Eigenvalues of a Natural Operator of Centro-affine and Graph Hypersurfaces

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Abstract. In this article we obtain optimal estimates for the eigenvalues of a natural operator $K_{T^*}$ for locally strongly convex centro-affine and graph hypersurfaces. Several immediate applications of our eigenvalue estimates are presented. We also provide examples to illustrate that our eigenvalue estimates are optimal.

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1. Introduction

Throughout this article we assume $n \geq 2$. An immersed hypersurface $f : M \to \mathbb{R}^{n+1}$ in an affine $(n+1)$-space $\mathbb{R}^{n+1}$ is called an affine hypersurface with relative normalization if there is a transversal vector field $\xi$ such that $D\xi$ has its image in $f^*(T_pM)$, where $D$ is the canonical flat connection on $\mathbb{R}^{n+1}$.

A hypersurface $f : M \to \mathbb{R}^{n+1}$ is called centro-affine if its position vector field is always transversal to $f^*(TM)$ in $\mathbb{R}^{n+1}$. In this case, for any vector fields $X,Y$ tangent to $M$, one can decompose $D_X f^*(Y)$ into its tangential and transverse components. This is written as

$$D_X f^*(Y) = f^*(\nabla_X Y) + h^f(X,Y)f,$$

where $h^f$ is a symmetric tensor of type $(0,2)$ and $\xi = f$. 

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Throughout this article, we assume that $h^f$ is definite, so $h^f$ defines a semi-Riemannian metric on $M$. In order to consider only a positive definite metric we now make the following changes: if $h^f$ is negative definite, we introduce a transversal vector field $\xi = -f$ and a $(0,2)$-tensor given by $h = -h^f$.

It is well-known that the centro-affine metric $h$ is definite if and only if the hypersurface is locally strongly convex. For this the following terminology is used:

(i) The centro-affine hypersurface $M$ is said to be of elliptic type if, for any point $f(p) \in \mathbb{R}^{n+1}$ with $p \in M$, the origin of $\mathbb{R}^{n+1}$ and the hypersurface are on the same side of the tangent hyperplane $f_* (T_p M)$; in this case the centro-affine normal vector field is given by $\xi = -f$.

(ii) The centro-affine hypersurface $M$ is said to be of hyperbolic type if, for any point $f(p) \in \mathbb{R}^{n+1}$, the origin of $\mathbb{R}^{n+1}$ and the hypersurface are on the different side of the tangent hyperplane $f_* (T_p M)$; in this case the centro-affine normal vector field is given by $\xi = f$.

An affine hypersurface $f: M \to \mathbb{R}^{n+1}$ is called a graph hypersurface if we choose as affine transversal field a constant vector field. For a graph hypersurface we also have the decomposition (1.1) as well. Again in case that $h$ is non-degenerate, it defines a semi-Riemannian metric, called the Calabi metric of the graph hypersurface.

Let $\hat{\nabla}$ denote the Levi-Civita connection of $h$ and let $K$ be the difference tensor $\nabla - \hat{\nabla}$ on $M$. Then, for each $X \in T_p M$, $K : Y \mapsto K(X,Y)$ is an endomorphism of $T_p M$. By taking the trace of $K$, one obtains a so-called Tchebychev form

$$T(X) := \frac{1}{n} \text{trace} \{ Y \mapsto K(X,Y) \}. \quad (1.2)$$

The Tchebychev vector field $T^\#$ can then be defined by

$$h(T^\#, X) = T(X). \quad (1.3)$$

The Tchebychev form and Tchebychev vector field play an important role in centro-affine differential geometry.

For each integer $k \in [2, n]$, we define an invariant $\hat{\theta}_k$ on the affine hypersurface $M$ in the same way as in [1] (see Section 3 for details).

The main results of this article are the following optimal estimates for the eigenvalues of the operator $K_{T^\#}$:

(I) For a locally strongly convex centro-affine hypersurface $M$ in $\mathbb{R}^{n+1}$ we have:

(I-a) If $\hat{\theta}_k \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $(\frac{n-1}{n})(\varepsilon - \hat{\theta}_k(p))$.

(I-b) If $\hat{\theta}_k = \varepsilon$ at a point $p$, every eigenvalue of $K_{T^\#}$ at $p$ is $\geq 0$, where $\varepsilon = 1$ or $-1$ according to $M$ is of elliptic or hyperbolic type.

(II) For a graph hypersurface $M$ in $\mathbb{R}^{n+1}$ we have:

(II-a) If $\hat{\theta}_k \neq 0$ at a point $p \in M$, every eigenvalue of the operator $K_{T^\#}$ at $p$ is greater than $(\frac{1-n}{n})\hat{\theta}_k(p)$.

(II-b) If $\hat{\theta}_k = 0$ at a point $p \in M$, every eigenvalue of $K_{T^\#}$ at $p$ is $\geq 0$. 
The proofs of the main results base on the equation of Gauss using the same idea introduced in earlier author’s articles [1, 2]. This is done in Section 4. Several immediate applications of our eigenvalue estimates of the operator $K_{T\#}$ are given in Section 5. In the last two sections, we provide some non-trivial examples to illustrate that our eigenvalue estimates are optimal for both centro-affine and graph hypersurfaces.

2. Preliminaries

We recall some basic facts about centro-affine and graph hypersurfaces. For the details, see [3, 4, 5, 6].

Let $f : M \to \mathbb{R}^{n+1}$ be a centro-affine hypersurface with centro-affine normal $\xi$. We assume that the centro-affine hypersurface is definite. As we already mentioned earlier, the centro-affine normal on the hypersurface is chosen in such way that the metric $h$ is positive definite.

The centro-affine structure equations are given by
\begin{align}
D_X f_*(Y) &= f_*(\nabla_X Y) + h(X, Y)\xi; \\
D_X \xi &= \mp f_*(X),
\end{align}
where $D_X \xi = -f_*(X)$ or $D_X \xi = f_*(X)$ according to $\xi = -f$ or $\xi = f$ respectively.

The corresponding equations of Gauss and Codazzi are given respectively by
\begin{align}
R(X, Y)Z &= h(Y, Z)X - h(X, Z)Y; \\
(\nabla_X h)(Y, Z) &= (\nabla_Y h)(X, Z).
\end{align}
The cubic form is the totally symmetric $(0, 3)$-tensor field $C(X, Y, Z) = (\nabla_X h)(Y, Z)$.

Let $\hat{\nabla}$, $\hat{K}$ and $\hat{R}$ denote the Levi-Civita connection, the sectional curvature and the curvature tensor of $h$, respectively. The difference tensor $K$ is then given by
\begin{equation}
K_X Y = K(X, Y) = \nabla_X Y - \hat{\nabla}_X Y,
\end{equation}
which is a symmetric $(1, 2)$-tensor field. The difference tensor $\hat{K}$ and the cubic form $C$ are related by
\begin{equation}
C(X, Y, Z) = -2h(K_X Y, Z).
\end{equation}

It is well-known that for centro-affine hypersurfaces we have
\begin{align}
h(K_X Y, Z) &= h(Y, K_X Z), \\
\hat{R}(X, Y)Z &= K_Y K_X Z - K_X K_Y Z + \varepsilon(h(Y, Z)X - h(X, Z)Y), \\
(\hat{\nabla} K)(X, Y, Z) &= (\hat{\nabla} K)(Y, Z, X) = (\hat{\nabla} K)(Z, X, Y),
\end{align}
where $\varepsilon = 1$ if $M$ is of elliptic type and $\varepsilon = -1$ if $M$ is of hyperbolic type. It follows from (2.7) that the endomorphism $K_X$ is self-adjoint with respect to $h$. 
When $f : M \to \mathbb{R}^{n+1}$ is a graph hypersurface, we have (1.1), (2.1), (2.4), (2.5), (2.6), (2.7) and (2.9) as well. However, (2.2), (2.3) and (2.8) shall be replaced by
\begin{align}
D_X \xi &= R(X,Y)Z = 0, \\
\hat{R}(X,Y)Z &= K_XK_YZ - K_YK_XZ.
\end{align}

3. Invariant $\hat{\theta}_k$ and relative $K$-null space

Let $M$ be a centro-affine or graph hypersurface with positive definite metric $h$. Denote by $\hat{K}(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM$ relative to $h$. The scalar curvature $\hat{\tau}$ at $p$ is then defined by
\begin{equation}
\hat{\tau}(p) = \sum_{1 \leq i < j \leq n} \hat{K}_{ij},
\end{equation}
where $\hat{K}_{ij} = \hat{K}(e_i \wedge e_j)$ and $e_1, \ldots, e_n$ is an $h$-orthonormal basis of $T_pM$.

Assume that $L^k$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L^k$ with respect to $h$. We choose an $h$-orthonormal basis $\{e_1, \ldots, e_k\}$ of $L^k$ with $e_1 = X$. Then the $k$-Ricci curvature $\hat{S}_{L^k}(X)$ and the scalar curvature $\hat{\tau}(L^k)$ are defined respectively by
\begin{align}
\hat{S}_{L^k}(X) &= \hat{K}_{12} + \cdots + \hat{K}_{1k}, \\
\hat{\tau}(L^k) &= \sum_{1 \leq i < j \leq k} K_{ij}.
\end{align}

Obviously, $\hat{S}_{L^2}$ and $\hat{\tau}(L^2)$ are nothing but the sectional curvature $\hat{K}(L^2)$. And $\hat{S}_{L^n}$ and $\hat{\tau}(L^n)$ are the Ricci and scalar curvatures relative to $h$.

For each integer $k \in [2, n]$, we define the invariant $\hat{\theta}_k$ on $M$ by (cf. [1, 2])
\begin{equation}
\hat{\theta}_k(p) = \left(\frac{1}{k-1}\right) \sup_{L^k, X} \hat{S}_{L^k}(X), \quad p \in T_pM,
\end{equation}
where $L^k$ runs over all linear $k$-subspaces in the tangent space $T_pM$ at $p$ and $X$ runs over all $h$-unit vectors in $L^k$.

The relative $K$-null space $\mathcal{N}^K_p$ of $M$ in $\mathbb{R}^{n+1}$ is defined by
\begin{equation}
\mathcal{N}^K_p = \{X \in T_pM : K(X,Y) = 0 \text{ for all } Y \in T_pM\}.
\end{equation}
When $\dim \mathcal{N}^K_p$ is constant, $\mathcal{N}^K = \bigcup_{p \in M} \mathcal{N}^K_p$ defines a subbundle of the tangent bundle, called the relative $K$-null subbundle.

4. Optimal estimates for eigenvalues of the operator

For centro-affine hypersurface in $\mathbb{R}^{n+1}$ we have the following result.

Theorem 4.1. Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbb{R}^{n+1}$. Then, for any integer $k \in [2, n]$, we have:
(1) If \( \hat{\theta}_k \neq \varepsilon \) at a point \( p \in M \), then every eigenvalue of \( K_{T^\#} \) at \( p \) is greater than \( \left( \frac{n-1}{n} \right) (\varepsilon - \hat{\theta}_k(p)) \).

(2) If \( \hat{\theta}_k(p) = \varepsilon \), every eigenvalue of \( K_{T^\#} \) at \( p \) is \( \geq 0 \).

(3) A nonzero vector \( X \in T_pM \) is an eigenvector of the operator \( K_{T^\#} \) with eigenvalue \( \left( \frac{n-1}{n} \right) (\varepsilon - \hat{\theta}_k(p)) \) if and only if \( \hat{\theta}_k(p) = \varepsilon \) and \( X \) lies in the relative \( K \)-null space \( N^p_{K} \) at \( p \), where \( \varepsilon = 1 \) or \(-1\) according to \( M \) is of elliptic or hyperbolic type.

Proof. Assume that \( f : M \to \mathbb{R}^{n+1} \) is a locally strongly convex centro-affine hypersurface in \( \mathbb{R}^{n+1} \). Let \( \{e_1, \ldots, e_n\} \) be an arbitrary \( h \)-orthonormal basis of \( T_pM \). From the definition of Tchebychev vector field, (2.8) and (3.1) we have

\[
2 \hat{\tau} = n(n-1)\varepsilon + h(K, K) - n^2 h(T^\#, T^\#). \tag{4.1}
\]

It is well-known that every endomorphism \( A \) of \( T_pM \) satisfies

\[
n h(A, A) \geq (\text{trace } A)^2, \tag{4.2}
\]

with equality holding if and only if \( A \) is proportional to the identity map \( I \). By applying (4.1) and (4.2), we obtain

\[
2 \hat{\tau} \geq n(n-1)\varepsilon - n(n-1) h(T^\#, T^\#) \tag{4.3}
\]

with the equality holding at \( p \) if and only if we have

(a) \( K_{T^\#} \) is proportional to the identity map and

(b) \( K_Z = 0 \) for \( Z \) perpendicular to \( T^\# \) at \( p \).

Let \( L_{i_1 \ldots i_k} \) be the \( k \)-plane section spanned by the orthonormal vectors \( e_{i_1}, \ldots, e_{i_k} \). It follows from (3.2) and (3.3) that

\[
\hat{\tau}(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \hat{S}_{L_{i_1 \ldots i_k}}(e_i), \tag{4.4}
\]

\[
\hat{\tau}(p) = \frac{(k-2)!(n-k)!}{(n-2)!} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \hat{\tau}(L_{i_1 \ldots i_k}). \tag{4.5}
\]

By combining (3.4), (4.4) and (4.5) we find

\[
\hat{\tau} \leq \frac{n(n-1)}{2} \hat{\theta}_k. \tag{4.6}
\]

Thus (4.3) and (4.6) ensure that

\[
h(T^\#, T^\#) \geq \varepsilon - \hat{\theta}_k. \tag{4.7}
\]

Hence the Tchebychev vector field \( T^\# \) vanishes at a point \( p \) only when \( \hat{\theta}_k(p) \geq \varepsilon \). Therefore, if \( T^\#(p) = 0 \), statements (1) and (2) of Theorem 4.1 hold automatically.
Next, let us assume that $T^*(p) \neq 0$. Since $K_{T^*}$ is self-adjoint with respect to $h$, we may choose an $h$-orthonormal basis $e_1, \ldots, e_n$ of $T_pM$ which diagonalizes the operator $K_{T^*}$. Let $e^*_1$ be the $h$-unit vector at $p$ in the direction of $T^*$ and let us choose $h$-orthonormal vectors $e^*_2, \ldots, e^*_n$ at $p$ perpendicular to $T^*$. Then we have

$$K_{e^*_1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and trace $K_{e^*_r} = 0$ for $r = 2, \ldots, n$.

Let us put $K^*_{ij} = h(K(e_i, e_j), e^*_r)$. Then (2.8) implies that

$$K_{ij} = \varepsilon - a_i a_j + \sum_{r=2}^n (K^*_{ij})^2 - \sum_{r=2}^n K^*_{ii} K^*_{jj}, \quad 1 \leq i \neq j \leq n. \tag{4.9}$$

Now, by applying the same argument as the proof of Theorem 1 of [1], we obtain

$$a_1(a_1 + \cdots + a_n) \geq (n-1)(\varepsilon - \hat{\theta}_k(p)) + a_1^2 \geq (n-1)(\varepsilon - \hat{\theta}_k(p)), \tag{4.10}$$

with both equality holding if and only if we have $\hat{S}_i(e_1) = \hat{\theta}_k(p)$ and $a_1 = K^*_{ij} = 0$ for $r = 2, \ldots, n; j = 2, \ldots, n$. The same inequality holds if the lower index 1 in (4.10) were replaced by any $j \in \{2, \ldots, n\}$. Hence, we have

$$K_{T^*} \geq \frac{n-1}{n}(\varepsilon - \hat{\theta}_k(p))I. \tag{4.11}$$

If $K_{T^*}X = \frac{n-1}{n}(\varepsilon - \hat{\theta}_k(p))X$ holds for some nonzero vector $X \in T_pM$, then $X$ is an eigenvector of $K_{T^*}$ with eigenvalue $(n-1)(\varepsilon - \hat{\theta}_k(p))/n$. Without loss of generality, we may choose $e_1 = X/\sqrt{h(X,X)}$. In this case we get

$$a_1(a_1 + \cdots + a_n) = (n-1)(\varepsilon - \hat{\theta}_k(p)). \tag{4.12}$$

On the other hand, from (4.10) and (4.12), we find $a_1 = 0$ and $\hat{\theta}_k(p) = \varepsilon$. Moreover, we know from (4.10) that $e_1$ lies in the relative $K$-null space $N^K_p$. Consequently, we obtain statements (1) and (2) of Theorem 4.1 and also one part of statement (3). The remaining part of statement (3) is obvious.

For graph hypersurfaces we have the following.

**Theorem 4.2.** Let $f : M \to \mathbb{R}^{n+1}$ be a graph hypersurface in $\mathbb{R}^{n+1}$ with positive definite Calabi metric. Then, for any integer $k \in [2, n]$, we have:

1. If $\hat{\theta}_k \neq 0$ at a point $p \in M$, then every eigenvalue of $K_{T^*}$ at $p$ is greater than $(\frac{1-n}{n}) \hat{\theta}_k(p)$.
2. If $\hat{\theta}_k = 0$ at $p$, then every eigenvalue of $K_{T^*}$ at $p$ is $\geq 0$. 


(3) A nonzero vector $X \in T_p M$ is an eigenvector of the operator $K_{T^*}$ with eigenvalue $\left(\frac{1-n}{n}\right) \hat{\theta}_k(p)$ if and only if we have $\hat{\theta}_k(p) = 0$ and $X \in N^K_p$.

**Proof.** For graph hypersurfaces in $\mathbb{R}^{n+1}$ we have

$$\hat{\theta}_k(p) = \beta(p).$$

Thus, by applying the same argument given in Theorem 4.1, we obtain Theorem 4.2.

5. Some applications

When $k = 2$, statement (1) of Theorem 4.1 implies immediately the following.

**Corollary 5.1.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbb{R}^{n+1}$. If $\sup \hat{\theta}_{\xi} \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^*}$ at $p$ is greater than $\left(\frac{n}{n-1}\right) \left(\varepsilon - \sup \hat{\theta}(p)\right)$.

Similarly, if we denote by $\sup S(p)$ the supremum of the Ricci curvature of $(M, h)$ at a point $p \in M$, then statement (1) of Theorem 4.1 with $k = n$ implies immediately the following.

**Corollary 5.2.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbb{R}^{n+1}$. If $\sup S \neq \varepsilon$ at a point $p \in M$, then every eigenvalue of the operator $K_{T^*}$ at $p$ is greater than $\left(\frac{n}{n-1}\right) \left(\varepsilon - \sup S(p)\right)$.

From Theorem 4.1 we also obtain the following.

**Corollary 5.3.** Let $f : M \to \mathbb{R}^{n+1}$ be a locally strongly convex centro-affine hypersurface in $\mathbb{R}^{n+1}$. If we have $\bigwedge^2 \hat{\theta} < \varepsilon$ on $M$ for some integer $k \in [2, n]$, then every eigenvalue of $K_{T^*}$ is positive.

Theorem 4.1 also gives rise to the following simple geometric characterization of hyper-ellipsoids and two-sheeted hyperboloids.

**Corollary 5.4.** An elliptic centro-affine hypersurface $M$ in $\mathbb{R}^{n+1}$ is centroaffinely equivalent to an open portion of a hyperellipsoid if and only if we have $nK_{T^*} = (n-1)(1 - \hat{\theta}_k)I$ on $M$ for some integer $k \in [2, n]$.

**Proof.** Let $f : M \to \mathbb{R}^{n+1}$ be an elliptic centro-affine hypersurface in $\mathbb{R}^{n+1}$. If $M$ is an open portion of a hyperellipsoid, then $K$ vanishes identically which implies that $K_{T^*} = 0$. Hence, according to (2.10), $(M, h)$ is of constant curvature one. Therefore we obtain $\hat{\theta}_2 = \cdots = \hat{\theta}_n = 1$. Consequently, we have $nK_{T^*} = (n-1)(1 - \hat{\theta}_k)I$ identically.

Conversely, let us assume that $nK_{T^*} = (n-1)(1 - \hat{\theta}_k)I$ holds identically for some integer $k \in [2, n]$, then statement (3) of Theorem 4.1 implies that every tangent vector of $M$ lies in the relative $K$-null subbundle. In this case $K$ vanishes identically on $M$. Consequently, by applying a theorem of Berwald [6, Section 7.1.1], we conclude that $M$ is centroaffinely equivalent to an open portion of a hyper-ellipsoid centered at the origin.
Corollary 5.5. A hyperbolic centro-affine hypersurface $M$ in $\mathbb{R}^{n+1}$ is centroaffinely equivalent to an open portion of a two-sheeted hyperboloid if and only if, for some integer $k \in [2, n]$, we have $nK_T = (1 - n)(1 + \theta_k)I$ identically on $M$.

Proof. This can be done in the same way as Corollary 5.4. \qed

Similarly Theorem 4.2 implies the following.

Corollary 5.6. Let $f : M \to \mathbb{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k < 0$ holds on $M$, then every eigenvalue of the operator $K_T$ is greater than $(1 - n/n)\sup K$ at $p$.

Corollary 5.7. Let $f : M \to \mathbb{R}^{n+1}$ be a graph hypersurface with positive definite Calabi metric. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k < 0$ holds on $M$, then every eigenvalue of $K_T$ is positive.

From Corollaries 5.3 and 5.7 we obtain the following.

Corollary 5.8. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 1$ at some point $p \in M$, then $M$ cannot be realized as an elliptic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Corollary 5.9. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < -1$ at some point $p \in M$, then $M$ cannot be realized as a hyperbolic proper affine hypersphere in $\mathbb{R}^{n+1}$.

Corollary 5.10. Let $M$ be a Riemannian $n$-manifold. If there exists an integer $k \in [2, n]$ such that $\hat{\theta}_k(p) < 0$ at some point $p \in M$, then $M$ cannot be realized as an improper affine hypersphere in $\mathbb{R}^{n+1}$.

6. Some examples of centro-affine hypersurfaces

In this section we provide some examples of locally strongly convex centro-affine hypersurfaces. From these examples we know that the eigenvalue estimates given in Theorem 4.1 are best possible.

Example 6.1. Let $M$ be the elliptic locally strongly convex centro-affine hypersurface defined by:

$$e^{bs} \left( e^{(b-1)b}s, \sin(ax_2), \ldots, \sin(ax_n) \prod_{j=2}^{n-1} \cos(ax_j), \prod_{j=2}^{n} \cos(ax_j) \right),$$

with $a = \sqrt{1 - b^2}$, $b \in (0, 1)$. Then the affine metric $h$ on $M$ is

$$h = ds^2 + dx_2^2 + \cos^2(ax_2)dx_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2(ax_j)dx_n^2.$$
The Levi-Civita connection of \( h \) satisfies
\[
\hat{\nabla}_{\partial/\partial s} \frac{\partial}{\partial s} = \hat{\nabla}_{\partial/\partial x_k} \frac{\partial}{\partial x_k} = \hat{\nabla}_{\partial/\partial x_2} \frac{\partial}{\partial x_2} = 0,
\]
\[
\hat{\nabla}_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = -a \tan(ax_i) \frac{\partial}{\partial x_j}, \quad 2 \leq i < j,
\]
\[
\hat{\nabla}_{\partial/\partial x_j} \frac{\partial}{\partial x_j} = a \sum_{k=2}^{j-1} \left( \frac{\sin(2ax_k)}{2} \prod_{l=k+1}^{j-1} \cos^2(ax_l) \right) \frac{\partial}{\partial x_k}, \quad j = 3, \ldots, n.
\] (6.3)

It follows from (6.1) and (6.2) that \( \hat{K}_{ij} = 0 \) and \( \hat{K}_{jk} = a^2 \) for \( 2 \leq j \neq k \leq n \). Hence we have
\[
\hat{\theta}_n = \left( \frac{n-2}{n-1} \right) (1 - b^2).
\] (6.4)

On the other hand, from (6.1) and a straight-forward computation, we find
\[
\nabla_{\partial/\partial s} \frac{\partial}{\partial s} = \left( b + \frac{1}{b} \right) \frac{\partial}{\partial s}, \quad \nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_j} = b \frac{\partial}{\partial x_j},
\]
\[
\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = -a \tan(ax_i) \frac{\partial}{\partial x_j}, \quad 2 \leq i < j \leq n,
\]
\[
\nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_j} = b \prod_{i=2}^{j-1} \cos^2(ax_i) \frac{\partial}{\partial s} + a \sum_{k=2}^{j-1} \left( \frac{\sin(2ax_k)}{2} \prod_{l=k+1}^{j-1} \cos^2(ax_l) \right) \frac{\partial}{\partial x_k}
\] (6.5)

for \( j = 2, \ldots, n \). By applying (2.5), (6.3) and (6.5) we find
\[
K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = \left( b + \frac{1}{b} \right) \frac{\partial}{\partial s}, \quad K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial x_j} \right) = b \frac{\partial}{\partial x_j},
\]
\[
K \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j} \right) = b \prod_{i=2}^{j-1} \cos^2(ax_i) \frac{\partial}{\partial s}, \quad K \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0
\] (6.6)

for \( 2 \leq i \neq j \leq n \). Therefore we obtain from (1.3), (6.2) and (6.6) that
\[
T^\# = \left( b + \frac{1}{nb} \right) \frac{\partial}{\partial s}, \quad K_{T^\#} \left( \frac{\partial}{\partial x_j} \right) = \lambda_j \frac{\partial}{\partial x_j}, \quad \lambda_j = b^2 + \frac{1}{n}
\] (6.7)

for \( j = 2, \ldots, n \). Consequently, we conclude that the eigenvalue \( \lambda_j \) of the operator \( K_{T^\#} \) associated with eigenvector \( \partial/\partial x_j \) satisfies
\[
\lambda_j - \frac{n-1}{n} (1 - \hat{\theta}_n) = \frac{2b^2}{n} \rightarrow 0 \quad \text{as} \quad b \rightarrow 0.
\]

**Example 6.2.** Consider the hyperbolic locally strongly convex centro-affine hypersurface defined by:
\[
e^{bs} \left( e^{-(b-1+b)s}, \sinh(ax_2), \ldots, \sinh(ax_n) \prod_{j=2}^{n-1} \cosh(ax_j), \prod_{j=2}^{n} \cosh(ax_j) \right)
\] (6.8)
with \( a = \sqrt{1 + b^2}, \ b \in (0, \infty). \) The induced affine metric \( h \) of this hypersurface is given by

\[
h = ds^2 + dx_2^2 + \cosh^2(ax_2)dx_3^2 + \cdots + \prod_{j=2}^{n-1} \cosh^2(ax_j)dx_n^2,
\]

which implies that \( \hat{K}_1 = 0, \hat{K}_jk = -a^2 \) for \( 2 \leq j \neq k \leq n. \) Hence we have

\[
\hat{\theta}_n = \left( \frac{2 - n}{n - 1} \right) (1 + b^2).
\]

From (2.1), (2.5), (6.8) and a straightforward computation we find

\[
K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = \left( b - \frac{1}{b} \right) \frac{\partial}{\partial s}, \ K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial x_j} \right) = b \frac{\partial}{\partial x_j},
\]

\[
K \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_j} \right) = b \prod_{i=2}^{n-1} \cosh^2(ax_i) \frac{\partial}{\partial s}, \ K \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = 0
\]

for \( 2 \leq i \neq j \leq n. \) Therefore we have

\[
T^\# = \left( b - \frac{1}{nb} \right) \frac{\partial}{\partial s}, \ K_{T^\#} \left( \frac{\partial}{\partial x_j} \right) = \left( b^2 - \frac{1}{n} \right) \frac{\partial}{\partial x_j}, \ j = 2, \ldots, n.
\]

Consequently, the eigenvalue \( \lambda_j \) of the operator \( K_{T^\#} \) associated with eigenvector \( \partial/\partial x_j \) satisfies

\[
\lambda_j + \frac{n - 1}{n} (1 + \hat{\theta}_n) = \frac{2b^2}{n} \rightarrow 0 \text{ as } b \rightarrow 0.
\]

Examples 6.1 and 6.2 show that the eigenvalue estimate given in statement (1) of Theorem 4.1 is optimal for locally strongly convex centro-affine hypersurfaces of both elliptic and hyperbolic types.

**Example 6.3.** Consider the following elliptic centro-affine locally strongly convex hypersurface:

\[
5 \left( \sin x_1, \sin x_2 \cos x_1, \ldots, \sin x_{n-1} \prod_{j=1}^{n-2} \cos x_j, \right.
\]

\[
e^{-\frac{b}{2} (b + \sqrt{b^2 - 4})} x_n \prod_{j=1}^{n-1} \cos x_j, \ e^{\frac{b}{2} (b + \sqrt{b^2 - 4})} x_n \prod_{j=1}^{n-1} \cos x_j \right)
\]

with \( b > 2. \) The affine metric of this hypersurface is given by

\[
h = dx_1^2 + \cos^2 x_1 dx_2^2 + \cdots + \prod_{j=1}^{n-1} \cos^2 x_j dx_n^2.
\]
It follows from (6.14) that \( \hat{\theta}_k = 1 \) for \( k = 2, \ldots, n \).

On the other hand, from (6.13) and a direct computation, we have

\[
K \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = K \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_n} \right) = 0, \quad K \left( \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n} \right) = b \frac{\partial}{\partial x_n}
\]  
(6.15)

for \( 1 \leq i, j \leq n - 1 \), which ensures that

\[
T^\# = \left( \frac{b}{n} \prod_{j=1}^{n-1} \sec^2 x_j \right) \frac{\partial}{\partial x_n}, \quad K_{T^\#} \left( \frac{\partial}{\partial x_j} \right) = 0, \quad j = 1, \ldots, n - 1.
\]  
(6.16)

**Example 6.4.** Let \( M \) be the hyperbolic locally strongly convex centro-affine hypersurface defined by

\[
\left( \sinh x_1, \sinh x_2 \cosh x_1, \ldots, \sinh x_{n-1} \prod_{j=1}^{n-2} \cos x_j, \right.
\]

\[
e^{\frac{1}{2}(b+\sqrt{b^2+4})x_n} \prod_{j=1}^{n-1} \cosh x_j, \left. e^{\frac{1}{2}(b-\sqrt{b^2+4})x_n} \prod_{j=1}^{n-1} \cosh x_j \right),
\]

with nonzero \( b \). Since the induced affine metric is given by

\[
h = dx_1^2 + \cosh^2 x_2 dx_2^2 + \cdots + \prod_{j=1}^{n-1} \cosh^2 x_j dx_n^2,
\]  
(6.18)

thus we have \( \hat{\theta}_2 = \cdots = \hat{\theta}_n = -1 \).

On the other hand, by (6.17) and a straightforward computation, we find

\[
K \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = K \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_n} \right) = 0, \quad K \left( \frac{\partial}{\partial x_n}, \frac{\partial}{\partial x_n} \right) = b \frac{\partial}{\partial x_n}
\]  
(6.19)

for \( 1 \leq i, j \leq n - 1 \). Hence we obtain

\[
T^\# = \frac{b}{n} \prod_{j=1}^{n-1} \sech^2 x_j \frac{\partial}{\partial x_n}, \quad K_{T^\#} \left( \frac{\partial}{\partial x_j} \right) = 0, \quad j = 1, \ldots, n - 1.
\]  
(6.20)

Clearly, Examples 6.3 and 6.4 illustrate that the estimate given in statement (2) of Theorem 4.1 is optimal for locally strongly convex centro-affine hypersurfaces of both elliptic and hyperbolic types.

**7. Examples of graph hypersurfaces**

**Example 7.1.** Consider the graph hypersurface \( M \) in \( \mathbb{R}^{n+1} \):

\[
\left( u_2, \ldots, u_n, \frac{s^4}{4} + \sum_{j=2}^{n} \frac{u_j^2}{4} \right)
\]  
(7.1)
with constant affine normal $\xi$ given by $(0, \ldots, 0, -1)$ and Calabi metric given by $h = ds^2 + s^{-2}(du_1^2 + \cdots + du_n^2)$.

A direct computation shows that $\hat{K}_{1j} = -s^{-2}$ and $\hat{K}_{ij} = -1$ for $2 \leq i \neq j \leq n$. Thus we get

$$\hat{\theta}_2 = \cdots = \hat{\theta}_n = \begin{cases} -\frac{1}{s^2} & \text{if } s^2 \geq 1; \\ -1 & \text{if } s^2 < 1. \end{cases} \quad (7.2)$$

From (7.1) and a straightforward computation, we find

$$K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 3 \frac{\partial}{\partial s}, \quad K \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial u_j} \right) = \frac{1}{s} \frac{\partial}{\partial u_j}, \quad 2 \leq i \neq j \leq n, \quad (7.3)$$

which yields

$$T^# = \frac{(n+2)}{ns} \frac{\partial}{\partial s}, \quad K_{T^#} \left( \frac{\partial}{\partial u_j} \right) = \lambda_j \frac{\partial}{\partial u_j}, \quad \lambda_j = \frac{(n+2)}{ns^2} \quad (7.4)$$

for $j = 2, \ldots, n$. Hence we obtain

$$\lambda_j - \left( 1 - \frac{n}{n} \right) \hat{\theta}_k = \frac{3}{ns^2} \rightarrow 0 \quad \text{as } s \rightarrow \infty. \quad (7.5)$$

This example shows that the estimate given in statement (1) of Theorem 4.2 is optimal.

**Example 7.2.** Consider the graph hypersurface $M$ in $\mathbb{R}^{n+1}$:

$$\left( u_2, \ldots, u_n, e^{u_1}, u_1 - \frac{1}{2} \sum_{j=2}^{n} u_j^2 \right) \quad (7.6)$$

with affine normal $\xi = (0, \ldots, 0, -1)$ and Calabi metric $h = du_1^2 + \cdots + du_n^2$. Obviously, we have $\theta_2 = \cdots = \theta_n = 0$. It follows from (7.6) that

$$K \left( \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1} \right) = 0, \quad K \left( \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = 0 \quad (7.7)$$

for $i, j = 2, \ldots, n$. Thus we have

$$T^# = \frac{1}{n} \frac{\partial}{\partial u_1}, \quad K_{T^#} \left( \frac{\partial}{\partial u_j} \right) = 0 \quad (7.8)$$

for $j = 2, \ldots, n$.

The last example illustrates that the estimate given in statement (2) of Theorem 4.2 is optimal as well.

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References


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