdots + b_nA \) of \( A \) such that \( \{a_i : i \in \mathbb{N}\} \subseteq F \subseteq I \).

For each \( i \in \mathbb{N} \), \( a_i = \sum_{j=1}^{n} a_{ij}b_j \), with \( a_{ij} \in A \). So 
\[
f = \sum_{i=0}^{\infty} (\sum_{j=1}^{n} a_{ij}b_j)X^i = \sum_{j=1}^{n} b_j \ast \left( \sum_{i=0}^{\infty} a_{ij}X^i \right) \in I \ast HA.
\]

Example. Let \((A, M)\) be a non-discrete valuation domain of rank one, defined by a valuation \( v \) with group \( G \). We can suppose that \( G \) is a dense subgroup of \( \mathbb{R} \). Let \((a_i)_{i \in \mathbb{N}}\) be a strictly decreasing sequence of elements of \( G \) converging to zero. For each \( i \in \mathbb{N} \), there is \( a_i \in M \), with \( v(a_i) = \alpha_i \). Let 
\[
f = \sum_{i=0}^{\infty} a_i X^i \in HM.
\]

Suppose that \( f \in M \ast HA \), there is \( b \in M \) and \( g = \sum_{i=0}^{\infty} c_i X^i \in HA \) such that 
\[
f = b \ast g.
\]
For each \( i \in \mathbb{N} \), \( a_i = bc_i \), so \( \alpha_i = v(a_i) = v(b) + v(c_i) \geq v(b) \), which is impossible.

3.2. Corollary. If \( I \) is a finitely generated ideal, then \( HI = I \ast HA \).

3.3. Proposition. The ring \( A \) is noetherian if and only if for each ideal \( I \) of \( A \), \( HI = I \ast HA \).

Proof. Suppose that \( A \) is not noetherian and let \((I_i)_{i \in \mathbb{N}}\) be a strictly increasing sequence of ideals of \( A \) and put 
\[
I = \bigcup_{i=0}^{\infty} I_i.
\]
For each \( i \in \mathbb{N}^* \), there is \( a_i \in I_i \setminus I_{i-1} \).

Since \( HI = I \ast HA \), there is a finitely generated ideal \( F = b_1A + \cdots + b_nA \) of \( A \) such that \( \{a_i : i \in \mathbb{N}^*\} \subseteq F \subseteq I \). Since the sequence \((I_i)_{i \in \mathbb{N}}\) is increasing, there is \( k \in \mathbb{N} \) such that \( b_1, \ldots, b_n \in I_k \) so \( F \subseteq I_k \) and \( \{a_i : i \in \mathbb{N}^*\} \subseteq I_k \), which is impossible.
Example. Let $K$ be a commutative field and $\{Y_i : i \in \mathbb{N}\}$ a sequence of indeterminates. The ring $A = K[Y_i : i \in \mathbb{N}]$ is not noetherian because its ideal $I = (Y_i : i \in \mathbb{N})$ is not finitely generated. Suppose that $HI = I \ast HA$, by Lemma 3.1, there is a finitely generated ideal $F$ of $A$ such that $\{Y_i : i \in \mathbb{N}\} \subseteq F \subseteq I$, so $I = F$, which is impossible.

3.4. Proposition. Let $I$ and $J$ be ideals of the ring $A$, with $HJ = J \ast HA$ and $J \subseteq \sqrt{J}$. Then there is $n \in \mathbb{N}^*$ such that $J^n \subseteq I$.

Proof. Suppose that for each $m \in \mathbb{N}^*$, $J^m \nsubseteq I$, there are $b_{m1}, \ldots, b_{mm} \in J$ such that the product $b_{m1} \cdots b_{mm} \notin I$. Let $C$ be the ideal of $A$ generated by the countably subset $\{b_{mi} : m \in \mathbb{N}^*, 1 \leq i \leq m\}$, then $C \subseteq J$ and $C^m \nsubseteq I$ for every $m \in \mathbb{N}^*$. Since $HJ = J \ast HA$, by Lemma 3.1, there is a finitely generated ideal $F$ of $A$ such that $C \subseteq F \subseteq J \subseteq \sqrt{J}$, so $F \subseteq \sqrt{J}$. But $F$ is finitely generated, there is $n \in \mathbb{N}^*$ such that $F^n \subseteq I$, so $C^n \subseteq I$, which is impossible.

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References


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