The Second Extension of the Thas-Walker Construction

Rolf Riesinger

Patrizigasse 7/14, A-1210 Vienna, Austria

Abstract. We extend the extended Thas-Walker construction by introducing the concept “flocklet” of the Klein quadric. Thus we have a tool to construct spreads with asymptotically complemented regulization.

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1. Introduction

Independently, M. Walker [15] and J. A. Thas (unpublished) discovered that to each flock of an elliptic or hyperbolic quadric or of a quadratic cone of PG(3, q) a spread of PG(3, q) and, consequently, a (finite) translation plane is constructable; compare [1, p. 8], [12, p. 441] or the surveys [3], [13, p.95-96], [14]. The Thas-Walker construction remains valid for flocks of PG(3, \( \mathbb{K} \)) with arbitrary commutative field \( \mathbb{K} \); cf. [6, p. 146-149]. In [7] the author used flocks of PG(3, \( \mathbb{K} \)) to construct spreads representing topological translation planes. The extended Thas-Walker construction exhibited in [8], hereafter called [ETW], starts with a flockoid of a Lie quadric of PG(4, \( \mathbb{K} \)) and yields also a spread; a Lie quadric \( L_4 \) is a hyperquadric of a Pappian projective 4-space such that \( L_4 \) has no vertex and contains a line; a collection \( \mathcal{D} \) of conics contained in a Lie quadric \( L_4 \) of a Pappian projective 4-space is called a flockoid of \( L_4 \), if the following two conditions hold:

\( \text{(FD1) } \) For each generatrix \( g \) of \( L_4 \) there exists exactly one conic \( k \in \mathcal{D} \) with \( g \cap k \neq \emptyset \).
\( \text{(FD2) } \) There are at most two improper conics in \( \mathcal{D} \).

In view of the present article, the paper [ETW] deals with the first extension of the Thas-Walker construction.
Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a Pappian projective 3-space with point set $\mathcal{P}$ and line set $\mathcal{L}$. As we deal with spreads composed of reguli and at most two exceptional lines, so we standardize by defining: A proper regulus $\mathcal{R}$ is the set of lines meeting three mutually skew lines; the directrices of $\mathcal{R}$ form the complementary (opposite) regulus $\mathcal{R}^c$; if $x \in \mathcal{L}$, then $\{x\}$ is called an improper regulus; $\{x\}^c := \{x\}$.

**Definition 1.** Let $\mathcal{S}$ be a spread of $\Pi$ and let $\Sigma$ be a collection of (proper or improper) reguli contained in $\mathcal{S}$. We call $\Sigma$ a regulization of $\mathcal{S}$, if the following hold:

(RZ1) Each line of $\mathcal{S}$ belongs to exactly one regulus of $\Sigma$ or to all reguli of $\Sigma$.

(RZ2) There are at most two improper reguli in $\Sigma$.

The set $\bigcup(\mathcal{R}^c | \mathcal{R} \in \Sigma) =: S^c_\Sigma$ is named complementary congruence of $\mathcal{S}$ with respect to $\Sigma$. If $S^c_\Sigma$ is an elliptic linear congruence of lines, then $\Sigma$ is called an elliptic regulization of $\mathcal{S}$. If $S^c_\Sigma$ belongs to a linear complex of lines, then we say that $\Sigma$ is a symplectically complemented regulization, otherwise we speak of an asymplyctically complemented regulization. If $S^c_\Sigma$ belongs to a single linear complex of lines, then $\Sigma$ is called a uni-symplectically complemented regulization.

The papers [ETW], [9], and [10] are devoted to the construction and investigation of spreads with symplectically complemented regulization. For the real projective 3-space $\text{PG}(3, \mathbb{R})$ an example of a non-regular spread with an asymplyctically complemented regulization is given in [6, 4.1.7]).

Let $\lambda$ be the well-known Klein mapping of $\mathcal{L}$ onto the Klein quadric $H_5$ which is embedded into a projective 5-space $\Pi_5$ with point set $\mathcal{P}_5$; cf. e.g. [4] and the translation table in [2, p. 29-30]. A Latin <Greek> plane on $H_5$ is the $\lambda$-image of a star of lines <a ruled plane>. If $\mathcal{R}$ is a proper or improper regulus, then $\lambda(\mathcal{R})$ is an irreducible conic or a point. For obvious reasons, we speak of proper or improper conics. If $\mathcal{S}$ is a spread of $\Pi$ with the symplectically complemented regulization $\Omega$, then $\{\lambda(\mathcal{R}^c) | \mathcal{R} \in \Omega\}$ is a flockoid of a uniquely determined Lie quadric $L_4$ with $L_4 \subset H_5$; cf. [ETW, Prop. 1].

Recall the extended Thas-Walker construction [ETW, Prop. 2]: If $D$ is a flockoid of a Lie quadric $L_4$ with $L_4 \subset H_5$, then $\bigcup((\lambda^{-1}(k))^c | k \in D)$ is a spread of $\Pi$ with the regulization $\{((\lambda^{-1}(k))^c | k \in D\}$ which is either unisymplectically complemented or elliptic.

In Proposition 1 we start with a spread $\mathcal{S}$ of $\Pi$ admitting an asymplyctically complemented regulization $\Gamma$ and investigate the set $\{\lambda(\mathcal{R}) | \mathcal{R} \in \Gamma\}$ =: $\mathcal{C}$ of conics. In the proof of Proposition 1 we shall find that $\mathcal{C}$ is a “flocklet” of the Klein quadric $H_5$; we define the concept “flocklet”, as follows.

**Definition 2.** A collection $\mathcal{B}$ of (proper or improper) conics contained in the Klein quadric $H_5$ is called a flocklet <flockling> of $H_5$, if the following two conditions hold:

(FT1)<(FG1)> For each Latin <Greek> plane $\gamma$ on $H_5$ there exists exactly one conic $k \in \mathcal{B}$ with $\gamma \cap k \neq \emptyset$.

(FT2)<= (FG2)> There are at most two improper conics in $\mathcal{B}$.

Depending on the intersection $M$ of the carrier planes of the participating proper conics we distinguish three types of flocklets:

\[^1\text{The assumption in Proposition 1 is more comprehensive; cf. Remark 1 and Definition 3.}\]
1) $M$ is empty (unbundled flocklet)
2) $M$ consists of exactly one point (star flocklet)
3) $M$ is a line (linear flocklet)\(^2\).

In the same way unbundled flocklings, star flocklings, and linear flocklings are defined.

The second extension of the Thas-Walker construction starts with a flocklet $\mathcal{A}$ of the Klein quadric $H_5$. Then $\cup((\lambda^{-1}(k))^{c}|k \in \mathcal{A})$ is a spread of $\Pi$ admitting the regulization $\{(\lambda^{-1}(k))^{c}|k \in \mathcal{A}\}$ which is either asympetically complemented or unisymplectically complemented or elliptic; cf. Proposition 2 and Remark 1. Each flockoid of a Lie quadric $L_4$ can be interpreted as flocklet and also as flockling of a Klein quadric $H_5$ containing $L_4$; cf. Remark 3. Each flock of an elliptic quadric $Q_e$ can be interpreted as flocklet and also as flockling of a Klein quadric $H_5$ containing $Q_e$; cf. Remark 4. Note, that a flock of a quadric $Q$ covers $Q$, but a flocklet of a Klein quadric $H_5$ is no covering of $H_5$.

In Section 3 we introduce Thas-Walker plane sets of Latin type to get further properties of the second extension of the Thas-Walker construction.

2. The second extension of the Thas-Walker construction

**Definition 3.** Let $\Sigma$ be an arbitrary regulization of a spread $\mathcal{S}$ of $\Pi$. We call

$$i(\Sigma):=\# \left(\bigcap_{\mathcal{R} \in \Sigma} \mathcal{R}\right)$$

the intersection number of $\Sigma$. If $i(\Sigma)=0$, then we say that the regulization $\Sigma$ is of intersection number 0.

By [6, Remark 2.4], $i(\Sigma) \in \{0,1,2\}$. From [6, Remark 2.5 and 2.6] we deduce:

**Remark 1.** A regulization $\Sigma$ is of intersection number 0 if, and only if, $\Sigma$ is asympetically complemented or unisymplectically complemented or elliptic.

In the following remark we sum up further properties of a regulization of intersection number 0; for (1) and (2) see [6, Remark 2.8 resp. 2.9], the statements (3) and (4) are evident.

**Remark 2.** Let $\mathcal{S}$ be a spread of $\Pi$ admitting the regulization $\Sigma$ of intersection number 0. Then following statements hold true:

1. Each element of $\mathcal{S}$ belongs to exactly one regulus of $\Sigma$.
2. The complementary congruence $\mathcal{S}_c^\Sigma=\cup(\mathcal{R}^c|\mathcal{R} \in \Sigma)$ of $\mathcal{S}$ with respect to $\Sigma$ is also a spread of $\Pi$.
3. $\{\mathcal{R}^c|\mathcal{R} \in \Sigma\}=:\Sigma^c$ is a regulization of $\mathcal{S}_c^\Sigma$ with $i(\Sigma^c)=0$.
4. The complementary congruence $(\mathcal{S}_c^\Sigma)^c_\Sigma$ of $\mathcal{S}_c^\Sigma$ with respect to $\Sigma^c$ coincides with $\mathcal{S}$; in symbols:

$$\left((\mathcal{S}_c^\Sigma)^c_\Sigma\right)_\Sigma^c = \mathcal{S}.$$

\(^2\)In contrast to a linear flock, a linear flocklet is not uniquely determined by the common line of the carrier planes of the participating proper conics.
The rest of this section generalizes [ETW, Section 3].

**Proposition 1.** Let \( S \) be a spread of \( \Pi \) and let \( \Gamma \) be a regulization of \( S \) with intersection number 0. Then \( \{ \lambda(\mathcal{R}^c) | R \in \Gamma \} \) is a flocklet of the Klein quadric \( H_5 \).

**Proof.** Clearly, (RZ2) implies (FT2). By Remark 2 (3), \( S^c \) is a spread admitting the regulization \( \Gamma^c \) with \( \delta(\Gamma^c) = 0 \); we can say that \( C \) is the \( \lambda \)-image of the regulization \( \Gamma^c \). Let \( \eta \) be an arbitrary Latin plane on \( H_5 \), then \( \lambda^{-1}(\eta) \) is a star of lines with a vertex, say \( Y \). A conic \( k \) of \( C \) has a non-empty intersection with \( \eta \) if, and only if, the regulus \( \lambda^{-1}(k) \) of \( \Gamma^c \) contains a line incident with \( Y \). In the spread \( S^c \) there exists exactly one line, say \( s_Y \), incident with \( Y \). Because of \( \delta(\Gamma^c) = 0 \) and Remark 2 (1), \( s_Y \) belongs to exactly one regulus, say \( \mathcal{R}_Y^c \), of \( \Gamma^c \). Now \( Y \in s_Y \in \mathcal{R}_Y^c \subseteq \Gamma^c \) implies \( \lambda(s_Y) \in \eta \cap \lambda(\mathcal{R}_Y^c) \) and \( \lambda(\mathcal{R}_Y^c) \subseteq C \), i.e., the conic \( \lambda(\mathcal{R}_Y^c) \) is the only element of \( C \) having at least one common point with \( \eta \). \( \square \)

**Remark 3.** Let \( D \) be a flockoid of the Lie quadric \( L_4 \) with \( L_4 \subseteq H_5 \). Then \( D \) is a flocklet and also a flocking of \( H_5 \).

**Proof.** Let \( \xi \) be an arbitrary (Latin or Greek) plane on \( H_5 \). Then \( \xi \cap \text{span} \ L_4 \) is always a line, say \( x \). Because of (FD1)\(^3\), there exists exactly one conic \( k_x \in D \) with \( k_x \cap x \neq \emptyset \) and thus \( k_x \cap \xi \neq \emptyset \). \( \square \)

By [ETW, Remark 4], each Lie quadric of \( \text{PG}(4, \mathbb{K}) \) is embeddable into the Klein quadric \( H_5 \) of \( \text{PG}(5, \mathbb{K}) \), hence each flockoid of a Lie quadric can be interpreted as flocklet and also as flocking of a suitable Klein quadric.

**Remark 4.** Let \( F \) be a flock of the elliptic quadric \( Q_e \) with \( Q_e \subseteq H_5 \). Then \( F \) is a flocklet and also as flocking of \( H_5 \).

**Proof.** Let \( \xi \) be an arbitrary (Latin or Greek) plane on \( H_5 \). Then \( \xi \cap \text{span} \ Q_e \) is always a point, say \( X \). In the flock \( F \) there exists exactly one conic \( k_X \) with \( X \in k_X \). \( \square \)

By [ETW, Remark 9], each elliptic quadric of \( \text{PG}(3, \mathbb{K}) \) is embeddable into a Lie quadric of \( \text{PG}(4, \mathbb{K}) \) which in turn is embeddable into the Klein quadric of \( \text{PG}(5, \mathbb{K}) \), by [ETW, Remark 4]. Hence each elliptic flock can be interpreted as flocklet and also as flocking of a suitable Klein quadric.

Before formulating and proving the converse of Proposition 1 in Proposition 2 we expose two lemmas about flocklets. The statements of the following Lemma 1 are immediate consequences of (FT1) and the properties of a plane section of a quadric.

**Lemma 1.** Let \( A \) be a flocklet of the Klein quadric \( H_5 \).

(i) Then different conics of \( A \) are disjoint.

(ii) If \( \{ P_1 \} \) and \( \{ P_2 \} \) are different improper conics of \( A \), then \( P_1 \cap P_2 \not\subseteq H_5 \).

(iii) If \( \eta \) is a Latin plane on \( H_5 \) and \( k \in A \) satisfies \( k \cap \eta \neq \emptyset \), then \( \eta \not\subseteq \text{span} \ k \), \( \eta \cap \text{span} \ k \) is no line, and \( \#(k \cap \eta) = 1 \).

**Lemma 2.** Let \( A \) be a flocklet of the Klein quadric \( H_5 \) and let \( k_1 \) be a proper conic of \( A \). If \( k_2 \in A \setminus \{ k_1 \} \), then there exists no tangent cone \( C_4 \) of \( H_5 \) with \( k_1 \cup k_2 \subseteq C_4 \).

\(^3\text{Compare [ETW, Definition 3].} \)
Proof. Let $C_4$ be a tangent cone of $H_5$ with a vertex, say $V$, and with $k_1 \subset C_4$. In span $C_4$ there exists a 3-dimensional subspace, say $S_3$, with $V \not\in S_3$. Now $C_4 \cap S_3 =: Q_h$ is a hyperbolic quadric and the Latin planes on $C_4$ give rise to a regulus $R_{Q_h}$ on $Q_h$. By $k_1^+$ we denote the image of $k_1$ under the projection from centre $V$ onto $S_3$. The proper conic $k_1^+ \subset Q_h$ meets each line of the regulus $R_{Q_h}$ in exactly one point. Consequently, $k_1$ and each Latin plane on $C_4$ have exactly one common point and, because of (FT1), $C_4$ cannot contain further (proper or improper) conics of $\mathcal{A}$.

Proposition 2. If $\mathcal{A}$ is a flocklet of the Klein quadric $H_5$, then

$$\cup((\lambda^{-1}(k))^c | k \in \mathcal{A}) =: T_{E_2}(\mathcal{A})$$ (3)

is a spread of $\Pi$ admitting the regularization

$$\{(\lambda^{-1}(k))^c | k \in \mathcal{A}\} =: T_{R_2}(\mathcal{A})$$ (4)

and $T_{R_2}(\mathcal{A})$ is of intersection number 0.

Proof. Let $X$ be an arbitrary point of $\Pi$ and denote the star of lines with vertex $X$ by $\mathcal{L}[X]$. In $T_{E_2}(\mathcal{A})$ there exists a line incident with $X$ if, and only if, there is a conic $k_X \in \mathcal{A}$ such that $X$ is on a line $h$ of the regulus $\lambda^{-1}(k_X)$, i.e., $h \in \mathcal{L}[X] \cap \lambda^{-1}(k_X)$ and thus $\lambda(h) \in \lambda(\mathcal{L}[X]) \cap k_X$. As $\lambda(\mathcal{L}[X])$ is a Latin plane on $H_5$, so there is a unique $k_X \in \mathcal{A}$ with $k_X \cap \lambda(\mathcal{L}[X]) \neq \emptyset$, by (FT1). Hence there is a unique regulus in $T_{E_2}(\mathcal{A})$, namely $(\lambda^{-1}(k_X))^c$, which contains a line incident with $X$. Consequently, $T_{E_2}(\mathcal{A})$ is a spread.

Next we prove the validity of (RZ1) and (RZ2) for $T_{R_2}(\mathcal{A})$. Clearly, (FT2) implies (RZ2). Instead of (RZ1) we show even more:

(RZ1*) Each line of $T_{E_2}(\mathcal{A})$ belongs to exactly one regulus of $T_{R_2}(\mathcal{A})$.

Let $b \in T_{E_2}(\mathcal{A})$ be arbitrary. We assume, to the contrary,

$$b \in (\lambda^{-1}(k_1))^c \cap (\lambda^{-1}(k_2))^c, \quad \{k_1, k_2\} \subset \mathcal{A}, \quad k_1 \neq k_2.$$ (5)

In the case that both $(\lambda^{-1}(k_1))^c$ and $(\lambda^{-1}(k_2))^c$ are improper reguli with $(\lambda^{-1}(k_i))^c = \{g_i\}$ and $g_i \in \mathcal{L}, \ (i = 1, 2)$, the lines $g_1$ and $g_2$ are skew and (5) yields the absurdity $b \in \{g_1\} \cap \{g_2\} = \emptyset$. Hence we may assume, without loss of generality, that $(\lambda^{-1}(k_1))^c$ is a proper regulus. Each line of $(\lambda^{-1}(k_1))^c \cup (\lambda^{-1}(k_2))^c$ meets $b$. Thus $k_1 \cup k_2$ is contained in the tangent cone of $H_5$ at the point $\lambda(b)$, a contradiction to Lemma 2. Thus $T_{R_2}(\mathcal{A})$ is a regulization and the validity of (RZ1*) implies $i(T_{R_2}(\mathcal{A})) = 0$.

The process of gaining a spread from a flocklet via formula (3) is called second extension of the Thas-Walker construction.

We combine Proposition 1 and 2 and get

Corollary 1. To each spread of PG(3, $K$) admitting a regulization of intersection number 0 there corresponds a flocklet of the Klein quadric of PG(5, $K$), and vice versa.
3. Thas-Walker plane sets of latin type

Let $Q$ be an elliptic or hyperbolic quadric of $PG(3,\mathbb{K})$ with bijective polarity $\pi_Q$. A point set $T$ of $PG(3,\mathbb{K})$ is called a Thas-Walker point set with respect to $Q$, if $\{\pi_Q(X) \cap Q \mid X \in T \land \pi_Q(X) \cap Q \neq \emptyset\}$ is a flock of $Q$. Let $L_4$ be a Lie quadric of $PG(4,\mathbb{K})$ with polarity $\pi_4$. A line set $T_e$ of $PG(4,\mathbb{K})$ is called a Thas-Walker line set with respect to $L_4$, if $\{\pi_4(X) \cap L_4 \mid X \in T_e \land \pi_4(X) \cap L_4 \neq \emptyset\}$ is a flock of $L_4$. In [7, Section 2.2], we considered a Thas-Walker point set $T$ with respect to an elliptic quadric $E \subset H_5$ and got the $\lambda$-image of a spread $\mathcal{T}$ by projecting $T$ from the line $e = \pi_5(span E)$, with $\pi_5$ the polarity defined by the Klein quadric $H_5$, into $H_5$, in symbols $\lambda(\mathcal{T}) = \bigcup ((X \lor e) \cap H_5 | X \in T)$. In [ETW, Section 4], we took a Thas-Walker line set $T_t$ with respect to a Lie quadric $L_4 \subset H_5$ and got the $\lambda$-image of a spread $\mathcal{T}_t$ by projecting $T_t$ from the point $Z = \pi_5(span L_4)$ into the Klein quadric $H_5$, in symbols $\lambda(\mathcal{T}_t) = \bigcup ((x \lor Z) \cap H_5 | x \in T_t)$. In the present section we continue this process: we shall take a Thas-Walker plane set $T_{La}$ of Latin type with respect to the Klein quadric $H_5$ and shall get the $\lambda$-image of a spread $\mathcal{T}_{La}$ by projecting $T_{La}$ from $\pi_5(span H_5) = \emptyset$ into the Klein quadric $H_5$. As this projection is the identity of $\mathcal{P}_5$, so the following statements and formulas concerning Thas-Walker plane sets of Latin type are of simpler appearance as their analogues in the case of Thas-Walker point resp. line sets.

A set $T_{La}$ of planes of $\Pi_5$ is called Thas-Walker plane set of Latin type with respect to $H_5$, if

$$D_2(T_{La}) := \{ \pi_5(\xi) \cap H_5 \mid \xi \in T_{La} \} \text{ with } T_{La} := \{ \xi \in T_{La} \mid \pi_5(\xi) \cap H_5 \neq \emptyset \} \tag{6}$$

is a flock of $H_5$. We put

$$T_{La}^p := \{ \xi \in T_{La} \mid \#(\pi_5(\xi) \cap H_5) > 1 \} \tag{7}$$

for the set of those planes of $T_{La}$ which yield proper conics.

**Remark 5.** Let $\{ P \} \subset H_5$ be an improper conic. In the case $\mathbb{K} = \mathbb{R}$ there are infinitely many planes $\alpha$ with $\pi_5(\alpha) \cap H_5 = \{ P \}$, since there are infinitely many planes incident with $P$ and contained in $\pi_5(P)$ and intersecting the tangent cone $\pi_5(P) \cap H_5$ only in $P$ (compare also the description of a tangent cone of $H_5$ in the proof of Lemma 2). In other words, if $T_{La_1}$ and $T_{La_2}$ are Thas-Walker plane sets of Latin type with respect to $H_5$, then $D_2(T_{La_1}) = D_2(T_{La_2})$ implies $T_{La_1} = T_{La_2}$, but not $T_{La_1}^p = T_{La_2}^p$.

For the discussion of Thas-Walker line sets in [ETW] we had to assume $\text{Char} \mathbb{K} \neq 2$ throughout Section 4. Here we need this additional assumption only in the following

**Lemma 3.** Assume $\text{Char} \mathbb{K} \neq 2$. Denote by $\mathcal{P}_L[H_5]$ the set of all Latin planes on the Klein quadric $H_5$, by the way, $\pi_5(\mathcal{P}_L[H_5]) = \mathcal{P}_L[H_5]$. A set $A$ of planes of $\Pi_5$ is a Thas-Walker plane set of Latin type with respect to $H_5$ if, and only if, the following three conditions hold true:

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4This concept has to be defined still.

5In order to have full conformity with the corresponding Lemma 5 from [ETW] we start the numbering of the conditions with 2.
Proof. If the intersection of the plane $\alpha \in A$ and the plane $\pi_5(\alpha)$ is empty, then $\pi_5(\alpha) \cap H_5$ is either a proper conic or empty, and conversely. We define $D_2(A)$ according to (6). Now (TWLa2) and (TWLa3) imply that all elements of $D_2(A)$ are proper or improper conics and that $D_2(A)$ satisfies (FT2), and vice versa. Finally, (TWLa4) $\Rightarrow$ (FT1).

If $k \subset H_5$ is a proper conic, then $\left(\lambda^{-1}(k)\right)^c = \lambda^{-1} \left(\pi_5(\text{span } k)\right)$. If $\alpha$ is a plane of $\Pi_5$ such that $\alpha \cap H_5$ is an improper conic, say $\{A\}$, then, as the reader proves easily, $\pi_5(\alpha) \cap H_5 = \{A\}$ and hence $\left(\lambda^{-1}(\{A\})\right)^c = \lambda^{-1} (\pi_5(\alpha))$. Thus we have the subsequent modification of the second extension of the Thas-Walker construction:

**Lemma 4.** If $T_{La}$ is a Thas-Walker plane set of Latin type with respect to the Klein quadric $H_5$, then

$$T_{La} := \cup (\lambda^{-1}(\xi) \mid \xi \in T_{La})$$

is a spread of $\Pi$ admitting the regulization

$$\Theta_{La} := \{\lambda^{-1}(\xi) \mid \xi \in T^r_{La}\}$$

wherein $T^r_{La}$ is defined by (6); $\Theta_{La}$ is of intersection number 0.

We say that $\Phi(T^r_{La}) := \cup \tau \mid \tau \in T^r_{La}$ is the 3-surface determined by $T^r_{La}$ and that each plane $\tau \in T^r_{La}$ is a $T^r_{La}$-generatrix of $\Phi(T^r_{La})$. The following lemma is evident.

**Lemma 5.** Suppose that the condition (and notations) of Lemma 4 hold. If each proper conic $k$ with $k \subset \Phi(T^r_{La}) \cap H_5$ is contained in a $T^r_{La}$-generatrix of $\Phi(T^r_{La})$, then

1. each proper regulus contained in the spread $T_{La}$ belongs to $\Theta_{La}$;
2. $T_{La}$ admits exactly one regulization, namely $\Theta_{La}$.

Put

$$v' := \dim \left( \bigvee_{\xi \in T^r_{La}} \xi \right) \quad \text{and} \quad v^p := \dim \left( \bigvee_{\xi \in T^r_{La}} \xi \right).$$

(A) If $v' = v^p = 5$, then $T_{La}$ is an asymptotic spread and $D_2(T_{La})$ is an unbundled flocklet.

In order to give a survey of all imaginable cases of $v' = v^p$ we exhibit the subsequent table. The first row of the table represents an abbreviation of statement (A). The first column of the table gives the assumption, middle and left column are the conclusions.

<table>
<thead>
<tr>
<th>$v' = v^p$</th>
<th>$T_{La}$</th>
<th>$D_2(T_{La})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>asymptotic</td>
<td>unbundled</td>
</tr>
<tr>
<td>4</td>
<td>unisymplectic</td>
<td>star</td>
</tr>
<tr>
<td>3</td>
<td>regular</td>
<td>linear</td>
</tr>
</tbody>
</table>

Table 1. Deductions from $v' = v^p$
Definition 4. If a flocklet \( B \) of the Klein quadric \( H_5 \) is neither a flockoid of a Lie quadric \( L_4 \subset H_5 \) nor a flock of an elliptic quadric \( Q_e \subset H_5 \), then we say that \( B \) is a genuine flocklet\(^9\).

Put
\[
d' := \dim( \bigcap_{\xi \in T_{L_a}} \xi ) \quad \text{and} \quad d'' := \dim( \bigcap_{\xi \in T'_{L_a}} \xi ).
\]

(B) If \( d' = d'' = -1 \), then \( D_2(T_{L_a}) \) is a genuine flocklet and \( \Theta_{L_a} \) is an asymptotically complemented regulization.

The following table gives a survey.

<table>
<thead>
<tr>
<th>( d' = d'' )</th>
<th>( \Theta_{L_a} )</th>
<th>( D_2(T_{L_a}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>asymptotically complemented</td>
<td>genuine flocklet</td>
</tr>
<tr>
<td>0</td>
<td>unisymplectically complemented</td>
<td>genuine flockoid</td>
</tr>
<tr>
<td>1</td>
<td>elliptic</td>
<td>elliptic flock</td>
</tr>
</tbody>
</table>

Table 2. Deductions from \( d' = d'' \)

4. Spreads with regulizations of intersection number 0: types

Given a spread \( S \) of \( \Pi \) with regulization \( \Sigma \) we can distinguish 9 combinations:

**Combination 1**: \( S \) regular and \( \Sigma \) elliptic: Examples are well-known.

**Combination 2**: \( S \) regular and \( \Sigma \) unisymplectically complemented: See [9, Section 5, Type 7].

**Combination 3**: \( S \) regular and \( \Sigma \) asymptotically complemented: See the subsequent Remark 6.

**Combination 4**: \( S \) unisymplectic and \( \Sigma \) elliptic: See [7, Theorem 3.3.1].

**Combination 5**: \( S \) unisymplectic and \( \Sigma \) unisymplectically complemented: See [9, Section 5, Type 3, Type 4, Type 5, and Type 6]. The case from [9, Section 5, Type 4] is investigated in [9, Section 6] shortly.

**Combination 6**: \( S \) unisymplectic and \( \Sigma \) asymptotically complemented: See [6, Remark 4.1.2 and (4.1.7)].

**Combination 7**: \( S \) asymplectic and \( \Sigma \) elliptic: See [7, Theorem 3.2.1].

**Combination 8**: \( S \) asymplectic and \( \Sigma \) unisymplectically complemented: See [9, Section 5, Type 1 and Type 2] which are investigated in [10] thoroughly.

**Combination 9**: \( S \) asymplectic and \( \Sigma \) asymptotically complemented: For sake of completeness, we construct an example in the subsequent Remark 7.

**Remark 6.** We use the concepts and notations of [6, Section 4]. Let \( \mathcal{E} \) be a regular spread of \( \text{PG}(3, \mathbb{R}) \). We decompose the elliptic quadric \( \lambda(\mathcal{E}) \) by two (proper) disjoint conics \( c_1, c_2 \) of \( \lambda(\mathcal{E}) \) into two elliptic caps \( U_1, U_2 \), and an elliptic zone \( V \). Put \( \text{span } c_1 \cap \text{span } c_2 =: v \). Choose lines \( g_j (j = 1, 2) \) with \( g_j \subset \text{span } c_j, g_j \cap c_j = \emptyset \), and \( g_j \neq v \) such that \( g_1 \) and \( g_2 \) are skew. Then
\[
\Phi := \lambda^{-1}(\mathcal{C}(U_1, g_1) \cup \mathcal{C}(V, v) \cup \mathcal{C}(U_2, g_2))
\]
is an asymptotically complemented regulization of \( \mathcal{E} \).

\(^9\)Compare also the definition of a genuine flockoid of a Lie quadric given in [9, Definition 1].
Remark 7. Based on [5, Satz 5] in [6, 4.3] two spreads $S_0$ and $S_1$ are defined; with the notations from [5, Satz 5] we have:

$$S_0 = E_1^1 \cup R_{1,1}^1 \cup E_2^2 \cup R_{2,1}^2 \cup E_3^3$$

(13)

wherein $R_{1,1}^1$ and $R_{2,1}^2$ are proper reguli described in [5, 3.2]; $\lambda(S_0)$ is composed of the elliptic caps $\lambda(E_1^1 \cup R_{1,1}^1) =: W_1$, $\lambda(R_{1,1}^1 \cup E_2^2) =: W_2$ and the elliptic zone $\lambda(R_{1,1}^1 \cup E_3^3 \cup R_{2,1}^2) =: Z$ with disjoint limiting conics. Consider the three solids

$$S_1 := \text{span}(W_1) = \{ p \in \mathcal{P} : p_0 + p_3 = p_1 - \varepsilon p_2 + p_4 + \varepsilon p_5 = 0 \}$$

$$S_2 := \text{span}(Z) = \{ p \in \mathcal{P} : p_0 + p_3 = p_1 + p_4 = 0 \}$$

$$S_3 := \text{span}(W_2) = \{ p \in \mathcal{P} : p_1 + p_4 = p_0 + 4\varepsilon'' p_2 + p_3 - \varepsilon'' p_5 = 0 \}$$

(14)

as $\dim(S_1 \cap S_2 \cap S_3) = 5$, so the spread $S_0$ is asymptotic. Choose lines $h_j$ ($j = 1, 2$) with $h_j \subseteq \text{span}(R_{j,1}^j)$, $h_j \cap \lambda(R_{j,1}^j) = \emptyset$, and $h_j \neq z$ such that $h_1$ and $h_2$ are skew. Then

$$\Psi := \bar{x}^{-1}(C(W_1, h_1) \cup C(Z, z) \cup C(W_2, h_2))$$

(15)

is an asymptotically complemented regulization of $S_0$.

The examples given for the Combinations 7, 8, and 9 show

**Corollary 2.** There exist genuine linear flocklets, genuine star flocklets, and genuine unbounded flocklets.

The spread $S_0$ of Remark 7 is like “patchwork”, therefore we ask for an explicit example of an algebraic asymptotic spread\(^{10}\) with asymptotically complemented regulization. Applying the second extension of the Thas-Walker construction we give in [11] such an example.

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References


\(^{10}\)A spread $S$ is called algebraic, if $S$ is an algebraic congruence of lines.


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