On the Classification of 16-dimensional Planes

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The concluding section Principles of classification of the book [18] on compact projective planes contains an outline of how a classification of 16-dimensional planes admitting a group $\Delta$ of dimension $\dim \Delta \geq h$ (where $h$ is at least 35) might be accomplished. In the second part, proofs of several claims (of Proposition 87.4 in particular) have been indicated only very sketchily; some details will be supplied in the following, compare Theorem A below. Implications of Theorem A will be discussed elsewhere.

The automorphism group $\Sigma$ of a compact plane $\mathcal{P}$ will always be taken with the compact-open topology. Only closed subgroups $\Delta$ of $\Sigma$ will be considered, $\Delta$ is then a locally compact transformation group of the point space $\mathcal{P}$.

**Theorem L.** If $\dim \Delta \geq 29$, or if $\Delta$ is connected and $\dim \Delta \geq 27$, then $\Delta$ is a Lie group.

This suffices for all classification purposes. A weaker result is given in [18] (87.1). For proofs see Salzmann [17] and Priwitzer-Salzmann [11].

From now on, assume that $\mathcal{P}$ is a compact 16-dimensional space, and that $\Delta$ is a connected Lie group. By the structure theory of Lie groups, there are 3 possibilities: $\Delta$ is semi-simple, or $\Delta$ contains a central torus subgroup, or $\Delta$ has a minimal normal vector subgroup $\Theta \cong \mathbb{R}^t$, compare [18] (94.26). In the first two cases, the results mentioned in [18] (87.2 and 3) have been improved in the meantime:

**Theorem S.** Let $\Delta$ be a semi-simple group of automorphisms of the 16-dimensional plane $\mathcal{P}$. If $\dim \Delta > 28$, then $\mathcal{P}$ is the classical Moufang plane, or $\Delta \cong \text{Spin}_3(\mathbb{R}, r)$ and $r \leq 1$, or $\Delta \cong \text{SL}_3\mathbb{H}$ and $\mathcal{P}$ is a Hughes plane as described in [18], §86.
The proof can be found in Priwitzer [9], [10].

**Theorem T.** Assume that $\Delta$ has a normal torus subgroup $\Theta \cong \mathbb{T}$. If $\dim \Delta > 30$, then $\Theta$ fixes a Baer subplane, $\Delta' \cong \text{SL}_2 \mathbb{H}$, and $\mathcal{P}$ is a Hughes plane.

This is proved in Sakmann [15].

Hence only the case $\mathbb{R}^t \cong \Theta < \Delta$ has to be considered. For convenience, the classical Moufang plane over the octonions will be excluded from the discussion. So-called stiffness theorems on the size of the stabilizer of a quadrangle play a decisive rôle:

(‡) Suppose that the fixed elements of the connected closed subgroup $\Lambda$ of $\Delta$ form a non-degenerate subplane $\mathcal{E}$.

(a) If $\dim \Lambda > 11$, or if $\mathcal{E}$ is a Baer subplane, then $\Lambda$ is compact.

(b) If $\Lambda$ is compact, or if $\Lambda$ is a Lie group and $\mathcal{E}$ is connected, then $\Lambda \cong G_2$, or $\Lambda \cong \text{SU}_3 \mathbb{C}$, or $\dim \Lambda \leq 7$.

(c) If $\Lambda$ is a compact Lie group and $\dim \Lambda < 8$, then $\Lambda \cong \text{SO}_4 \mathbb{R}$ or $\dim \Lambda \leq 4$.

(d) If $\Lambda$ is a Lie group and $\mathcal{E}$ is a Baer subplane, then $\Lambda$ is isomorphic to $\text{SU}_2 \mathbb{C}$ or $\dim \Lambda \leq 1$.

The first results are essentially due to Bödi [1], [2]. For (c) and (d) see Salzmann [13] and [18] (83.22).

**Lemma 0.** If $\mathbb{R}^t \cong \Theta < \Delta$ and if $\dim \Delta \geq 24$, then $\Delta$ fixes a point or a line, say a line $W$.

**Proof.** Grundhöfer-Salzmann [8], Proposition XI.10.19.

**Theorem A.** Assume that $\Delta$ is not semi-simple and that $\mathcal{P}$ is not a Hughes plane. If $\dim \Delta \geq 33$, then, up to duality, $\Delta$ has a minimal normal subgroup $\Theta \cong \mathbb{R}^t$ consisting of axial collineations with common axis $W$. Either $\Theta \leq \Delta_{[a,W]}$ is a group of homologies and $t = 1$, or $\Theta$ is contained in the group $\mathcal{T} = \Delta_{[W,W]}$ of elations with axis $W$.

**Remarks.** This has been stated in [18], p. 587 under the stronger hypothesis $\dim \Delta \geq 36$. The theorem does not assert that every given minimal normal subgroup is axial. The proof is fairly easy for $t < 8$ and rather involved for even $t \geq 8$. The different cases will be treated in separate propositions. The result may well be true for even smaller dimensions of $\Delta$, but a proof would become unreasonably complicated.

A group $\Xi$ of collineations is called *straight* if each point orbit $x^\Xi$ is contained in some line. The following result (Stroppel [19] Lemma 3 or Priwitzer-Salzmann [11], Th. B) is a clue to the proof of the existence of axial collineations:

**Baer’s theorem.** If $\Xi$ is a straight subgroup of $\Delta$, then $\Xi$ is contained in a group $\Delta_{[z]}$ of central collineations with common center $z$, or the fixed elements of $\Xi$ form a Baer subplane $\mathcal{F}_\Xi$ of $\mathcal{P}$ and $\Xi$ is compact by (‡).
Corollary 1. If $\Pi \cong \mathbb{R}$ and $\Pi$ is straight, then $\Pi \leq \Delta_{[z,A]}$ for some center $z$ and axis $A$.

Proof. Note that $\Pi$ is not compact. By the dual of [18] (61.8), all elements in $\Pi$ have the same axis. \hfill \Box

Corollary 2. If $\Theta \cong \mathbb{R}^t$, and if each one-parameter subgroup $\Pi$ of $\Theta$ is straight, then $\Theta$ satisfies the assertions of Theorem A.

Proof. By [18] (61.7), the center map $\Theta \setminus \{\mathbb{I}\} \to P$ is continuous, and the centers of all one-parameter subgroups of $\Theta$ form a compact and connected set $Z$. Commutativity of $\Theta$ implies that either $Z$ is a single point, or $Z$ is contained in the common axis $W$ of all elements of $\Theta$. The dual is also true. If there exist homologies in $\Theta$, then $t = 1$ by [18] (61.2). \hfill \Box

Corollary 3. If $\Theta \cong \mathbb{R}^t$ is a minimal normal subgroup of $\Delta$ and if some one-parameter subgroup $\Pi$ of $\Theta$ is straight, then $\Theta$ satisfies the assertions of Theorem A.

Proof. Let $\Pi \leq \Delta_{[z,A]}$ as in Corollary 1, and assume that $\Delta$ fixes the line $W$. Commutativity of $\Theta$ implies that $z^\Delta = z$ or $z^\Delta \subseteq A$. If $A = W$, then $\Theta = \Theta_{[A]}$ by minimality of $\Theta$. If $A \neq W$, then $z^\Delta \subseteq W$ and hence $z^\Delta = z$. This case is dual to the first one. \hfill \Box

Lemma 1. Assume that the one-parameter subgroup $\Pi$ of $\Theta$ is not straight. Then there is an orbit $b^\Pi$ which generates a connected subplane; its closure will be denoted by $\mathcal{E} = \langle b^\Pi \rangle$. For $g \in \Pi \setminus \{\mathbb{I}\}$, the stabilizer $\Delta_g$ in the action of $\Delta$ on $\Theta$ is the centralizer of $\Pi$, and the connected component $\Lambda$ of $\Delta_{b,\mathcal{E}}$ induces the identity on $\mathcal{E}$. Hence (\dagger) applies, and the dimension formula [18] (96.10) gives

\[
(*) \quad \dim \Delta = \dim b^\Delta + \dim g^{\Delta b} + \dim \Lambda \leq 16 + t + \dim \Lambda. \hfill \Box
\]

Proposition 1. If $t < 8$ and if some one-parameter subgroup $\Pi$ of $\Theta$ is not straight, then $\dim \Delta \leq 32$.

Proof. Use Lemma 1. If $\Lambda \cong G_2$, then $\dim \mathcal{E} = 2$ by [18] (83.24), and $\mathcal{E}$ consists of all fixed elements of $\Lambda$. Moreover, $\Lambda$ acts trivially on $\Theta$ because $\Lambda$ fixes $g$ and each non-trivial representation of $G_2$ is at least 7-dimensional [18] (95.10). The connected component $\Theta_b^{1}$ of the stabilizer $\Theta_b$ is contained in the compact group $\Lambda$, hence $\Theta_b^{1} = \mathbb{I}$, $\dim \Theta_b = 0$, and $\dim b^\Theta = t$. Since $\Theta$ centralizes the group $\Lambda$, the fixed plane $\mathcal{E}$ of $\Lambda$ is $\Theta$-invariant. Consequently $b^\Theta \subseteq \mathcal{E}$, and it follows that $t \leq 2$ and $\dim \Delta \leq 32$. In all other cases $\dim \Lambda \leq 8$ by (\dagger), and $\dim \Delta < 32$. \hfill \Box

Proposition 1 and Corollary 2 imply that Theorem A is true in the case $t < 8$ and $\dim \Delta > 32$.

Lemma 2. Let $\mathbb{R}^t \cong \Theta < \Delta$ and $\mathbb{R} \cong \Pi \leq \Theta$. Assume that $\dim \Delta > 32$, and that the orbit $b^\Theta$ is not contained in any line. Then $\Theta_b = \mathbb{I}$ and $8 \leq t \leq 16$. Moreover, the orbit $b^\Theta$ is not contained in any proper closed subplane of $\mathcal{P}$, and $\Delta_b$ acts effectively on $\Theta$. 

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Proof. Proposition 1 shows that \( t \geq 8 \). Let \( \mathcal{F} = \langle b^\Theta \rangle \) be the smallest closed subplane containing \( b^\Theta \). Note that \( \mathcal{F} \) is connected and \( \Delta_b \)-invariant, and that \( \Theta_b \) fixes \( \mathcal{F} \) pointwise. From (\(*\)) it follows that \( 17 - t \leq \dim \Lambda \). Obviously, \( b^\Theta \subseteq \mathcal{F} \) and \( \Theta_b^1 \leq \Lambda \). If \( \dim \mathcal{F} = 2 \), then \( t - 2 \leq \dim \Lambda \) and \( 15 \leq 2 \dim \Lambda \). Therefore, \( \dim \Lambda \geq 8 \), and \( \Lambda \) is compact by (\( \dagger \)). Now \( \Theta_b^1 = \1 \) and \( \dim b^\Theta = t \leq 2 \), a contradiction. If \( \dim \mathcal{F} = 4 \), then \( \Delta_b \) induces on \( \mathcal{F} \) a group \( \Delta_b/\Phi \) of dimension at most 8 (see [18] (72.8) and note that \( \Delta_b \leq \Delta_W \) and \( b \notin W \)). The kernel \( \Phi \) of this action satisfies \( \dim \Phi > 8 \). The connected component of \( \Phi \) is isomorphic to \( G_2 \) by (\( \dagger \)), but this would imply \( \dim \mathcal{F} = 2 \), see [18] (83.24). Hence \( \dim \mathcal{F} \geq 8 \) and \( \mathcal{F} \) is a Baer subplane or \( \mathcal{F} = \mathcal{P} \), for short, \( \mathcal{F} \leq \mathcal{P} \). According to [18] (83.6), the group \( \Theta_b \) is compact, and then \( \Theta_b = \1 \). Suppose, finally, that \( \mathcal{F} \not\leq \mathcal{P} \). Then \( t = \dim \mathcal{F} = 8 \), and \( \Delta_b \) induces on \( \mathcal{F} \) a group \( \Gamma \cong \Delta_b/\Phi \) of dimension \( \dim \Gamma \geq 14 \) (note that \( \dim \Phi \leq 3 \) by (\( \dagger \)),(d)). Hence \( \dim \Gamma \Theta \geq 22 \), \( \mathcal{F} \) is isomorphic to the quaternion plane, and \( \Theta \) acts on \( \mathcal{F} \) as a group of translations, see [18] (84.13) or Salzmann [14]. This contradicts the assumption that \( b^\Theta \) is not contained in a line. \( \square \)

Note. For the proof of Theorem A in the case \( t \geq 8 \) it is essential that \( \Theta \) is chosen as a minimal normal subgroup. Write \( \Xi = \text{Cs} \Theta \) for the centralizer of \( \Theta \). Then \( \Gamma = \Delta/\Xi \) is an irreducible subgroup of \( \text{G}_L\mathbb{R} \). The structure of such groups is well known, compare [18] (95.6). In particular, the commutator subgroup \( \Gamma' \) is semi-simple, and \( \Gamma \) is the product of \( \Gamma' \) and the center \( Z \) of \( \Gamma \). The irreducible representations of almost simple groups in dimension at most 16 are listed in [18] (95.10). Extensive use will be made of this list, compare also Bödi-Joswig [3]. Whenever \( X \) is an almost simple factor of \( \Gamma \), the dimension of a minimal \( X \)-invariant subgroup of \( \Theta \) divides \( t \) by Clifford’s Lemma [18] (95.5). If \( t \) is odd, then \( \Gamma' \) is also irreducible and \( \dim Z \leq 1 \). Hence these cases are less difficult, they will be discussed next. A special argument is needed for large values of \( t \).

Proposition 2. If \( t < 15 \) and \( t \) is odd, and if \( \dim \Delta > 32 \), then \( \Theta \) satisfies the assertions of Theorem A.

Proof. As in Lemma 2, assume that the orbit \( b^\Theta \) is not contained in a line. Since \( \Delta_b \) acts effectively on \( \Theta \), the Note shows that \( \dim \Gamma' \geq 16 \), and \( \Lambda \) is mapped injectively into \( \Gamma \). Now let \( t = 9 \). From (\( * \)) it follows that \( \dim \Lambda \geq 8 \), and (\( \dagger \)) implies that \( \Lambda \cong \text{SU}_3 \mathbb{C} \) or \( \Lambda \cong G_2 \). Therefore, \( \dim \Delta \leq 39 \) and \( \dim \Gamma \leq 30 \). Moreover, \( \Lambda \) acts on a 6- or 7-dimensional subspace of \( \Theta \) and fixes a complement, see the List [18] (95.10). By Clifford’s Lemma, \( \Lambda \) is properly contained in an almost simple factor \( \Upsilon \) of \( \Gamma \), and \( \Upsilon \) acts effectively and irreducibly on \( \Theta \). Noting that \( 8 < \dim \Upsilon \leq 30 \), the List shows that \( \Upsilon \cong \text{PSL}_3 \mathbb{C} \), but this group does not contain \( \Lambda \). Hence \( t \in \{11,13\} \), and \( t \) is a prime number. Clifford’s Lemma implies that \( \Gamma' \) is almost simple. Since \( \dim \Gamma' > 3 \), it follows from the List that \( \dim \Gamma' \geq 55 \), an obvious contradiction. \( \square \)

Lemma 3. A group \( \Delta \) of dimension \( \geq 31 \) fixes at most one point \( a \notin W \). Assume that \( t \geq 12 \) and that \( \Theta_W \neq \1 \). Let \( L \) be a line such that \( L \cap W = u \neq u^\Theta \). If \( L \) does not contain the exceptional point \( a \), then \( \Theta_L = \1 \) and \( \dim L^\Theta \geq 12 \). Hence \( L^\Theta \) is not contained in any proper closed subplane of \( \mathcal{P} \), and \( \Delta_L \) acts effectively on \( \Theta \).
Proof. The first assertion follows immediately from (†) and the dimension formula. Either \( \langle L^\Theta \rangle = Q \) is a closed subplane, or \( L^\Theta \) is contained in a pencil \( L_x \), and \( x^\Theta = x \). Suppose that \( x^\Delta \neq x \). Then \( \Theta_u \) induces the identity on the connected subplane \( D = \langle x^\Delta, u^\Theta \rangle \).

By the dimension formula, \( 12 \leq t = \dim u^\Theta + \dim \Theta_u \), and \( \dim \Theta_u \geq 4 \). Since \( \Theta_u \) is not compact, (†) implies \( \dim \Theta_u \leq 7 \), and \( \dim u^\Theta \geq 5 \). Consequently, \( D = P \) and \( \Theta_u = 1 \).

This contradiction shows that \( x^\Delta = x \) is the unique point \( a \). Hence \( L \neq au \) leads to the first alternative, and \( \Theta_L \) acts trivially on \( Q \). Either \( \dim \Theta_L \geq 8 \) or \( \dim L^\Theta > 4 \) and \( Q \leq P \). In both cases, \( \Theta_L \) is compact by (†), and then \( \Theta_L = 1 \). Consequently, \( Q = P \) and \( \Delta_L \cap \text{Cs} \Theta = 1 \).

The lemma leads to a useful modification of condition (⋆):

**Proposition 3.** In the situation of Lemma 3, each one-parameter subgroup \( \Pi \) of \( \Theta \) has some orbit \( b^\Pi \) which is not contained in any line. If \( u \) and \( L \) are chosen as above, let \( \varrho \in \Pi \leq \Theta_u \), \( \varrho \neq 1 \). Then \( \Theta_u < \Pi < \Delta_L \Theta \), \( \dim \varrho^\Pi \leq \dim \Theta_u \leq 8 \), and

\[
(\star \star) \quad H < \Delta, \quad \dim \Delta/H \leq 16 - t, \quad \text{and} \quad \dim H \leq 16 + \dim \varrho^H + \dim K \leq 24 + \dim \Lambda,
\]

where \( K \) denotes the connected component of \( H_{b,\varrho} = \Delta_b \cap \text{Cs}_H \Pi \) and \( \Lambda \) has the same meaning as in Lemma 1.

**Proof.** Because \( \Delta_L \) fixes the point \( u \) and \( \Theta \) is commutative, \( \Theta_u \) is invariant in \( H = \Delta_L \Theta \). By assumption, \( \Theta_u < \Theta \) and \( \Theta \) is a minimal normal subgroup of \( \Delta \). Moreover, \( \dim \Theta_u \geq t - 8 \geq 4 \). Consequently, \( H \neq \Delta \). From \( \Delta_L \cap \Theta = 1 \) it follows that \( \dim H = \dim \Delta_L + t \), and the dimension formula shows \( \dim \Delta = \dim \Delta_L + \dim L^\Delta \leq \dim \Delta_L + 16 \). Therefore, \( \dim \Delta - \dim H \leq 16 - t \). Since \( \dim \varrho^H \leq 16 \), the last inequality in (⋆⋆) is immediate from the dimension formula.

**Proposition 4.** Suppose that \( \dim \Delta \geq 31 \) and that \( u^\Theta \neq u \in W = W^\Delta \). If \( \dim \Lambda > 8 \), then \( \Lambda \cong G_2 \) by (†), and the centralizer \( X = \text{Cs}_\Theta \Lambda \) has dimension at most 2. Moreover, \( t \in \{1, 2, 8, 9\} \).

**Proof.** Use the same notation as in Proposition 3. By assumption, \( \Pi \leq X \) and \( \dim X \geq 1 \). The orbit \( b^X \) is contained in the 2-dimensional subplane \( E \) of the fixed elements of \( \Lambda \). Obviously, \( X_{b,1} \leq \Theta \cap \Lambda = 1 \). Therefore, \( \dim X \leq 2 \). From [18] (95.3 and 10) it follows that the complement of \( X \) in \( \Theta \) has a dimension divisible by 7. Hence the proposition is true unless \( t \geq 15 \). Then Proposition 3 applies, and the first statements of (⋆⋆) exclude the possibility \( t = 16 \). In the case \( t = 15 \), condition (⋆⋆) implies that \( \dim \Delta \leq 39 \). Hence \( \Delta \) induces on \( \Theta \) an irreducible group \( \Gamma \) of dimension at most 24. According to the note, \( \Gamma' \) is irreducible on \( \Theta \), and \( \Lambda \) is properly contained in \( \Gamma' \). By Clifford’s Lemma, \( \Gamma' \) is almost simple. Inspection of the List [18] (95.10) shows that there is no group with these properties.

Propositions 3 and 4 imply:
Corollary 4. If \( t \geq 12 \), then \( \dim \Delta \leq 48 - t \) or \( \Theta \) satisfies the assertions of Theorem A.

Corollary 5. If \( t = 16 \) and \( \dim \Delta > 32 \), then \( \Theta = T \) is a transitive elation group.

Proposition 5. If \( t = 15 \) and \( \dim \Delta > 32 \), then \( \Theta \) is a group of elations.

Proof. Assume that \( u^\Theta \neq u \) for some point \( u \) on the fixed line \( W \) of \( \Delta \). Then \( \dim \Delta = 33 \) by Corollary 4, and Propositions 3 and 4 give \( \dim H = 32 \) and \( \Lambda \cong SU_3 \mathbb{C} \). Moreover, (**) implies \( \dim g^H = 8 \) for each admissible \( g \). Hence \( \Delta_L \) is transitive on \( \Theta_u \cong \mathbb{R}^8 \). With [18] (96.22) or the List (95.10) it follows that the commutator subgroup \( \Upsilon \) of \( \Delta_L \) is isomorphic to \( SU_4 \mathbb{C} \). According to the Note and to Clifford’s Lemma in particular, \( \Upsilon \) is properly contained in an almost simple factor of \( \Gamma \cong \Delta/Cs \Theta \). This implies \( \dim \Gamma \geq 21 \) and \( \dim \Delta \geq 36 \), a contradiction.

Lemma 4. Let \( p \) and \( q \) be prime numbers. A semi-simple irreducible subgroup \( G \) of \( SL_{pq} \mathbb{R} \) has not more than two almost simple factors.

Proof. \( G \) is either almost simple or a product of two proper semi-simple factors \( A \) and \( B \) such that \( B \leq Cs A \). Let \( U \) be a minimal \( A \)-invariant subspace of \( V = \mathbb{R}^{pq} \). If \( U = V \), then \( B \leq H^X \) by Schur’s Lemma [18] (95.4). Hence \( 4|pq = 4 \) and \( G \cong SO_4 \mathbb{R} \). If \( U < V \), however, then \( A \) acts effectively on \( U \) (because the fixed elements of the kernel of the action of \( A \) on \( U \) form a \( G \)-invariant subspace of \( V \)). Clifford’s Lemma shows that \( \dim U \in \{p, q\} \) and that \( A \) is almost simple.

Alternative proof (suggested by the referee). All semi-simple irreducible subgroups \( G \) of \( SL_{t} \mathbb{R} \) have been determined by Dynkin [5/6] Th. 1.5: either \( G \) is maximal in \( SL_{t} \mathbb{R} \) or in a symplectic or orthogonal group, and the claim follows from Theorems 1.3 and 1.4, or \( G \) belongs to a long list of exceptions, but these are even almost simple.

The following well-known theorem will be needed several times:

Complete reducibility. If a semi-simple group \( G \) acts on a real vector space \( V \), then each \( G \)-invariant subspace of \( V \) has a \( G \)-invariant complement in \( V \).

This follows directly from an analogous result about representations of semi-simple Lie algebras, see e.g. Bourbaki [4] I.6.2 Theorem 2, p. 52 or Freudenthal-de Vries [7] § 35 or § 50.

Proposition 6. If \( t = 14 \) and \( \dim \Delta > 32 \), then \( \Theta \) is a group of elations.

Proof. Assume that \( \Theta \) does not act trivially on \( W \). Put again \( \Xi = Cs \Theta \).

(a) Corollary 4 shows that \( \dim \Delta \leq 34 \). According to Lemma 2 and the Note, the stabilizer \( \Delta_b \) acts effectively on \( \Theta \) and may be considered as a subgroup of \( \Gamma = \Delta/\Xi \), and the semi-simple commutator group \( \Gamma' \) satisfies \( 15 \leq \dim \Gamma' \leq 20 \). No almost simple group with dimension between 15 and 20 has an irreducible representation in dimension 7 or 14.
Therefore, Lemma 4 implies that $\Gamma'$ is a product of exactly two almost simple subgroups $A$ and $B$. Let $\dim A \leq \dim B$. Then $\dim B \geq 8$, and $B$ acts effectively and irreducibly on $\mathbb{R}^7$. By the List, $\dim B = 14$, and $B$ is one of the two simple groups of type $G_2$. (Later it will be seen that $B$ is in fact the compact form.) As in the proof of Lemma 4, it follows that $A$ acts effectively on $\mathbb{R}^2$. Consequently, $A \cong \text{SL}_2 \mathbb{R}$ and $\dim \Gamma' = 17$, moreover, the center $Z$ of $\Gamma$ consists of real dilatations of $\Theta$, see [18] (95.6).

(b) The last statement gives $\dim \Gamma \leq 18$ and $\dim \Xi \geq 15$. On the other hand, $\dim \Xi \leq 16$ since $\dim \Delta / \Xi \geq \dim \Delta_b \geq \dim \Delta - 16$. Therefore, $\Xi$ is contained in the radical $\sqrt{\Delta}$, and a maximal semi-simple subgroup $\Psi$ of $\Delta$ is locally isomorphic to $\Gamma'$.

(c) The group $\Delta$ has a subgroup $\hat{\Delta}$ of codimension 3 which induces on $\Theta$ the group $BZ$ and acts on a 7-dimensional subgroup $N$ of $\Theta$. Note that $\dim \hat{\Delta}_b \geq 14$. If $\hat{\Delta}_b$ is transitive on $N$, then it is also transitive on the 6-sphere consisting of the rays in $N$, and $\hat{\Delta}_b$ contains the compact group $G_2$, compare [18] (96.19 and 22). If $\hat{\Delta}_b$ is not transitive on $N$, then for some $\varrho \in N$ the connected component $\hat{\Lambda}$ of $\hat{\Delta}_{b,\varrho}$ is at least 8-dimensional, and (‡) shows that $\hat{\Lambda} \cong \text{SU}_3 \mathbb{C}$. This group is not contained in the non-compact group $G_2(2)$. Therefore $B$ is compact, and steps (a) and (b) imply $\Gamma' = A \times B \cong \text{SL}_2 \mathbb{R} \times \mathbb{H} \cong \Psi$.

(d) Because $\Delta_b \leftrightarrow \Gamma$ and $\dim \Gamma / \Delta_b \leq 1$, it follows that $G_2 \cong B \leftrightarrow \Delta_b$. One can now conclude that $\Delta_b$ acts irreducibly on $\Theta$. In fact, the action of $B$ and complete reducibility force a proper $\Delta_b$-invariant subgroup $N$ of $\Theta$ to be 7-dimensional. Lemma 1 with $\varrho \in N$ gives $\dim \Delta_{b,\varrho} \geq 10$ and then $\hat{\Lambda} \cong G_2$, but this contradicts the fact $\varrho^\hat{\Lambda} = \varrho$. As a consequence of [18] (95.6(b)), even the action of the semi-simple group $\Delta_b'$ on $\Theta$ is irreducible, hence $\dim \Delta_b' = 17$ and $\Delta_b' \cong \Psi$.

(e) In particular, the involution in $A$ corresponds to an involution $\alpha$ in the center of $\Delta_b$. By [18] (84.9), the group $B \cong G_2$ cannot act on a Baer subplane. Consequently, $\alpha$ is a reflection, see [18] (55.29). Because of (‡), the group $A$ acts effectively on the 2-dimensional plane $E = F_B$ of the fixed elements of $B$. If $A$ would fix a flag in $E$, then $A$ would be solvable by [18] (33.8). Therefore, $b$ and $W$ are the only fixed elements of $A$ in $E$, and $\alpha \in D_{[b,W]}$. Since $\Theta_b = I$, it follows from [18] (61.19b) that $\alpha^\Theta \alpha$ is a 14-dimensional subset of $T = D_{[W,W]}$. Note that $\Theta T \leq \sqrt{\Delta}$ and that $\dim \sqrt{\Delta} \leq 17$. Minimality of $\Theta$ implies $\Theta \leq T$.

\textbf{Proposition 7.} If $t = 10$ and $\dim \Delta > 32$, then $\Theta$ is a group of elations.

\textit{Proof.} From (§) and Proposition 4 one obtains $17 \leq \dim \Delta_b \leq \dim \varrho^\Delta_\varrho + \dim \Lambda \leq 10 + 8$. Either $\Delta_b$ is transitive on $\Theta$, or $\Lambda \cong \text{SU}_3 \mathbb{C}$ and $\dim \varrho^\Delta_\varrho = 9$ for some $\varrho \in \Theta$. In the first case, $\Delta_b$ would contain the group $\text{SU}_5 \mathbb{C}$, and $\dim \Delta_b$ would be to large, compare [18] (96.16–22). Similarly, $\Delta_b$ cannot be transitive on a 9-dimensional subspace of $\Theta$. Hence $\Delta_b$ acts effectively and irreducibly on $\Theta$. Since each non-trivial representation of $\text{SU}_3 \mathbb{C}$ on $\Theta$ is either 6- or 8-dimensional, it follows from Clifford’s Lemma that $\Lambda$ is not normal in $\Delta_b$. Therefore, $\Lambda$ is properly contained in an almost simple and irreducible factor $X$ of $\Delta_b'$, and $X = \Delta_b'$ by Schur’s Lemma [18] (95.4). The List shows that $\Delta_b'$ must be locally isomorphic to $\text{SL}_4 \mathbb{R}$ or to $\text{SL}_2 \mathbb{H}$, but these groups do not contain $\text{SU}_3 \mathbb{C}$. \qed
The only remaining cases $t = 8$ and $t = 12$ are more difficult. If the given group $\Theta$ does not consist of elations, it will be shown that some other normal vector group $\Theta$ of $\Delta$ satisfies the conditions of Theorem A.

**Proposition 8.** If $t = 8$ and $\dim \Delta > 32$, then either $\Theta$ or some minimal normal subgroup $\Theta \cong \mathbb{R}^7$ consists of elations.

**Proof.** (a) According to Corollary 3, we may assume that for each one-parameter subgroup $\Pi < \Theta$ there is some point $b$ such that $b^\Pi$ generates a subplane. Lemma 1 then shows that the connected component $\Lambda$ of $\Delta_b \cap C_s \Pi$ has dimension at least 9, and $\Lambda \cong G_2$ by (1). Consider the action of the connected component $B$ of $\Delta_b$ on the 7-sphere $S$ consisting of the rays in $\Theta$, and let $r, r'$ denote the two opposite rays contained in $\Pi$. Since $\dim B \geq 17$ and $\dim B / \Lambda \leq 1$, it follows that $r^B$ is a connected set of positive dimension. For each $s \in S \setminus \{r, r'\}$, the orbit $s^A$ is a 6-sphere, and $r^B$ is a connected union of $\Lambda$-orbits. Consequently, $r^B$ contains an open neighbourhood of $r$ in $S$, and $r^B$ is open in $S$, see also [18] (96.25). The dimension formula implies $\dim B / \Lambda \geq 7$ and $21 \leq \dim \Delta_b \leq 22$.

(b) In particular, $r^A$ is open in $S$, and this is true for each ray $r \in S$ because step (a) is valid for an arbitrary choice of $r$. Therefore, $\Delta$ acts transitively on $S$. Put $\Xi = C_s \Theta$ as in the Note. Then $\Gamma = \Delta / \Xi$ is the effective group induced by $\Delta$ on $\Theta$. Remember from Lemma 2 that $\Delta_b$ is embedded into $\Gamma$. Step (a) and Lemma 1 imply $21 \leq \dim \Gamma \leq 30$. According to [18] (96.19), a maximal compact subgroup $\Phi$ of $\Gamma$ acts transitively on $S$, and from [18] (96.20) it follows that $\Phi$ is isomorphic to a subgroup of $SO_8 \mathbb{R}$. Moreover, $\Phi$ has a subgroup $\Lambda \cong G_2$. There are only two groups $\Phi$ which satisfy these conditions, viz. $\Phi \cong \text{Spin}_7 \mathbb{R}$ and $\Phi \cong SO_8 \mathbb{R}$, see [18] (96.21 and 22). The centralizer of $\Phi$ in $GL_8 \mathbb{R}$ is isomorphic to $\mathbb{R}^8$, and the remarks in the Note show that $\Gamma$ is the product of $\Phi = \Gamma'$ and the center $Z$ of $\Gamma$. By [18] (94.27), the group $\Delta$ contains a subgroup $\Psi$ which is a covering group of $\Gamma'$. In fact, $\Psi$ is simply connected, since $\Delta$ cannot contain a group $SO_7 \mathbb{R}$, see [18] (55.34 or 40).

(c) Assume that $\dim \Gamma' > 21$. Then $\Psi \cong \text{Spin}_8 \mathbb{R}$, and the center of $\Psi$ contains 3 reflections $\alpha, \sigma$, and $\alpha \sigma$ with centers $a, u$, and $v$. By (1), the stabilizer $\nabla$ of the triangle $\{a, u, v\}$ satisfies $\dim \nabla \leq 30$, and $\Psi / \nabla < \Delta$. It suffices to consider the case $\Delta = \Psi \Theta$. Note that $\Delta$ is not transitive on $W$ (otherwise $\Delta$ would induce on $W$ the group $SO_9 \mathbb{R}$). On $W \setminus v$ the action of $\Psi$ is equivalent to a linear action, and for each $z \neq u, v$ the orbit $z^\Psi$ is a 7-sphere, compare [18] (96.36). Hence $z^\Delta$ is open in $W$ whenever $z^\Delta \neq z^\Psi$, see also [18] (96.25). If $v^\Delta = v$, then $v^\Delta = u^\Theta$ is open in $W$, and $\sigma^\Theta \sigma = \{\vartheta^{-1} \vartheta^\sigma \mid \vartheta \in \Theta\}$ would generate a transitive group of elations with axis $au$ in $\Theta$. Therefore $u^\Theta \neq u$ and $v^\Theta \neq v$, and $W$ contains some orbit $z^\Delta = Z \approx S_7$. This leads to a contradiction as follows: Since $z^\Theta$ is $\Psi_x$-invariant, the argument of [18] (96.25) shows that either $z^\Theta = z$ or $z^\Theta$ is open in $Z$. Because $\Psi$ is transitive on $Z$ and $\Theta$ is normal, all $\Theta$-orbits in $Z$ are equivalent. The commutative group $\Theta$ cannot be transitive on $Z$. Therefore the orbits are points, $\Theta | Z = 1$, and $\Theta$ would act freely and transitively on $az \setminus \{a, z\} \approx \mathbb{R}^8 \setminus \{0\}$ which is impossible.

(d) The last steps imply $\Psi \cong \Gamma' \cong \text{Spin}_7 \mathbb{R}$. Since $21 \leq \dim \Delta_b \leq \dim \Gamma \leq 22$, and since $\text{Spin}_7 \mathbb{R}$ has no subgroup of codimension 1, the covering group $\Psi$ of $\Gamma'$ can be chosen.
in $\Delta_b$. Hence $\Delta_b' \cong \text{Spin}_2\mathbb{R}$. Now it is not difficult to determine the structure of $\Delta$. Put again $\Xi = Cs\Theta$. Lemma 2 implies $\Xi_b = 1$ and $\dim \Xi \leq 16$. On the other hand, the bound on $\dim \Gamma$ gives $\dim \Xi \geq 11$. Consider the group $\Upsilon = \Xi \cap Cs\Psi$ and note that $\Lambda \leq \Psi$. The orbit $b^\Upsilon$ is contained in the 2-dimensional fixed plane $\mathcal{F}_\Lambda$. Consequently, $\dim \Upsilon \leq 2$. Let $\Psi$ act on the Lie algebra $\Xi$. By complete reducibility, $\Theta$ has an invariant complement $n$ in $\Xi$, and $\dim n > 2$. If $\dim n < 7$, the representation of $\text{Spin}_2\mathbb{R}$ on $n$ is trivial and $n = 1\Upsilon$, a contradiction. Hence $\dim n \geq 7$ and $\dim \Xi \in \{15, 16\}$. Because $\Psi \cap \Xi \leq \Xi_b = 1$, we may assume that $\Delta = \Psi \Xi$.

(e) The central involution $\sigma$ of $\Psi$ is a reflection, its axis $K$ is different from $W$ (or else $\sigma^2 = \Theta$ would consist of elations). If $\dim \Xi = 15$, then $n \cong \mathbb{R}^7$ and $\Psi$ induces on $n$ the group $\text{SO}_7\mathbb{R}$. Therefore $n$ is the Lie algebra of $N = \Xi \cap Cs\sigma$, moreover, $\Xi$ is a vector group and $N$ is $\Delta$-invariant. Proposition 1 shows that $\Theta = N$ is an elation group as required.

(f) Finally, let $\dim \Xi = 16$. Because $\Xi_b = 1$, the orbit $b^\Xi$ is open in $P$ by [18] (96.11(a)). Note that $b^\sigma = b$. If $\xi \in \Xi$ and $b^\xi \in K$, then $\xi^\sigma \xi^{-1} \sigma \in \Xi_b = 1$ and $\sigma^\xi = \xi^\sigma$. This gives $\Xi_K = \Xi \cap Cs\sigma$ and $\dim \Xi_K = 8$, moreover, $\Xi = \Xi_K \times \Theta$. Under the action of $\Psi$, the group $\Xi_K$ splits into a one-parameter group and a 7-dimensional vector group $\Theta$ on which $\Psi$ induces a group $\text{SO}_7\mathbb{R}$. Obviously, $\Theta$ is $\Psi\Xi$- and hence $\Delta$-invariant, and $\Theta$ is an elation group, again by Proposition 1 and Corollary 2.

\begin{proposition}
If $t = 12$ and $\dim \Delta > 32$, then $\Delta$ has a normal vector subgroup $\bar{\Theta}$ consisting of elations. ($\bar{\Theta}$ may be different from $\Theta$).
\end{proposition}

\begin{proof}
Assume that $\Theta$ is not contained in the elation group $T = \Delta_{[W,W]}$, and use the notation introduced in Proposition 3.

(a) Propositions 3 and 4 imply that $29 \leq \dim H \leq 32$. Consider a minimal $H$-invariant subgroup $M$ of $\Theta_u$ and let $\Pi \neq \rho \in \Pi \leq M$. Then $H$ and $\Delta_L$ act irreducibly on $M \ominus \rho^H$, and $\dim \rho^H \geq 5$ by (**). Remember from Lemma 2 that $\Theta_b = 1$. Consequently, $\dim \rho^M > 4$ and $\langle \rho^M \rangle = B$ is at least 8-dimensional ($B \leq \mathcal{P}$). Statement (d) of (i) shows that $\dim (H_b \cap CsM) \leq 3$, and $H$ induces on $M$ a group of dimension at least 10. If $K$ denotes again the connected component of $H_b \cap Cs\Pi$, then $\dim K \leq 8$ by Proposition 4. The following will be shown in the next steps: $M = \Theta_u \cong \mathbb{R}^8$ and $\Delta_L$ does not act effectively on $M$.

(b) If $M \cong \mathbb{R}^5$, then (** and (i)) imply $K \cong \text{SU}_3\mathbb{C}$. This group does not admit a non-trivial representation in dimension $< 6$. Hence $K \leq Cs M$ contrary to what has been stated in (a).

(c) In the case $M \cong \mathbb{R}^6$, the same argument shows that $\dim K < 8$ (note that the action of $K$ on $M$ is not irreducible, since $\Pi$ is $K$-invariant). Consequently, $\dim H_b = 13$ by (**), moreover, $\dim \rho^{H_b} = 6$ for each choice of $\rho$, and $H_b$ acts transitively on $M \setminus \{\Pi\}$, see [18] (96.11(a)). Let $\Psi = H_b|_M$ denote the effective group induced on $M$. Since $\Psi$ is irreducible and $\dim \Psi \leq 13$, the List of representations shows $\Psi' \cong \text{SU}_3\mathbb{C}$, and $\dim \Psi = 10$. Hence $H_b$ has a normal subgroup $\Phi \cong \text{SU}_2\mathbb{C}$ acting trivially on $M$, cf. step (a) and (i)(d). The involution $\omega \in \Phi$ fixes a Baer subplane $\mathcal{F}_\omega$ and the orbit $b^M$ is contained in the point-set
$F$ of $\mathcal{F}_\omega = B$. Since $H_0 \leq CS_\omega$, the group $\Psi'$ acts non-trivially on $B$ and, therefore, on $S = F \cap W \approx S_4$, but this contradicts Richardson’s theorem [18] (96.34).

(d) Finally, let $M \cong \mathbb{R}^7$. Steps (b) and (c) imply that $H_0$ acts irreducibly on $M$, and (**) shows that $\dim H_0 \leq 15$. Again, $\Psi = H_0|_M$ is at least 10-dimensional. According to the Note, $\Psi'$ is an almost simple group of type $G_2$. By complete reducibility, $M$ has a $\Psi'$-invariant complement $N \cong \mathbb{R}^5$ in $\Theta$. If $\langle b^M \rangle = B$ is a Baer subplane, then $\Psi'M$ is a 21-dimensional automorphism group of $B$, and $B$ is isomorphic to the quaternion plane $H$, see Salzmann [12], cp. also [18] (84.21(b)) or Salzmann [14], but $H$ does not admit a group of type $G_2$. Hence $\langle b^M \rangle = \mathcal{P}$ and $\Psi'$ is a subgroup of $H_0$. The List shows that $\Psi'$ centralizes $N$. Therefore, $\Psi'$ induces the identity on $b^M$. Because $\langle b^M \rangle = \mathcal{P}$, there is some point $c \in b^M$ such that $\langle b^M, c \rangle = \mathcal{P}$. Consequently, $\Psi'_c = 1$, but $\dim \Psi'_c \geq 7$, a contradiction proving the first statement at the end of step (a).

(e) Suppose now that $\Delta_L$ acts effectively on $M \cong \mathbb{R}^8$. By assumption, this action is irreducible. The structure theorem [18] (95.6(b)) shows that the commutator subgroup $T = \Delta_L'$ is semi-simple and that the center $Z$ of $\Delta_L$ consists of real or complex dilatations of $M$. There are two possibilities: (i) the action of $T$ on $M$ is irreducible, and (ii) $M$ is a direct sum of two 4-dimensional subspaces $M_\nu$ and $T$ acts equivalently and effectively on the spaces $M_\nu$. Note that $\dim \Delta_L \geq 17$ and $\dim T \geq 15$. The action of the semi-simple group $T$ on $\Theta$ is completely reducible. Hence there is a $T$-invariant complement $N \cong \mathbb{R}^4$ of $M$ in $\Theta$. In case (i), the group $N$ is even $\Delta_L$-invariant: in fact, for each $\zeta \in Z$ the subspace $N \zeta N$ is invariant under $T$ and at most 8-dimensional, hence it has trivial intersection with $M$. Choose a one-parameter group $E$ in $N$ and a point $p$ such that $\langle p^E \rangle = \mathcal{E}$ is a subplane. Remember that $H = \Delta_L \Theta$ has dimension at least 29. Consider the connected component $K$ of $H_p \cap CS E$. The dimension formula gives $\dim K \geq 9$, and (i) shows that $K \cong G_2$, but then Proposition 4 would imply $t \leq 9$. Case (ii) also leads to a contradiction: $T$ is semi-simple and effective on $\mathbb{R}^4$, therefore, $\Theta \cong SL_4 \mathbb{R}$ and $\dim T = 15$, moreover, $Z \cong \mathbb{C}^4$ and $Z$ has a subgroup $P \cong \mathbb{R}$ consisting of real dilatations. The group $H \cong TP\Theta$ acts on $M_\nu \cong \mathbb{R}^4$. Choose $\varrho \in M_\nu$ and write $K$ for the connected component of $H_{\varrho \nu}$. Then $\dim K \geq 8$, and $K \cong SU_3 \mathbb{C}$ by (i), but the latter group is not contained in the maximal semi-simple subgroup $T$ of $\hat{H}$. This proves the last claim of step (a).

(f) The group $M \cong \mathbb{R}^8$ acts transitively on the affine line pencil $L_u \setminus \{W\} = L_u^* \cong \mathbb{R}^8$; in Lemma 3 it has been shown that $\Theta_L = 1$ for each line $L \in L_u^*$ with at most one exception $au$. By [18] (96.11), each non-trivial orbit $L^M$ is open in $L_u^*$ and homeomorphic to $\mathbb{R}^8$. Hence $M$ is sharply transitive on $L_u^*$ or on $L_u^* \setminus \{au\}$, but the latter space is not contractible.

(g) According to (e) there is an element $\alpha \neq 1$ in $\Delta_L \cap CS M$, and (f) implies that $\alpha$ fixes each line in $L_u$. Consequently, $\alpha$ is an axial collineation with center $u$ and some axis $A$. If $\alpha$ is an elation, i.e. if $u \in A$, then $A = W$ (since $\alpha^M = \alpha$), and $\alpha^\Theta$ is a 4-dimensional subset of $T$. Because the elements in $\alpha^\Theta$ have different centers, $T$ is commutative. Any minimal invariant subgroup of $T$ may be chosen as the group $\Theta$.

(h) Now let $\alpha \in \Delta_{[u,A]}$ be a homology. Since $u^A \neq u$, Hähl’s results on the generation of elations by homologies can be applied. In its simplest form, Hähl’s theorem says the
following:

(●) If \( \Gamma \) is a Lie subgroup of \( \Sigma_A \), if \( \Gamma_{[c,A]} \neq \{1\} \) for some center \( c \notin A \), and if \( E \) is the group of elations in \( \Gamma \) with axis \( A \), then \( \dim E = \dim c^\Gamma \), see [18] (61.20) for a proof. Note that commutativity of \( E \) is not known if \( c^\Gamma \) is contained in a line.

(i) Suppose that \( \Theta \leq \Gamma = \Delta_A \), and put \( A \cap W = v \). Then (●) shows that \( \dim \Delta_{[v,A]} \geq 4 \). Moreover, \( \dim u^\Theta = 4 \) implies that the commutator set \( \{\alpha,\Theta\} = \{\alpha^{-1}\alpha\theta \mid \theta \in \Theta\} \subseteq \Theta_{[A]} \) is at least 4-dimensional. Since \( \Theta \) is commutative, all elements in \( \Theta_{[A]} \) have the same center \( z \in W \), and \( \dim \Theta_{[A]} \leq 8 \). By [18] (61.2), a group of homologies has a compact subgroup of codimension at most 1, but \( \Theta \) is a vector group. Hence \( \Theta_{[A]} = \Theta_{[v,A]} \) consists of elations. Because \( \Theta \) is a minimal normal subgroup of \( \Delta \), the group \( \Theta_{[A]} \) is not normal, and, therefore, \( A^\Delta \neq A \). Since \( A^\Theta = A \) and \( \Theta \) is normal, \( \Theta \) fixes each line in the orbit \( A^\Delta \). On the other hand, all fixed lines of the elation group \( \Theta_{[A]} \) pass through the center \( v \). Consequently, \( A^\Delta \subseteq \mathcal{L}_v \) and \( \Theta = \Theta_{[v,A]} \) is an invariant elation group.

(j) Only the possibility \( A^\Theta \neq A \) remains. The strategy in this case is to apply the dual (●) of (●) to the group \( \Omega \) generated by \( \alpha \) and the connected component of \( \Delta_u \). Assume first that \( A^\Omega = A \). Because \( \dim (\Theta_u \cup (\Theta_{[A]} \leq 12) \), there is an element \( \vartheta \in \Theta \) such that \( u^\vartheta = z \neq u \) and \( \Theta^u \neq A \). Remember that \( \Theta \leq \Delta \) and that \( \dim u^\Theta = 4 \). Therefore, \( z^\Theta \subseteq u^\Omega \) and \( \dim \Omega_z \geq 21 \). Let \( a \in A \setminus W \), \( a \notin A^\Theta \). Then the connected component \( \Lambda \) of \( \Omega_{a,z} \) satisfies \( \dim \Lambda \geq 13 \), moreover, \( \Lambda \leq \Omega^\vartheta \leq \Delta_{A^\vartheta} \), and \( \Lambda \) fixes a non-degenerate quadrangle. Now (‡) implies \( \Lambda \cong G_2 \). By assumption, \( \Gamma = \Delta/Cs\Theta \) is an irreducible subgroup of \( \text{GL}_{12}\mathbb{R} \), and \( \dim \Gamma \leq 24 \) by Corollary 4. Since \( \Lambda \leftrightarrow \Gamma' \), it follows from [18] (95.6) that \( \Gamma' \) is almost simple and irreducible on \( \Theta \). Hence \( \Gamma' \cong \text{Spin}_7\mathbb{R} \), but, according to the List, this group does not have an irreducible representation in dimension 12.

(k) Consequently, \( A^\Omega \neq A \), and then \( \dim \Delta_{[u,u]} \geq \dim A^\Omega > 0 \) by (●). If some one-parameter group in \( \Delta_{[u,u]} \) has an affine axis, then, because of (f), all groups \( \Delta_{[u,L]} \) with \( L \in \mathcal{L}_u \setminus \{W\} \) have the same positive dimension, and the dual of [18] (61.12) implies \( \Delta_{[u,L]} \cong \mathbb{R}^8 \). Since \( u^\Delta \neq u \), it follows that \( \Delta \) is transitive. If each one-parameter subgroup of \( \Delta_{[u,u]} \) has axis \( W \) however, then \( \dim \Delta_{[z,W]} > 0 \) for each center \( z \in u^\Delta \). Hence \( \dim \Gamma > 4 \), and \( \Gamma \) contains a normal subgroup \( \Theta \) as claimed.

Theorem A is now an immediate consequence of the propositions.

References


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