Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 46 (2005), No. 2, 513-521.

Biharmonic Curves in the Generalized Heisenberg Group

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Abstract. We find the conditions under which a curve in the generalized Heisenberg group is biharmonic and non-harmonic. We give some existence and non-existence examples of such curves.

MSC 2000: 53C43, 53C42
Keywords: Biharmonic curves, Heisenberg group

1. Introduction

First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds, as they are presented in [2], [9] and in [5].

Let $f : M \to N$ be a smooth map between two Riemannian manifolds (M, g) and (N, h). Let $f^{-1}(TN)$ be the induced bundle over M of the tangent bundle, TN, defined as follows. Denote by $\pi : TN \to N$ the projection. Then

$$f^{-1}(TN) = \{(x, u) \in M \times TN, \pi(u) = f(x), x \in M\} = \bigcup_{x \in M} T_{f(x)}N.$$

The set of all C^∞-sections of $f^{-1}(TN)$, denoted by $\Gamma(f^{-1}(TN))$, is $\Gamma(f^{-1}(TN)) = \{V : M \to TN, C^\infty\text{-map}, V(x) \in T_{f(x)}N,\ x \in M\}$. Denote by ∇^M, ∇^N, the Levi-Civita connections on (M, g) and (N, h) respectively. For a smooth map f between (M, g) and (N, h), we define the induced connection ∇ on the induced bundle $f^{-1}(TN)$ as follows. For $X \in \chi(M)$, $V \in \Gamma(f^{-1}(TN))$, define $\nabla_X V \in \Gamma(f^{-1}(TN))$ by $\nabla_X V = \nabla^N_{f_* X} V$.

*The author wants to thank Professor Vasile Oproiu for his many valuable advices

The differential of the smooth map f can be viewed as a section of the bundle $\Lambda^1(f^{-1}(TN)) = T^*M \otimes f^{-1}(TN)$, and we denote by $|df|$ its norm at a point $x \in M$.

Suppose that M is a compact manifold. Define the energy density of f by $e(f) = \frac{1}{2}|df|^2$, and the energy of f by $E(f) = \int_M e(f) *1$, where $*1$ is the volume form on M. The map f is a harmonic map if it is a critical point of the energy, $E(f)$. In [9] it is proved that a map $f : M \to N$ is a harmonic map if and only if it satisfies the Euler-Lagrange equation $\tau(f) = 0$, where $\tau(f) = \operatorname{trace} \nabla df$ is an element of $\Gamma(f^{-1}(TN))$ called the tension field of f. The Laplacian acting on $\Gamma(f^{-1}(TN))$, induced by the connection ∇, is given by the Weitzenböck formula

$$\Delta V = -\operatorname{trace} \nabla^2 V,$$

for some $V \in \Gamma(f^{-1}(TN))$.

The bienergy of f is defined by $E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 * 1$. We say that f is a biharmonic map if it is a critical point of the bienergy, $E_2(f)$. It is proved in [5] that a map $f : M \to N$ is a biharmonic map if and only if it satisfies the equation $\tau_2(f) = 0$, where

$$\tau_2(f) = -\Delta\tau(f) - \operatorname{trace} R^N(df(\cdot), \tau(f))df(\cdot), \tag{1.1}$$

where R^N denotes the curvature tensor field on (N, h).

Note that any harmonic map is a biharmonic map and, moreover, an absolute minimum of the bienergy functional.

2. Generalized Heisenberg group

Consider $\mathbb{R}^{2n+1}$ with the elements of the form $X = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, z)$. Define the product on $\mathbb{R}^{2n+1}$ by

$$X\widetilde{X} = (x_1 + \widetilde{x}_1, y_1 + \widetilde{y}_1, \ldots, x_n + \widetilde{x}_n, y_n + \widetilde{y}_n, z + \widetilde{z} + \frac{1}{2}\sum_{i=1}^{n}(\widetilde{x}_i y_i - \widetilde{y}_i x_i)),$$

where $X = (x_1, y_1, \ldots, x_n, y_n, z)$, $\widetilde{X} = (\widetilde{x}_1, \widetilde{y}_1, \ldots, \widetilde{x}_n, \widetilde{y}_n, \widetilde{z})$.

Let $\mathbb{H}_{2n+1} = (\mathbb{R}^{2n+1}, g)$ be the generalized Heisenberg group endowed with the Riemannian metric g which is defined by

$$g = \sum_{i=1}^{n}(dx_i^2 + dy_i^2) + \Big[dz + \frac{1}{2}\sum_{i=1}^{n}(y_i dx_i - x_i dy_i)\Big]^2. \tag{2.1}$$

Note that the metric g is left invariant.

We can define a global orthonormal frame field in $\mathbb{H}_{2n+1}$ by

$$E_{2i-1} = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z}, \quad E_{2i} = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \quad E_{2n+1} = \frac{\partial}{\partial z},$$

for $i = 1, \ldots, n$. The Levi-Civita connection of the metric g is given by, (see [7] for the

3-dimensional case),

$$\begin{cases} \nabla_{E_{2i-1}}E_{2j-1} = 0, & \nabla_{E_{2i-1}}E_{2j} = \frac{1}{2}\delta_{ij}E_{2n+1}, \\ \nabla_{E_{2i}}E_{2j} = 0, & \nabla_{E_{2i}}E_{2j-1} = -\frac{1}{2}\delta_{ij}E_{2n+1}, \\ \nabla_{E_{2n+1}}E_{2i-1} = -\frac{1}{2}E_{2i}, & \nabla_{E_{2i-1}}E_{2n+1} = -\frac{1}{2}E_{2i}, \\ \nabla_{E_{2n+1}}E_{2i} = \frac{1}{2}E_{2i-1}, & \nabla_{E_{2i}}E_{2n+1} = \frac{1}{2}E_{2i-1}, \\ \nabla_{E_{2n+1}}E_{2n+1} = 0, & \end{cases} \tag{2.2}$$

for $i, j = 1, \ldots, n$. We have too

$$\begin{cases} [E_{2i-1}, E_{2j-1}] = 0, & [E_{2i}, E_{2j}] = 0, \\ [E_{2i-1}, E_{2n+1}] = 0, & [E_{2i}, E_{2n+1}] = 0, \\ [E_{2i-1}, E_{2j}] = \delta_{ij}E_{2n+1}. & \end{cases}$$

The curvature tensor field of ∇ is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

and Riemann-Christoffel tensor field is

$$R(X,Y,Z,W) = g(R(X,Y)W, Z),$$

where $X, Y, Z, W \in \chi(\mathbb{R}^{2n+1})$. We will use the notations

$$R_{abc} = R(E_a, E_b)E_c, \quad R_{abcd} = R(E_a, E_b, E_c, E_d),$$

where $a, b, c, d = 1, \ldots, 2n+1$. Then the non-zero components of the curvature tensor field and of the Riemann-Christoffel tensor field are, respectively

$$\begin{cases} R_{(2i-1)(2j-1)(2k)} = -\frac{1}{4}\delta_{jk}E_{2i} + \frac{1}{4}\delta_{ik}E_{2j}, \\ R_{(2i-1)(2j)(2k-1)} = \frac{1}{4}\delta_{jk}E_{2i} + \frac{1}{2}\delta_{ij}E_{2k}, \\ R_{(2i-1)(2j)(2k)} = -\frac{1}{4}\delta_{ik}E_{2j-1} - \frac{1}{2}\delta_{ij}E_{2k-1}, \\ R_{(2i-1)(2n+1)(2j-1)} = -\frac{1}{4}\delta_{ij}E_{2n+1}, \\ R_{(2i-1)(2n+1)(2n+1)} = \frac{1}{4}\delta_{ij}E_{2i-1}, \\ R_{(2i)(2j)(2k-1)} = -\frac{1}{4}\delta_{jk}E_{2i-1} + \frac{1}{4}\delta_{ik}E_{2j-1}, \\ R_{(2i)(2n+1)(2j)} = -\frac{1}{4}\delta_{ij}E_{2n+1}, \\ R_{(2i)(2n+1)(2n+1)} = \frac{1}{4}\delta_{ij}E_{2i}, \end{cases} \tag{2.3}$$

$$\begin{cases} R_{(2i-1)(2j-1)(2i)(2k)} = -\frac{1}{2}\delta_{jk} + \frac{1}{4}\delta_{ik}\delta_{ij}, \\ R_{(2i-1)(2j)(2i)(2k-1)} = \frac{1}{4}\delta_{jk} + \frac{1}{2}\delta_{ik}\delta_{ij}, \\ R_{(2i-1)(2j)(2k)(2k-1)} = \frac{1}{2}\delta_{ij} + \frac{1}{4}\delta_{ik}\delta_{jk}, \\ R_{(2i)(2j-1)(2j-1)(2k)} = \frac{1}{4}\delta_{ik} - \frac{1}{4}\delta_{jk}\delta_{ij}, \\ R_{(2i-1)(2n+1)(2n+1)(2j-1)} = -\frac{1}{4}\delta_{ij}, \\ R_{(2i)(2n+1)(2n+1)(2j)} = -\frac{1}{4}\delta_{ij}, \end{cases} \tag{2.4}$$

for $i, j, k = 1, \ldots, n$.

3. Biharmonic curves in $\mathbb{H}_{2n+1}$

Let $\gamma : I \to \mathbb{H}_{2n+1}$ be a non-inflexionar curve, parametrized by its arc length. Let $\{T, N_1, \dots, N_{2n}\}$ be the Frenet frame in $\mathbb{H}_{2n+1}$ defined along γ, where $T = \gamma'$ is the unit tangent vector field of γ, N_1 is the unit normal vector field of γ, with the same direction as $\nabla_T T$ and the vectors $N_1, \dots, N_{2n}$ are the unit vectors obtained from the following Frenet equations for γ.

$$\begin{cases} \nabla_T T & = & \chi_1 N_1, \\ \nabla_T N_1 & = & -\chi_1 T + \chi_2 N_2, \\ \dots & \dots & \dots \\ \nabla_T N_{2n-1} & = & -\chi_{2n-2} N_{2n-2} + \chi_{2n-1} N_{2n}, \\ \nabla_T N_{2n} & = & -\chi_{2n-1} N_{2n-1}, \end{cases} \tag{3.1}$$

where $\chi_1 = \|\nabla_T T\| = \|\tau(\gamma)\|$, and $\chi_2 = \chi_2(s), \dots, \chi_{2n} = \chi_{2n}(s)$ are real valued functions, where s is the arc length of γ. If $\chi_k \in \mathbb{R}$, $k = 1, \dots, 2n+1$ we say that γ is a helix.

The biharmonic equation of γ is

$$\tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T = 0. \tag{3.2}$$

Using the Frenet equations one obtains

$$\nabla_T^3 T = (-3\chi_1\chi_1')T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2)N_1 + (2\chi_1'\chi_2 + \chi_1\chi_2')N_2 + \chi_1\chi_2\chi_3 N_3. \tag{3.3}$$

Using (2.3) we get

$$R(T, \nabla_T T)T = \sum_{i=1}^{n} (\xi_{2i-1}E_{2i-1} + \xi_{2i}E_{2i}) + \xi_{2n+1}E_{2n+1},$$

with

$$\xi_{2i-1} = \frac{3}{4}T_{2i}\sum_{j=1}^{n}(-T_{2j-1}N_1^{2j} + T_{2j}N_1^{2j-1}) + \frac{1}{4}T_{2i-1}T_{2n+1}N_1^{2n+1} - \frac{1}{4}T_{2n+1}^2 N_1^{2i-1},$$

$$\xi_{2i} = \frac{3}{4}T_{2i-1}\sum_{j=1}^{n}(T_{2j-1}N_1^{2j} - T_{2j}N_1^{2j-1}) + \frac{1}{4}T_{2i}T_{2n+1}N_1^{2n+1} - \frac{1}{4}T_{2n+1}^2 N_1^{2i},$$

$$\xi_{2n+1} = \frac{1}{4}\sum_{j=1}^{n}(-T_{2j-1}^2 N_1^{2n+1} - T_{2j}^2 N_1^{2n+1} + T_{2j-1}T_{2n+1}N_1^{2j-1} + T_{2j}T_{2n+1}N_1^{2j}),$$

where $T = \sum_{a=1}^{2n+1} T_a E_a$ and $N_1 = \sum_{a=1}^{2n+1} N_1^a E_a$. After a straightforward computation, we have

$$R(T, \nabla_T T)T = \sum_{k=1}^{2n} \eta_k N_k, \tag{3.4}$$

with

$$\eta_1 = \frac{3}{4}\Big[\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\Big]^2 - \frac{1}{4}T_{2n+1}^2 - \frac{1}{4}(N_1^{2n+1})^2, \tag{3.5}$$

$$\eta_k = \frac{3}{4}\Big[\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\Big]\Big[\sum_{i=1}^{n}(T_{2i}N_k^{2i-1} - T_{2i-1}N_k^{2i})\Big] - -\frac{1}{4}N_1^{2n+1}N_k^{2n+1}, \quad (3.6)$$

where $N_k = \sum_{a=1}^{2n+1} N_k^a E_a$.

From (3.2), (3.3) and (3.4) it follows that the biharmonic equation of γ is

$$\tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T = (-3\chi_1\chi_1')T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2 - \chi_1\eta_1)N_1 +$$

$$(2\chi_1'\chi_2 + \chi_1\chi_2' - \chi_1\eta_2)N_2 + (\chi_1\chi_2\chi_3 - \chi_1\eta_3)N_3 - \chi_1\sum_{i=4}^{2n}\eta_k N_k,$$

where $\eta_a,\ a = 1, \ldots, 2n$ are given by (3.5) and (3.6). Hence

Theorem 3.1. *Let $\gamma : I \to \mathbb{H}_{2n+1}$ be a curve, parametrized by its arc length. Then γ is a biharmonic and non-harmonic curve if and only if*

$$\begin{cases} \chi_1 \in \mathbb{R} \setminus \{0\}, \\ \chi_1^2 + \chi_2^2 = -\eta_1, \\ \chi_2' = \eta_2, \\ \chi_2\chi_3 = \eta_3, \\ \eta_k = 0,\ k = 4, \ldots, 2n, \end{cases} \quad (3.7)$$

where $\eta_k,\ k = 1, \ldots, 2n$, are given by (3.5) and (3.6).

Corollary 3.2. *If $\chi_1 \in \mathbb{R} \setminus \{0\}$ and $\chi_2 = 0$ for a curve $\gamma : I \to \mathbb{H}_{2n+1}$, parametrized by its arc length, then γ is a biharmonic and non-harmonic curve if and only if $\chi_1^2 = -\eta_1$ and $\eta_k = 0,\ k = 2, \ldots, 2n$.*

Corollary 3.3. *Let $\gamma : I \to \mathbb{H}_{2n+1}$ be a curve, parametrized by its arc length. If $\eta_1 \geq 0$ then γ cannot be a biharmonic and non-harmonic curve.*

In [7] the following two results for the usual Heisenberg group, $\mathbb{H}_3$, are proved.

Theorem 3.4. *Let γ be the helix given by*

$$\gamma(s) = (r\cos(as), r\sin(as), c\ a\ s),$$

where $r > 0$, $\frac{1}{a^2} = r^2(1 + \frac{1}{4}r^2)$. Then γ is a biharmonic and non-geodesic curve.

Remark 3.5. In the case above if $r = \sqrt{\frac{1+\sqrt{5}}{2}}$, then γ is a biharmonic and non-harmonic curve with $\chi_2 = 0$.

In the case of the higher dimensions, we find a similar example related to Theorem 3.1. We consider a curve in $\mathbb{R}^{2n+1}$, given by

$$\gamma(s) = (c_1\cos(a_1 s), c_1\sin(a_1 s), \ldots, c_n\cos(a_n s), c_n\sin(a_n s), cs),$$

where $c_i > 0,\ a_i \neq 0,\ c \neq 0,\ i = 1, \dots, n$. Then one obtains

$$T(s) = \gamma'(s) = \sum_{i=1}^{n}[-c_i a_i \sin(a_i s)E_{2i-1} + c_i a_i \cos(a_i s)E_{2i}] + AE_{2n+1},$$

where $A = c - \frac{1}{2}\sum_{i=1}^{n} c_i^2 a_i$. From $\|T(s)\| = 1$ we have $A^2 + \sum_{i=1}^{n} c_i^2 a_i^2 = 1$. After a straightforward computation, using (2.2), one obtains

$$\nabla_T T = \sum_{i=1}^{n}[c_i a_i (A - a_i)\cos(a_i s)E_{2i-1} + c_i a_i (A - a_i)\sin(a_i s)E_{2i}].$$

From the first equation in (3.1) and from $\|N_1\| = 1$, we have

$$\chi_1 = \Big[\sum_{i=1}^{n} c_i^2 a_i^2 (A - a_i)^2\Big]^{1/2} \in \mathbb{R},$$

and

$$N_1 = \sum_{i=1}^{n} \frac{c_i a_i |A - a_i|}{\Big[\sum_{j=1}^{n} c_j^2 a_j^2 (A - a_j)^2\Big]^{1/2}}[\cos(a_i s)E_{2i-1} + \sin(a_i s)E_{2i}].$$

Note that $N_1^{2n+1} = 0$. Next, one obtains

$$\nabla_T N_1 + \chi_1 T = \frac{1}{2\chi_1}\sum_{i=1}^{n}\Big\{c_i a_i [|A - a_i|(A - 2a_i) - 2\chi_1^2]\Big\}[\sin(a_i s)E_{2i-1} -$$

$$\cos(a_i s)E_{2i}] + \frac{1}{2\chi_1}\Big[2\chi_1^2 A - \sum_{j=1}^{n} c_j^2 a_j^2 |A - a_i|\Big]E_{2n+1}.$$

From (3.1), using $\|N_2\| = 1$ we have $|\chi_2|^2 = \|\chi_2 N_2\|^2$.

In order to find a curve which satisfies conditions of Corollary 3.2 we assume that $\chi_2 = 0$ and $\chi_1^2 = -\eta_1$. From this conditions, after a straightforward computation we get

Proposition 3.6. *Let $\gamma : I \to \mathbb{H}_{2n+1}$ be the curve defined by*

$$\gamma(s) = (c_1\cos(a_1 s), c_1\sin(a_1 s), \dots, c_n\cos(a_n s), c_n\sin(a_n s), cs),$$

where $c_i > 0,\ a_i \neq 0,\ c \neq 0$. If $a_i = a = A - \frac{1}{2A}$, $\sum_{i=1}^{n} c_i^2 = \frac{4A^2(1-A^2)}{(2A^2-1)^2}$ and $c = \frac{A^3}{2A^2-1}$, where

$$A = \pm\Big[\frac{3n^2 - 1 + (9n^4 + 6n^2 + 5)^{1/2}}{6n^2 + 2}\Big]^{1/2},$$

then γ is a biharmonic and non-harmonic curve in $\mathbb{H}_{2n+1}$.

Note that, for such a curve and for $k \neq 1$, one obtains

$$\sum_{i=1}^{n}(T_{2i}N_k^{2i-1} - T_{2i-1}N_k^{2i}) = \frac{\chi_1}{|A-a|}\sum_{i=1}^{n}(N_1^{2i-1}N_k^{2i-1} + N_1^{2i}N_k^{2i}) = -\frac{\chi_1}{|A-a|}N_1^{2n+1}N_k^{2n+1} = 0.$$

That is $\eta_k = 0$, for any $k = 2, \dots, 2n$.

Also, note that the Proposition 3.6 is a generalization of the result in the Remark 3.5.

Next, one obtains

Proposition 3.7. *Let $\gamma : I \to \mathbb{H}_{2n+1}$ be the curve defined by*

$$\gamma(s) = (r\cos(as), r\sin(as), \ldots, r\cos(as), r\sin(as), cs),$$

where $r > 0$, $a_i \neq 0$, $c \neq 0$. If $\chi_2 = 0$, $\chi_1 \neq 0$ and γ is biharmonic then $n = 1$.

In the following we obtain a class of biharmonic and non-harmonic curves for which the second curvature does not necessarily vanish, (see[1] for the similar result in 3-dimensional case).

Proposition 3.8. *Let $\gamma : I \to \mathbb{H}_{2n+1}$, $\gamma(s) = (x_1(s), y_1(s), \ldots, x_n(s), y_n(s), z(s))$, be the curve with the parametric equations*

$$\begin{cases} x_i(s) &= \frac{1}{\beta}\frac{\sin\alpha}{\sqrt{n}}\sin(\beta s + a_i) + b_i, \\ y_i(s) &= -\frac{1}{\beta}\frac{\sin\alpha}{\sqrt{n}}\cos(\beta s + a_i) + c_i, \\ z(s) &= \Big(\cos\alpha + \frac{(\sin\alpha)^2}{2\beta}\Big)s - \sum_{i=1}^n \frac{b_i}{2\beta\sqrt{n}}\sin\alpha\cos(\beta s + a_i) \\ & \quad - \sum_{i=1}^n \frac{c_i}{2\beta\sqrt{n}}\sin\alpha\cos(\beta s + a_i) + d, \end{cases} \tag{3.8}$$

with $i = 1, \ldots, n$, where $\beta = \frac{cos\alpha \pm \sqrt{5(\cos\alpha)^2 - 4}}{2}$, $\alpha \in (0, \arccos\frac{2\sqrt{5}}{5}] \cup [\arccos(-\frac{2\sqrt{5}}{5}), \pi)$ and a_i, b_i, c_i, $d \in \mathbb{R}$. Then γ is a biharmonic and non-harmonic curve.

Proof. The covariant derivative of the unit tangent vector field, T, of γ, is

$$\nabla_T T = \sum_{i=1}^n [(T'_{2i-1} + T_{2i}T_{2n+1})E_{2i-1} + (T'_{2i} - T_{2i-1}T_{2n+1})E_{2i}] + T'_{2n+1}E_{2n+1},$$

and T is given by

$$T(s) = \gamma'(s) = \frac{\sin\alpha}{\sqrt{n}}\sum_{i=1}^n [\cos(\beta s + a_i)E_{2i-1} + \sin(\beta s + a_i)E_{2i}] + \cos\alpha E_{2n+1}.$$

Taking into account the first Frenet equation one obtains

$$\chi_1 = |\sin\alpha(\cos\alpha - \beta)|$$

and, since we can assume, without loss of generality, that $\sin\alpha(\cos\alpha - \beta) > 0$, we have

$$N_1 = \sum_{i=1}^n \Big(\frac{\sin(\beta s + a_i)}{\sqrt{n}}E_{2i-1} - \frac{\cos(\beta s + a_i)}{\sqrt{n}}E_{2i}\Big).$$

After a straightforward computation one obtains that $\eta_1 = (\sin\alpha)^2 - \frac{1}{4}$ and $\eta_k = 0$, for any $k = 2, \ldots, 2n$.

In order to find χ_2 we obtain, using the equations (2.2),

$$\nabla_T N_1 + \chi_1 T = \sum_{i=1}^n \Big\{\frac{\cos(\beta s + a_i)}{\sqrt{n}}\Big[\beta - \frac{1}{2}\cos\alpha + (\cos\alpha - \beta)(\sin\alpha)^2\Big]E_{2i-1}$$

$$+\frac{\sin(\beta s+a_i)}{\sqrt{n}}\Big[\beta-\frac{1}{2}\cos\alpha+(\cos\alpha-\beta)(\sin\alpha)^2\Big]E_{2i}\Big\}-\sin\alpha\Big[\frac{1}{2}-(\cos\alpha-\beta)\cos\alpha\Big]E_{2n+1}.$$

Then, using Frenet equations, we have

$$\chi_2^2=\|\nabla_T N_1+\chi_1 T\|^2=\beta^2-\beta\cos\alpha+\frac{1}{4}-(\sin\alpha)^2(\cos\alpha-\beta)^2.$$

Hence χ_2 is a constant and, from hypothesis, one obtains

$$\chi_1^2+\chi_2^2=\frac{1}{4}-(\sin\alpha)^2=-\eta_1.$$

Since $\chi_2 N_2=\nabla_T N_1+\chi_1 T$ and $\chi_3^2=\|\nabla_T N_2+\chi_2 N_1\|^2$, we obtain, after a straightforward computation, that $\chi_3=0$.

Hence, all conditions from Theorem 3.1 are verified by γ and then γ is a biharmonic and non-harmonic curve.

Remark 3.9. In the same way as above it is easy to see that all biharmonic and non-harmonic curves in $\mathbb{H}_{2n+1}$ with constant second curvature and with the unit tangent vector field, T, of the form

$$T(s)=\frac{\sin\alpha}{\sqrt{n}}\sum_{i=1}^{n}[\cos(f_i(s))E_{2i-1}+\sin(f_i(s))E_{2i}]+\cos\alpha E_{2n+1},$$

where f_i are some smooth functions of the arc length, such that $f_i'=f_j'$, for any $i,j=1,\ldots,n$, and $\alpha\in\mathbb{R}$, are given by Proposition 3.8.

Finally, we have

Proposition 3.10. *Let $\gamma : I\to\mathbb{H}_{2n+1}$ be the curve defined by*

$$\gamma(s)=(c_1 s,c_2 s,\ldots,c_{2n+1}s)$$

with $\sum_{j=1}^{2n+1}c_j^2=1$. Then γ is biharmonic if and only if is harmonic.

Proof. We have

$$T(s)=\gamma'(s)=\sum_{j=1}^{2n+1}c_jE_j,\ \|T\|=1,\ \nabla_T T=c_{2n+1}\sum_{i=1}^{n}(c_{2i}E_{2i-1}-c_{2i-1}E_{2i}).$$

It follows that $\chi_1=c_{2n+1}\sqrt{\sum_{i=1}^{n}(c_{2i-1}^2+c_{2i}^2)}$, and

$$N_1=\sum_{i=1}^{n}\Bigg[\frac{c_{2i}}{\sqrt{\sum_{j=1}^{n}(c_{2j-1}^2+c_{2j}^2)}}E_{2i-1}-\frac{c_{2i-1}}{\sqrt{\sum_{j=1}^{n}(c_{2j-1}^2+c_{2j}^2)}}E_{2i}\Bigg].$$

One obtains

$$\nabla_T N_1+\chi_1 T=\frac{\sum_{k=1}^{n}(c_{2k-1}^2+c_{2k}^2)-c_{2n+1}^2}{2}\cdot\Bigg[\sum_{i=1}^{n}\Bigg(\frac{c_{2i-1}c_{2n+1}}{\sqrt{\sum_{j=1}^{n}(c_{2j-1}^2+c_{2j}^2)}}E_{2i-1}+$$

$$\frac{c_{2i}c_{2n+1}}{\sqrt{\sum_{j=1}^{n}(c_{2j-1}^2+c_{2j}^2)}}E_{2i}\Bigg) - \sqrt{\sum_{j=1}^{n}(c_{2j-1}^2+c_{2j}^2)}E_{2n+1}\Bigg].$$

That means

$$\chi_2 = \frac{\sum_{k=1}^{n}(c_{2k-1}^2+c_{2k}^2) - c_{2n+1}^2}{2}.$$

Thus $\chi_1^2+\chi_2^2 = \frac{1}{4}$. But, one obtains that $\eta_1 = \frac{3}{4} - c_{2n+1}^2$, and from (3.7) we have that if γ is biharmonic then $\chi_1^2+\chi_2^2 = -\eta_1 = -\frac{3}{4}+c_{2n+1}^2$. Thus if γ is biharmonic then $\chi_1 = 0$, and then γ is a harmonic curve.

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Received March 23, 2004