Ikeda-Nakayama Modules*

Robert Wisbauer  Mohamed F. Yousif  Yiqiang Zhou

Heinrich-Heine-University, 40225 Düsseldorf, Germany
e-mail: wisbauer@math.uni-duesseldorf.de

The Ohio State University, Lima Campus, Ohio 45804, USA
e-mail: yousif.1@osu.edu

Memorial University of Newfoundland, St.John’s, NF A1C 5S7, Canada
e-mail: zhou@math.mun.ca

Abstract. Let $S M_R$ be an $(S, R)$-bimodule and denote $l_S(A) = \{s \in S : sA = 0\}$ for any submodule $A$ of $M_R$. Extending the notion of an Ikeda-Nakayama ring, we investigate the condition $l_S(A \cap B) = l_S(A) + l_S(B)$ for any submodules $A, B$ of $M_R$. Various characterizations and properties are derived for modules with this property. In particular, for $S = \text{End}(M_R)$, the $\pi$-injective modules are those modules $M_R$ for which $S = l_S(A) + l_S(B)$ whenever $A \cap B = 0$, and our techniques also lead to some new results on these modules.

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1. Annihilator conditions

Let $R$ and $S$ be rings and $S M_R$ be a bimodule. For any $X \subseteq M$ and any $T \subseteq S$, denote

\[ l_S(X) = \{s \in S : sX = 0\} \quad \text{and} \quad r_M(T) = \{m \in M : Tm = 0\}. \]

There is a canonical ring homomorphism $\lambda : S \to \text{End}(M_R)$ given by $\lambda(s)(x) = sx$ for $x \in M$ and $s \in S$. For any submodules $A$ and $B$ of $M_R$ and any $t \in l_S(A \cap B)$, define

\[ \alpha_t : A + B \to M, \quad a + b \mapsto ta. \]

Clearly, $\alpha_t$ is a well-defined $R$-homomorphism.

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Lemma 1. Let $sM_R$ be a bimodule and $A, B$ be submodules of $M_R$. The following are equivalent:

1. $l_S(A \cap B) = l_S(A) + l_S(B)$.
2. For any $t \in l_S(A \cap B)$, the diagram

$$
\begin{array}{ccc}
0 & \to & A + B & \to & M \\
& & \downarrow \alpha_t & & \\
& & M & & \\
\end{array}
$$

can be extended commutatively by $\lambda(s)$, for some $s \in S$.

Proof. $(1) \Rightarrow (2)$. Suppose $(1)$ holds. For $A, B, t$ given as in $(2)$, write $t = u + v$ where $u \in l_S(A)$ and $v \in l_S(B)$. Then, for all $a \in A$ and $b \in B$,

$$
\alpha_t(a + b) = ta = (u + v)a = va = v(a + b) = \lambda(v)(a + b).
$$

$(2) \Rightarrow (1)$. It is clear that $l_S(A \cap B) \supseteq l_S(A) + l_S(B)$. Let $t \in l_S(A \cap B)$. Define $\alpha_t : A + B \to M$ as above. By $(2)$, there exists $s \in S$ such that $\lambda(s)$ extends $\alpha_t$.

Thus, for all $a \in A$ and $b \in B$, $ta = \alpha_t(a + b) = \lambda(s)(a + b) = s(a + b)$. It follows that $(t - s)a + (-s)b = 0$ for all $a \in A$ and $b \in B$. So, $t - s \in l_S(A)$ and $-s \in l_S(B)$, and hence $t = (t - s) - (-s) \in l_S(A) + l_S(B)$. \hfill \Box

Lemma 2. Let $sM_R$ be a bimodule and $A, B$ be submodules of $M_R$ such that $A \cap B = 0$. The following are equivalent:

1. $S = l_S(A) + l_S(B)$.
2. The diagram

$$
\begin{array}{ccc}
0 & \to & A + B & \to & M \\
& & \downarrow \alpha_1 & & \\
& & M & & \\
\end{array}
$$

can be extended commutatively by $\lambda(s)$, for some $s \in S$.

Proof. $(1) \Rightarrow (2)$. Apply Lemma 1 with $t = 1$.

$(2) \Rightarrow (1)$. It suffices to show that $1 \in l_S(A) + l_S(B)$. Note that $\alpha_1 : A + B \to M$ is given by $\alpha_1(a + b) = a$ ($a \in A$ and $b \in B$). By $(2)$, there exists $s \in S$ such that $\lambda(s)$ extends $\alpha_1$. Arguing as in the proof of ‘$(2) \Rightarrow (1)$’ of Lemma 1, we have $1 = (1 - s) - (-s) \in l_S(A) + l_S(B)$. \hfill \Box

Lemma 3. Let $sM_R$ be a bimodule such that $sM$ is faithful and $A, B$ be complements of each other in $M_R$. The following are equivalent:

1. $S = l_S(A) + l_S(B)$.
2. $S = l_S(A) \oplus l_S(B)$.
3. $M = A \oplus B$ and, for the projection $f$ of $M$ onto $A$ along $B$, $f = \lambda(s)$ for some $s \in S$. 

Proof. (1) ⇒ (3). By (1), we have \( S = 1_S(A) + 1_S(B) \). Write \( S = u + v \) where \( u \in 1_S(A) \) and \( v \in 1_S(B) \). It follows that \( a = uv \) for all \( a \in A \), \( b = ub \) for all \( b \in B \) and \( vB = uA = 0 \). Thus, \( B \subseteq r_M(v^2) \subseteq r_M(v^2) \cap A = 0 \). Since \( B \) is complement of \( A \) in \( M_R \), we have \( B = r_M(v) = r_M(v^2) \). Similarly, \( A = r_M(u) = r_M(u^2) \). Next we show that \((vu)M \cap (A + B) = 0\). For any \( z \in (vu)M \cap (A + B) \), write \( z = vu = a + b \), where \( x \in M \), \( a \in A \) and \( b \in B \). Noting that \( vu = uv \), we have that \( v^2u^2x = (vu)(a + b) = 0 \). So, \( u^2x \in r_M(v^2) = r_M(v) \), and this gives that \( u^2vx = vu^2x = 0 \). So, \( vu \in r_M(u^2) = r_M(u) \). Thus, \( z = vu = uB = 0 \). So, \((vu)M \cap (A + B) = 0\). Since \( A + B \) is essential in \( M_R \), \((vu)M = 0 \), and hence \( vu = 0 \) since \( S \) is faithful. So, \( uM \subseteq r_M(v) = B \) and \( vM \subseteq r_M(u) = A \), and hence \( M = vM + uM = A + B = A \oplus B \).

Let \( f \) be the projection of \( M \) onto \( A \) along \( B \). Then \( f(M) = A \) and \( (1 - f)(M) = B \). Noting that \( S \) is faithful, we have \( 1_S(A) = 1_S(f(M)) = \{ s \in S : \lambda(s)f(M) = 0 \} = \{ s \in S : \lambda(s)f = 0 \} \) and \( 1_S(B) = 1_S((1 - f)(M)) = \{ s \in S : \lambda(s)(1 - f) = 0 \} \). Thus, \( \lambda(u)f = 0 \) and \( \lambda(v)(1 - f) = 0 \). It follows that

\[
0 = \lambda(v)(1 - f) = \lambda(1 - u)(1 - f) = (1 - \lambda(u))(1 - f) = 1 - f - \lambda(u),
\]

and thus \( f = 1 - \lambda(u) = 1 - u = \lambda(v) \).

(3) ⇒ (2). By (3), \( M = A \oplus B \). Let \( f \) be the projection of \( M \) onto \( A \) along \( B \). Then \( f^2 = f \in End(M_R) \), \( A = f(M) \) and \( B = (1 - f)(M) \). By (3), \( f = \lambda(s) \) for some \( s \in S \). It follows that \( (s^2 - s)M = \lambda(s^2 - s)(M) = (f^2 - f)(M) = 0 \). So, \( s^2 = s \), since \( S \) is faithful. And so,

\[
1_S(A) = 1_S(f(M)) = 1_S(s)(M) = 1_S(s) = S(1 - s),
\]

and, similarly, \( 1_S(B) = Ss \). Thus, \( S = 1_S(A) \oplus 1_S(B) \).

(2) ⇒ (1). Obvious. \( \square \)

A module \( M_R \) is called \( \pi \)-injective (or \( \text{quasi-continuous} \)) if every submodule is essential in a direct summand (C1) and, for any two direct summands \( M_1, M_2 \) with \( M_1 \cap M_2 = 0 \), \( M_1 \oplus M_2 \) is also a direct summand (C3) (see [8]). It is known that \( M_R \) is \( \pi \)-injective if and only if \( M = A \oplus B \) whenever \( A \) and \( B \) are complements of each other in \( M_R \) (see [8, Theorem 2.8])

**Corollary 4.** Let \( S \) be a bimodule such that \( S \) is faithful. The following are equivalent:

1. For any submodules \( A \) and \( B \) of \( M_R \) with \( A \cap B = 0 \), \( S = 1_S(A) + 1_S(B) \).
2. If \( A \) and \( B \) are complements of each other in \( M_R \), then \( S = 1_S(A) + 1_S(B) \).
3. If \( A \) and \( B \) are complements of each other in \( M_R \), then \( S = 1_S(A) \oplus 1_S(B) \).
4. \( M \) is \( \pi \)-injective and, for any \( f^2 = f \in End(M_R) \), \( f = \lambda(s) \) for some \( s \in S \).

Proof. (1) ⇔ (2) is obvious, and (2) ⇔ (3) ⇔ (4) is by Lemma 3. \( \square \)

For submodules \( A, B \) of \( M_R \), let

\[
\pi : M/(A \cap B) \rightarrow M/A \oplus M/B, \quad m + (A \cap B) \mapsto (m + A, m + B)
\]

be the canonical \( R \)-homomorphism. The next lemma can easily be verified.
Lemma 5. Let $M_R$ be an $R$-module with $S = \text{End}(M_R)$ and $A, B$ be submodules of $M_R$. The following are equivalent:

1. $l_S(A \cap B) = l_S(A) + l_S(B)$.

2. For any $R$-homomorphism $f : M/(A \cap B) \rightarrow M$, the diagram

$$
\begin{array}{ccc}
0 & \rightarrow & M/(A \cap B) \\
& & \pi \\
& & \downarrow f \\
& & M
\end{array}
$$


\[
\text{can be extended commutatively by some } g : M/A \oplus M/B \rightarrow M.
\]

2. Ikeda-Nakayama modules

A well known result of Ikeda and Nakayama [6] says that every right self-injective ring $R$ satisfies the so called Ikeda-Nakayama annihilator condition, i.e., $l_R(A \cap B) = l_R(A) + l_R(B)$ for all right ideals $A, B$ of $R$. Rings with the Ikeda-Nakayama annihilator condition, called right Ikeda-Nakayama rings, were studied in [2]. Extending this notion we call $M_R$ an Ikeda-Nakayama module (IN-module) if

$$l_S(A \cap B) = l_S(A) + l_S(B)$$

for any submodules $A$ and $B$ of $M_R$ where $S = \text{End}(M_R)$. Clearly, every quasi-injective module is an IN-module (Lemma 1) and every IN-module is $\pi$-injective (Corollary 4).

Proposition 6. The following are equivalent for a module $M_R$ with $S = \text{End}(M_R)$:

1. $M_R$ is an IN-module.

2. For any finite set $\{A_i : i = 1, \ldots, n\}$ of submodules of $M_R$,

$$l_S(A_1 \cap \cdots \cap A_n) = l_S(A_1) + \cdots + l_S(A_n).$$

3. For any submodules $A, B$ of $M_R$ and any $f \in S$ with $f(A \cap B) = 0$, the diagram

$$\begin{array}{ccc}
0 & \rightarrow & A + B \\
& & \downarrow \alpha_f \\
& & M
\end{array}$$

\[
\text{can be extended commutatively by some } g : M \rightarrow M.
\]

4. For any submodules $A, B$ of $M_R$ and any $R$-homomorphism $f : M/(A \cap B) \rightarrow M$, the diagram

$$\begin{array}{ccc}
0 & \rightarrow & M/(A \cap B) \\
& & \pi \\
& & \downarrow f \\
& & M
\end{array}$$

\[
\text{can be extended commutatively by some } g : M/A \oplus M/B \rightarrow M.
\]
Proof. (1) ⇒ (2) can be easily proved by using induction on \( n \); (2) ⇒ (1) is obvious; (1) ⇔ (3) is by Lemma 1; and (1) ⇔ (4) is by Lemma 5.

Remark 7. The equivalences (1) ⇔ (2) ⇔ (3) in Proposition 6 can be proved to hold for an arbitrary bimodule \( _SM_R \).

Many characterizations of \( \pi \)-injective modules are given in [13, 41.21 & 41.23]. In particular, the equivalence “(1) ⇔ (2)” of the next theorem is contained in [13, 41.21].

Theorem 8. The following are equivalent for a module \( M_R \) with \( S = \text{End}(M_R) \):

1. \( M \) is \( \pi \)-injective.
2. For any submodules \( A \) and \( B \) of \( M_R \) with \( A \cap B = 0 \), \( S = l_S(A) + l_S(B) \).
3. For any submodules \( A \) and \( B \) of \( M_R \) with \( A \cap B = 0 \) and any \( f \in S \), the diagram
   
   \[
   \begin{array}{ccc}
   0 & \rightarrow & A + B \\
   & \downarrow \alpha_f & \\
   & M & \\
   \end{array}
   \]

   can be extended commutatively by some \( g : M \rightarrow M \).
4. For any submodules \( A, B \) of \( M_R \) with \( A \cap B = 0 \), the diagram
   
   \[
   \begin{array}{ccc}
   0 & \rightarrow & A + B \\
   & \downarrow \alpha_1 & \\
   & M & \\
   \end{array}
   \]

   can be extended commutatively by some \( g : M \rightarrow M \).
5. For any submodules \( A, B \) of \( M_R \) with \( A \cap B = 0 \) and any \( f \in S \), the diagram
   
   \[
   \begin{array}{ccc}
   0 & \rightarrow & M \\
   & \downarrow f & \\
   & M/A \oplus M/B & \\
   \end{array}
   \]

   can be extended commutatively by some \( g : M/A \oplus M/B \rightarrow M \).
6. For any submodules \( A \) and \( B \) of \( M_R \) with \( A \cap B = 0 \), \( S_0 = l_{S_0}(A) + l_{S_0}(B) \) where \( S_0 \) is the subring of \( S \) generated by all idempotents of \( S \).
7. If \( A \) and \( B \) are complements of each other in \( M_R \), then \( S = l_S(A) \oplus l_S(B) \).

In each of the conditions (2)–(6), the pair \( A, B \) of submodules with \( A \cap B = 0 \) can be replaced by a pair \( A, B \) of submodules such that they are complements of each other in \( M_R \).

Proof. (2) ⇔ (3) ⇔ (4) ⇔ (5): By Lemmas 1, 2 and 5.

(1) ⇔ (2) ⇔ (7): By Corollary 4.

(1) ⇔ (6): Apply Corollary 4 to the bimodule \( _{S_0}M_R \).

One condition in the equivalence list of Theorem 8 says that, if \( A, B \) are complements of each other in \( M_R \), then the map \( \alpha_1 : A \oplus B \rightarrow M \) given by \( \alpha_1(a + b) = a \) extends to \( M \). This is an improvement of a result of Smith and Tercan [11, Thm.4] where it was proved that \( M_R \) is \( \pi \)-injective if and only if \( M \) satisfies \((P_2)\), i.e., if \( A \) and \( B \) are complement submodules of \( M \) with \( A \cap B = 0 \), then every map from \( A \oplus B \) to \( M \) extends to \( M \).
**Remark 9.** Two modules $X$ and $Y$ are said to be *orthogonal* and written $X \perp Y$ if they have no nonzero isomorphic submodules. A submodule $N$ of the module $M$ is called a *type submodule* if, whenever $N \subseteq P \subseteq M$, there exists $0 \neq X \subseteq P$ such that $N \perp X$. Two submodules $X$ and $Y$ of $M$ are said to be *type complements of each other* in $M$ if they are complements of each other in $M$ such that $X \perp Y$. The module $M$ is called TS if each of its type submodules is a direct summand of $M$. The module $M$ is said to satisfy $(T_3)$ if, whenever $X$ and $Y$ are type submodules as well as direct summands such that $X \perp Y$ is essential in $M$, $X \oplus Y = M$. As shown in [14], a module $M$ satisfies both TS and $(T_3)$ if and only if, whenever $A, B$ are type complements of each other in $M$, $M = A \oplus B$. The module satisfying TS and $(T_3)$ can be regarded as the ‘type’ analogue of the notion of $\pi$-injective modules. Several characterizations of this ‘type’ analogue of $\pi$-injective modules have been obtained in [14]. Some new characterizations of this notion can be obtained by restating Theorem 8 with ‘$A \cap B = 0$’ being replaced by ‘$A \perp B$’, ‘$A, B$ are complements of each other in $M$’ replaced by ‘$A, B$ are type complements of each other in $M$’, and “all idempotents of $S$” by “all idempotents $f$ with $f(M) \perp \text{Ker}(f)$”.

**Proposition 10.** Let $C$ be the center of $\text{End}(M_R)$. The following are equivalent:

1. For any submodules $A, B$ of $M_R$ with $A \cap B = 0$, $C = \text{I}_C(A) + \text{I}_C(B)$.
2. $M_R$ is $\pi$-injective and every idempotent of $\text{End}(M_R)$ is central.
3. $M_R$ is $\pi$-injective and every direct summand of $M_R$ is fully invariant.

**Proof.** (1) $\iff$ (2). Apply Corollary 4 to the bimodule $CM_R$.

(2) $\Rightarrow$ (3). Let $X$ be a direct summand of $M_R$. Then $X = f(M)$ for some $f^2 = f \in \text{End}(M_R)$. For any $g \in \text{End}(M_R)$, since $f$ is central by (2), $g(X) = g(f(M)) = f(g(M)) \subseteq f(M) = X$. This shows that $X$ is a fully invariant submodule of $M_R$.

(3) $\Rightarrow$ (2). Let $f, g \in \text{End}(M_R)$ with $f^2 = f$. By (3), $g(f(M)) \subseteq f(M)$ and $g((1 - f)(M)) \subseteq (1 - f)(M)$. It follows that $fgf = gf$ and $(1 - f)g(1 - f) = g(1 - f)$. Thus, $g - gf = g(1 - f) = (1 - f)g(1 - f) = g - gf - fg + fg = g - gf - fg + gf = g - fg$. This shows that $fg = gf$. \hfill $\square$

### 3. Applications

In the rest of the paper, we discuss some applications of Theorem 8. Recall that a module $M$ is called *continuous* if (C1) holds and every submodule isomorphic to a direct summand is itself a direct summand of $M$ (C2). As a generalization of (C2)-condition, a module $M_R$ is called $GC^2$ if, for any submodule $N$ of $M_R$ with $N \cong M$, $N$ is a summand of $M$. Note that if $R$ is the $2 \times 2$ upper triangular matrix ring over a field, then $R_R$ satisfies both (C1) and (GC2) but it does not satisfy (C3).

**Proposition 11.** Let $M_R$ be a module with $S = \text{End}(M_R)$. The following are equivalent:

1. For any family $\{A_i : i \in I\}$ of submodules of $M_R$ with $\cap_{i \in I} A_i = 0$, $S = \Sigma_{i \in I} \text{I}_S(A_i)$.
2. $M_R$ is finitely cogenerated and, for any finite family $\{A_i : i = 1, \ldots, n\}$ of submodules of $M_R$ with $\cap_{i = 1}^n A_i = 0$, the map

$$M \overset{h}{\to} \bigoplus_{i = 1}^n M/A_i, \quad m \mapsto (m + A_1, \ldots, m + A_n),$$
Proof. It is straightforward to verify the equivalences (1) ⇔ (2) ⇔ (3).

Suppose that $M_R$ satisfies both (1) and (GC2). By Theorem 8, $M_R$ is $\pi$-injective. Thus, by [8, Lemma 3.14], $M$ is continuous. To show that $S$ is semilocal, let $\sigma : M \rightarrow M$ be a monomorphism. Then $M = \sigma(M) \oplus N$ for some $N \subseteq M$ (by the GC2-condition). It must be that $N = 0$ since $M$ is finite dimensional (indeed, finitely cogenerated). So, $\sigma$ is an isomorphism. Therefore, $M$ satisfies the assumptions in Camps-Dicks [3, Thm.5], and so $\text{End}(M)$ is semilocal. But, by [8, Prop.3.5 & Lemma 3.7], idempotents of $S/J(S)$ lift to idempotents of $S$, and thus $S$ is semiperfect. □

A ring $R$ is called right Kasch if every simple right $R$-module embeds in $R_R$, or equivalently if $I(I) \neq 0$ for any maximal right ideal $I$ of $R$.

Corollary 12. If $R$ satisfies the condition that, for any set $\{A_i : i \in I\}$ of right ideals such that $\cap_{i \in I} A_i = 0$, $R = \Sigma_{i \in I} R(A_i)$ and $R_R$ satisfies (GC2), then $R$ is a semiperfect right continuous ring with a finitely generated essential right socle. In particular, $R$ is left and right Kasch.

Proof. The first part follows from Theorem 11. The second part is by [9, Lemma 4.16]. □

A ring $R$ is called strongly right IN if, for any set $\{A_i : i \in I\}$ of right ideals, $I_R(\cap_{i \in I} A_i) = \Sigma_{i \in I} I_R(A_i)$. The ring $R$ is called right dual if every right ideal of $R$ is a right annihilator. It is well-known that every two-sided dual ring is strongly left and right IN.

Corollary 13. The following are equivalent for a ring $R$:

1. $R$ is a two-sided dual ring.
2. $R$ is strongly left and right IN, and left (or right) GC2.
3. $R$ is left and right finitely cogenerated, left and right IN, and left (or right) GC2.


(2) ⇒ (3): It is clear by Corollary 12.

(3) ⇒ (1): Suppose $\cap_{i \in I} A_i = 0$ where all $A_i$ are right ideals $R$. Since $R$ is right finitely cogenerated, $\cap_{i \in F} A_i = 0$ where $F$ is a finite subset of $I$. Thus, $R = I_R(\cap_{i \in F} A_i) = \Sigma_{i \in F} I_R(A_i)$ because of the IN-condition, and hence $R = \Sigma_{i \in I} I_R(A_i)$. By Corollary 12, $R$ is left and right Kasch. Since $R$ is left and right IN, it follows from [2, Lemma 9] that $R$ is a two-sided dual ring. □

The GC2-condition in Corollary 12 and in Corollary 13(3) can not be removed. To see this, let $R$ be the trivial extension of $\mathbb{Z}$ and the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$. Then $R$ has an essential minimal ideal, so $R$ is finitely cogenerated and, for any set $\{A_i : i \in I\}$ of right ideals of $R$, $R = \Sigma_{i \in I} I_R(A_i)$. Moreover, $R$ is IN. But $R$ contains non-zero divisors which are not invertible, so $R$ is not GC2. Clearly, $R$ is not Kasch, so it is not semiperfect by Corollary 12. We do not know if the GC2-condition can be removed in Corollary 13(2).
Proposition 14. Suppose every finitely generated left ideal of $R$ is a left annihilator. Then the following are equivalent:

1. Every closed right ideal of $R$ is a right annihilator of a finite subset of $R$.
2. $R_R$ satisfies (C1).
3. $R$ is right continuous.

Proof. (3) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (1): If $I_R$ is closed in $R_R$, then $I = eR$ for some $e^2 = e \in R$. Hence $I = r(1 - e)$.
(1) $\Rightarrow$ (2): Let $I_R$ and $K_R$ be complements of each other in $R_R$. Then, by (1), $I = r_R(a_1, \ldots, a_n)$ and $K = r_R(b_1, \ldots, b_m)$ where $a_i, b_j \in R$. Thus,

$$R = I_R(I \cap K) = I_R[r_R(a_1, \ldots, a_n) \cap r_R(b_1, \ldots, b_m)] = I_R[r_R(\sum_{i=1}^n Ra_i + \sum_{j=1}^m Rb_j)] = \sum_{i=1}^n Ra_i + \sum_{j=1}^m Rb_j = I_R(I) + I_R(K).$$

Thus, by Theorem 8, $R_R$ is $\pi$-injective, and in particular $R_R$ satisfies (C1).
(2) $\Rightarrow$ (3): Since $r_R(I_R(F)) = F$ for all finitely generated left ideals $F$ of $R$, $R$ is right P-injective, and hence satisfies the right C2-condition. Thus, $R$ is right continuous. $\square$

A ring $R$ is called a right CF-ring (resp. right FGF-ring) if every cyclic (resp. finitely generated) right $R$-module embeds in a free module. The ring $R$ is called right FP-injective if every $R$-homomorphism from a finitely generated submodule of a free right $R$-module $F$ into $R$ extends to $F$. Note that every right self-injective ring is right FP-injective, but not conversely. Also every finitely generated left ideal of a right FP-injective ring is a left annihilator (see [7]). The well known GFG problem asks whether every right GFG-ring is QF. It is known that every right self-injective, right FGF-ring is QF. In fact, Björk [1] and Tolskaya [12] independently proved that every right self-injective, right CF-ring is QF. On the other hand, Nicholson-Yousif [10, Theorem 4.3] shows that every right FP-injective ring for which every 2-generated right module embeds in a free module is QF. Our next corollary extends the two results.

Corollary 15. Suppose $R$ is a right CF-ring such that every finitely generated left ideal is a left annihilator. Then $R$ is a QF-ring.

Proof. Since $R$ is right CF, every right ideal is a right annihilator of a finite subset of $R$. By Proposition 14, $R_R$ is $\pi$-injective. Then, by [5, Corollary 2.9], $R$ is right artinian. Clearly, $R$ is two-sided mininjective. So, $R$ is QF by [9, Cor.4.8]. $\square$

Corollary 16. Every right CF, right FP-injective ring is QF. In particular, every right FGF, right FP-injective ring is QF.

A ring $R$ is called right FPF-ring if every finitely generated faithful right $R$-module is a generator of Mod-$R$, the category of all right $R$-modules. A ring is left (resp. right) duo if every left (resp. right) ideal is two sided. We conclude by noticing that every right FPF-ring which is left or right duo is $\pi$-injective. The next corollary follows from Theorem 8 and the proof of [4, 3.1A2, p.3.2].

Corollary 17. Let $R$ be a right FPF-ring. If $R$ is a left or right duo ring, then $R_R$ is $\pi$-injective. In particular, every commutative FPF-ring is $\pi$-injective.
References


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