Generators of Order Two for $S_n$
and its Two Double Covers

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1. Introduction

This paper considers the minimum number of involutions, $i(G)$, required to generate both of
the double covers $G$ of the symmetric group. In particular, explicit generators, of order two,
for each of the groups are also introduced. These generators may, for example, be useful for
implementation in Magma-Cayley or GAP.

As a starting point we observe that if $i(G) = 1$ then $G$ is cyclic and if $i(G) = 2$ then $G$ is
dihedral. Hence for any group of order greater than two that is not isomorphic to a dihedral
group we have $i(G) \geq 3$.

It is also well-known that the symmetric group $S_n$, $n \geq 3$, may be generated by the
two cycles $(1, 2)$ and $(1, 2, 3, \ldots, n)$. But as we may write $(1, 2, 3, \ldots, n)$ as the product of,
multiplying from the left,

$$(1, n-1)(2, n-2) \ldots (r, n-r) \text{ and } (1, n)(2, n-1) \ldots (t, n+1-t)$$

where $r$ is the integer part of $(n-1)/2$ and $t$ the integer part of $n/2$, it is clear that $S_n$,
$n \geq 4$, may be generated by three involutions. Moreover, for $n \geq 4$, $i(S_n) = 3$.

2. $i(\tilde{S}_n)$

This double cover of $S_n$, which lifts a transposition of $S_n$ to an element of order 4, will be
denoted by $\tilde{S}_n$. So that $\tilde{S}_n$ is the group with generators $z, r_1, r_2, \ldots, r_{n-1}$ and relations

$$z^2 = 1,$$
\[ zr_i = r_i z, \quad r_i^2 = z \quad \text{for} \quad 1 \leq i \leq n - 1, \]
\[ (r_jr_{j+1})^3 = z \quad \text{for} \quad 1 \leq j \leq n - 2, \]
\[ r_hr_k = zr_hr_k \quad \text{for} \quad |h - k| > 1 \quad \text{and} \quad 1 \leq h, k \leq n - 1. \]

For computations in \( \tilde{S}_n \) we will use a method first described by Conway and others at Cambridge (Atlas [2]). This method is outlined in a paper by David B. Wales [5]. In this the elements of \( \tilde{S}_n \) are products of the form \( \pm[\sigma_i] \), where the \( \sigma_i \) are disjoint cycles in \( S_n \) and \( \pm[\sigma_i] \) are the corresponding lifts in \( \tilde{S}_n \).

**Definition 2.1.** For distinct elements \( a_1, \ldots, a_m \) we define \([a_1 a_2 \ldots a_m] = a_1 a_2 \ldots a_m a_1\). We call \( \pm[a_1 a_2 \ldots a_k] \) signed cycles in \( \tilde{S}_n \). Each is a lift of the cycle \( (a_1 a_2 \ldots a_k) \) in \( S_n \).

In fact each \( a_i \) corresponds to an element of a subgroup of a Clifford algebra which is isomorphic to \( \tilde{S}_n \). But the following rules are sufficient to enable the calculation of products of disjoint signed cycles in \( \tilde{S}_n \) (these appear as 2.3 and 2.4 in [5]).

\[
[a_i] = -1, \\
[a_1 a_2 \ldots a_m] = (-1)^{m+1}[a_2 a_3 \ldots a_m a_1], \\
[a_1 a_2 \ldots a_{m-1}] a_m = (-1)^m a_m [a_1 a_2 \ldots a_{m-1}].
\]

In particular, these are used in [1] to prove the following proposition.

**Proposition 2.2.** Any product of \( k \) disjoint signed transpositions in \( \tilde{S}_n \) has order two if the integer part of \((k + 1)/2\) is a multiple of two, and order four otherwise.

Hence in \( \tilde{S}_n \) an involution is of the form \( \pm \pi \) where \( \pi \) is a signed cycle consisting of \( k \) disjoint transpositions and \( k \) is congruent to 0 or 3 modulo 4. Also as the only other element of order two in \( \tilde{S}_n \) is \( -1 \), we have immediately that \( \tilde{S}_4 \) and \( \tilde{S}_5 \) may not be generated by involutions.

Also, as we will be making repeated use of this fact, it is convenient at this point to note that factoring out \( \tilde{S}_n \) by the subgroup \( Z = \langle 1, -1 \rangle \) recovers \( S_n \). That is \( \tilde{S}_n / Z \cong S_n \).

**Proposition 2.3.** For \( n \geq 3 \), \( \tilde{S}_n \) may be generated by \( a = \pm[1, 2] \) and \( b = \pm[1, 2, 3, \ldots, n] \).

**Proof.** As \( S_n \) is generated by \( (1, 2) \) and \( (1, 2, 3, \ldots, n) \) it follows that \( a \) and \( b \) will generate at least one, up to parity of sign, of every type of signed cycle. Hence we only need show that we may also generate \( -1 \). But \( a^2 = -1 \).

**Proposition 2.4.** \( i(\tilde{S}_6) = 5 \) and \( i(\tilde{S}_7) = 3 \).

**Proof.** That \( i(\tilde{S}_6) = 5 \) is proved in Section 3. However, in order to give specific generators, we note that \( (1,2)(3,4)(5,6), (1,3)(2,4)(5,6), (1,4)(2,3)(5,6), (1,5)(2,6)(3,4) \) and \( (1,6)(2,3)(4,5) \) generate \( \tilde{S}_6 \). Thus, as in Proposition 2.3, the corresponding signed cycles will generate \( \tilde{S}_6 \).

For \( \tilde{S}_7 \) we only need note that
\[
(1, 2)(3, 4)(5, 6), \quad (1, 4)(3, 5)(2, 7) \quad \text{and} \quad (2, 3)(3, 7)(1, 6)
\]
generate \( S_7 \). Hence the corresponding signed cycles generate \( \tilde{S}_7 \).
Proposition 2.5. For $8 \leq n \leq 12$, the following involutions generate $S_n$. When $n = 8$

$$a = [1, 2][3, 8][4, 7][5, 6], \quad b = [1, 3][4, 8][5, 7] \quad \text{and} \quad c = [3, 8][4, 7][5, 6].$$

For $9 \leq n \leq 12$  
$$a = [1, 2][3, 9][4, 8][5, 7], \quad b = [1, 3][4, 9][5, 8][6, 7]$$

and  
$$c = \begin{cases} 
[3, 9][4, 8][5, 7] & \text{when } n = 9, \\
[3, 10][4, 8][5, 7] & \text{when } n = 10, \\
[3, 10][4, 11][5, 7] & \text{when } n = 11, \\
[3, 10][4, 11][5, 12] & \text{when } n = 12.
\end{cases}$$

Proof. For $n = 8$ and $9$ we have $ac = \pm[1, 2]$ and $ab = \pm[1, 2, \ldots, n]$ from which the result follows.

When $n = 10, 11$ or $12$, $(ac)^3 = \pm[1, 2]$ and $ab = \pm[1, 2, \ldots, 9]$ so for each $n$ we may generate the subgroup $S_n$, in particular the signed cycles $d = [1, 3], e = [1, 4]$ and $f = [1, 5]$.

It only remains to show that $[1, 2, \ldots, n]$ can also be generated for each $n$. But when $n = 10$ we have $abcde = \pm[1, 2, \ldots, 10]$, when $n = 11$, $abcdec = \pm[1, 2, \ldots, 11]$ and when $n = 12$, $abcdefc = \pm[1, 2, \ldots, 12]$. \qed

Proposition 2.6. $i(S_{13}) = 4$.

Proof. Note that in $S_{13}$ the only involutions are $-1$ and signed cycles of type $2^3$ and $2^4$. Also we require at least $12$ signed transpositions in our generators to ensure that all the numbers from $1$ to $13$ have some link. However we cannot use three signed cycles of type $2^4$, the minimum needed, as they are all even and thus cannot generate any odd signed cycles. Thus $i(S_{13}) > 3$. That $i(S_{13}) = 4$ follows by noting that $(1, 12)(2, 11)(3, 10)(4, 9), (5, 8)(6, 7)(1, 13)(2, 12), (3, 11)(4, 10)(5, 9)(6, 8)$ and $(1, 2)(4, 11)(5, 6)$ will generate $S_{13}$. Thus the corresponding signed cycles generate $S_{13}$. \qed

Proposition 2.7. For $14 \leq n \leq 16$, the following involutions generate $S_n$.

$$n = 14 \quad a = [1, 2][3, 14][4, 13][5, 12][6, 11][7, 10][8, 9],$$

$$b = [1, 3][4, 14][5, 13][2, 9],$$

$$c = [6, 12][7, 11][8, 10][2, 9].$$

$$n = 15 \quad a = [2, 15][3, 14][4, 13][5, 12][6, 11][7, 10][8, 9],$$

$$b = [1, 2][3, 15][4, 14][5, 13][6, 12][7, 11][8, 10],$$

$$c = [3, 14][4, 13][6, 11][7, 10].$$

$$n = 16 \quad a = [2, 16][3, 15][4, 14][5, 13][6, 12][7, 11][8, 10],$$

$$b = [1, 2][3, 16][4, 15][5, 14][6, 13][7, 12][8, 11][9, 10],$$

$$c = [3, 16][4, 15][5, 14][6, 13][7, 12][8, 11][9, 10].$$

Proof. For $n = 16$, $bc = \pm[1, 2]$ and $ab = \pm[1, 2, \ldots, 16]$ from which the result follows. Now for $n = 14$, $abc = \pm[1, 2, \ldots, 14]$ and for $n = 15$, $ab = \pm[1, 2, \ldots, 15]$. Thus in both cases we need only show that we may also generate $\pm[1, 2]$. But when $n = 15$, $(ab)^4((abc)^3bab)^3(ab)^{-4} = \pm[2, 9]$ and $(b, \pm[2, 9])^3 = \pm[1, 2]$. While for $n = 14$, $d = ((ba)^2(ca)^2bc)^3 = \pm[1, 5][7, 13]$ and $f = (abcdebd)^5 = \pm[5, 6]$, from which we obtain $(abc)^4f(abc)^10 = \pm[1, 2]$. \qed
It is convenient at this point to introduce some notation.

**Definition 2.8.** We will denote the following product of $\delta + 1$ signed transpositions 
$[\alpha, \beta][\alpha + 1, \beta - 1][\alpha + 2, \beta - 2]\ldots[\alpha + \delta, \beta - \delta]$ by $T(\alpha, \alpha + \delta, \alpha + \beta)$.

As an example of this notation, in the previous proposition, we could express the generators for $\tilde{S}_{16}$ as 
$a = T(2, 8, 18), \quad b = [1, 2]T(3, 9, 19)$ and $c = T(3, 9, 19)$.

**Proposition 2.9.** For $n \geq 17$, the following involutions, which are dependent on the value of $n$ modulo 8, generate $\tilde{S}_n$.

When $n \equiv 1$,

- $a = [1, 2]T(3, (n + 1)/2, n + 3)$,
- $b = [1, 3]T(4, (n + 3)/2, n + 4)$,
- $c = T(3, (n + 1)/2, n + 3)$.

When $n \equiv 2$,

- $a = [1, 2]T(3, n/2, n + 2)$,
- $b = [1, 3]T(4, (n + 2)/2, n + 3)$,
- $c = [3, n]T(4, n/2, n + 2)$.

When $n \equiv 3$,

- $a = [1, 2]T(3, (n - 1)/2, n + 1)$,
- $b = [1, 3]T(4, (n + 1)/2, n + 2)$,
- $c = [3, n][4, n - 1]T(5, (n - 1)/2, n + 1)$.

When $n \equiv 4$,

- $a = [1, 2]T(3, (n - 2)/2, n)$,
- $b = [1, 3]T(4, n/2, n + 1)$,
- $c = T(3, 5, n + 3)T(6, (n - 2)/2, n)$.

When $n \equiv 5$,

- $a = [1, 2]T(3, (n - 3)/2, n - 1)$,
- $b = [1, 3]T(4, (n - 1)/2, n)$,
- $c = T(3, 6, n + 3)T(7, (n - 3)/2, n - 1)$.

When $n \equiv 6$,

- $a = [1, 2]T(3, (n - 4)/2, n - 2)$,
- $b = [1, 3]T(4, (n - 2)/2, n - 1)$,
- $c = T(3, 7, n + 3)T(8, (n - 4)/2, n - 2)$.

When $n \equiv 7$,

- $a = [1, 2]T(3, (n - 5)/2, n - 3)$,
- $b = [1, 3]T(4, (n - 3)/2, n - 2)$,
- $c = T(3, 8, n + 3)T(9, (n - 5)/2, n - 3)$.

When $n \equiv 0$,

- $a = [1, 2]T(3, (n - 6)/2, n - 4)$,
- $b = [1, 3]T(4, (n - 4)/2, n - 3)$,
- $c = T(3, 9, n + 3)T(10, (n - 6)/2, n - 4)$.

Note that when $n = 24$ we have $(n - 6)/2 < 10$. So we define $T(10, (n - 6)/2, n - 4)$ to be 1.

**Proof.** For $n \neq 1$ we have $(ac)^3 = \pm[1, 2]$ and when $n \equiv 1, ac = \pm[1, 2]$. Hence we only need show that $\pm[1, 2, \ldots, n]$ is also generated in each case. When $n \equiv 1$, we have directly that
\[ ab = \pm[1, 2, \ldots, n] \]. For the remaining values of \( n \) we note that when

\[
\begin{align*}
    n &\equiv 2 \quad ab = \pm[1, 2, \ldots, n-1] \text{ so } \tilde{S}_{n-1} \text{ is generated}, \\
    n &\equiv 3 \quad ab = \pm[1, 2, \ldots, n-2] \text{ so } \tilde{S}_{n-2} \text{ is generated}, \\
    n &\equiv 4 \quad ab = \pm[1, 2, \ldots, n-3] \text{ so } \tilde{S}_{n-3} \text{ is generated}, \\
    n &\equiv 5 \quad ab = \pm[1, 2, \ldots, n-4] \text{ so } \tilde{S}_{n-4} \text{ is generated}, \\
    n &\equiv 6 \quad ab = \pm[1, 2, \ldots, n-5] \text{ so } \tilde{S}_{n-5} \text{ is generated}, \\
    n &\equiv 7 \quad ab = \pm[1, 2, \ldots, n-6] \text{ so } \tilde{S}_{n-6} \text{ is generated}.
\end{align*}
\]

In particular, as \( n \geq 17 \), we may generate for each \( n \) the signed cycles \( d = [1, 9], e = [1, 8], f = [1, 7], g = [1, 6], h = [1, 5], j = [1, 4], k = [1, 3] \). Thus we may obtain \( \pm[1, 2, \ldots, n] \) from \( abekc \) when \( n \equiv 2 \), \( abekjc \) when \( n \equiv 3 \), \( abchjkc \) when \( n \equiv 4 \), \( abcghjkc \) when \( n \equiv 5 \), \( abcfghjkc \) when \( n \equiv 6 \), \( abcfghjkce \) when \( n \equiv 7 \) and \( abcdedefghjkce \) when \( n \equiv 0 \).

The previous propositions give directly the following theorem.

**Theorem 2.10.** For \( n \geq 7 \) and \( n \neq 13 \) \( i(\tilde{S}_n) = 3 \).

3. \( i(\tilde{S}_n) \)

We denote by \( \tilde{S}_n \) the double cover of \( S_n \) that lifts a transposition of \( S_n \) to an element of order 2. Hence \( \tilde{S}_n \) is the group with generators \( z, r_1, r_2, \ldots, r_{n-1} \) and relations

\[
\begin{align*}
    z^2 &= 1, \\
    zr_i &= r_i z, \quad r_i^2 = 1 \quad \text{for } 1 \leq i \leq n-1, \\
    (r_j r_{j+1})^3 &= 1 \quad \text{for } 1 \leq j \leq n-2, \\
    r_k r_h &= zr_h r_k \quad \text{for } |h-k| > 1 \text{ and } 1 \leq h, k \leq n-1.
\end{align*}
\]

For computations in \( \tilde{S}_n \) we multiply each of the generators of the Clifford Algebra \( C(\Omega) \), where \( \Omega = \{1, 2, \ldots, n\} \cup \{\delta\} \) (see [5]), by the complex number \( i \) to obtain an algebra over \( \mathbb{C} \) generated by \( f_1, f_2, \ldots, f_n, f_\delta \) where \( f_j^2 = 1 \) and \( f_j f_k = -f_k f_j \) for \( j \neq k \). The subgroup of this complex algebra generated by

\[
(f_1 - f_2)/\sqrt{2}, \ (f_2 - f_3)/\sqrt{2}, \ldots, (f_n - f_\delta)/\sqrt{2}
\]

is isomorphic to \( \tilde{S}_n \).

By identifying \( D_{a_j} \) with \( (f_j - f_\delta)/\sqrt{2} \), where \( a_j \) are distinct in \( \Omega \setminus \{\delta\} \), the following two relations are readily verified.

\[
D_{a_j} D_{a_j} = 1 \quad \text{and} \quad D_{a_1} D_{a_2} \ldots D_{a_m} D_{a_1} = (-1)^{m+1} D_{a_2} D_{a_3} \ldots D_{a_m} D_{a_1} D_{a_2}.
\]

Now, misusing notation, as in [5], we may write these as

\[
a_j a_j = 1 \quad \text{and} \quad a_1 a_2 \ldots a_m a_1 = (-1)^{m+1} a_2 a_3 \ldots a_m a_1 a_2.
\]

We are now in a position to define a signed cycle in \( \tilde{S}_n \).
Definition 3.1. For distinct elements \( a_1, \ldots, a_m \) we define \( \langle a_1 a_2 \ldots a_m \rangle = a_1 a_2 \ldots a_m a_1 \). We call \( \pm \langle a_1 a_2 \ldots a_k \rangle \) signed cycles in \( \hat{S}_n \). Each is a lift of the cycle \( (a_1 a_2 \ldots a_k) \) in \( S_n \).

Using this definition with the above relations we obtain the following rules for multiplying signed cycles in \( \hat{S}_n \).

\[
\langle a_j \rangle = 1, \\
\langle a_1 a_2 \ldots a_m \rangle = (-1)^{m+1} \langle a_2 a_3 \ldots a_m a_1 \rangle, \\
\langle a_1 a_2 \ldots a_{m-1} \rangle a_m = (-1)^m a_m \langle a_1 a_2 \ldots a_{m-1} \rangle.
\]

As an example of the multiplication of signed cycles we recover our presentation for \( \hat{S}_n \) as follows: Let \( r_1 = \langle 1, 2 \rangle, r_2 = \langle 2, 3 \rangle, \ldots, r_{n-1} = \langle n-1, n \rangle \) and \( z = -1 \). Clearly \( z^2 = 1 \) and

\[
z r_j = -\langle j, j + 1 \rangle = r_j z, \quad \text{for} \quad 1 \leq j \leq n-1, \\
r_j^2 = \langle j, j + 1 \rangle \langle j, j + 1 \rangle = j, j + 1, j, j + 1, j = 1, \\
(r_j r_{j+1})^3 = \langle j, j + 1 \rangle (j + 1, j + 2)^3 = (-j + 1, j) \langle j + 1, j + 2 \rangle^3 \\
= -j + 1 \langle j + 1, j + 2, j + 1 \rangle = j + 1 \langle j + 2, j \rangle j + 1 \\
= j + 1, j + 1 = 1 \quad \text{for} \quad 1 \leq j \leq n - 2
\]

and

\[
r_j r_k = \langle j, j + 1 \rangle \langle k, k + 1 \rangle = -\langle k, k + 1 \rangle \langle j, j + 1 \rangle \\
= z r_k r_j \quad \text{for} \quad | j - k | > 1 \quad \text{and} \quad 1 \leq j, k \leq n - 1.
\]

Note that, for \( n \geq 4 \), \( \hat{S}_n \not\cong \hat{S}_n \) if \( n \neq 6 \), see [3]. Also note that we will again be taking advantage of the fact that by factoring out \( \hat{S}_n \) by the subgroup \( Z = \langle 1, -1 \rangle \) we recover \( S_n \). That is \( \hat{S}_n / Z \cong S_n \).

Proposition 3.2. Any product of \( k \) disjoint signed transpositions in \( \hat{S}_n \) has order two if the integer part of \( k / 2 \) is a multiple of two, and order four otherwise.

Proof. Let \( S \) denote a product of \( k \) disjoint signed transpositions so that we have, for \( k \geq 2 \),

\[
S = \pm \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \ldots \langle a_k, b_k \rangle,
\]

where \( a_i \) and \( b_i \) are distinct integers in the signed transpositions of \( \hat{S}_n \). Then a straightforward induction proof gives, for \( k > 1 \),

\[
S^2 = (-1)^{k-1}(-1)^{k-2} \ldots (-1)^2(-1).
\]

While for \( k = 1 \), \( \langle a_1, b_1 \rangle \langle a_1, b_1 \rangle = 1 \).

So in \( \hat{S}_n \) an involution is of the form \( \pm \pi \) where \( \pi \) is a signed cycle consisting of \( k \) disjoint transpositions and \( k \) is congruent to 0 or 1 modulo 4.

Proposition 3.3. For \( n \geq 4 \), \( \hat{S}_n \) may be generated by \( a = \pm \langle 1, 2 \rangle \) and \( b = \pm \langle 1, 2, \ldots, n \rangle \).
Proof. As in Proposition 2.3 we need only show that $-1$ is generated. Now in $\hat{S}_n$ we may generate either
$$g = + (1, 2) \langle 3, 4 \rangle \quad \text{or} \quad h = - (1, 2) \langle 3, 4 \rangle,$$
but in either case $g^2 = h^2 = -1$. \hfill $\square$

The investigation into the value of $i(\hat{S}_n)$ closely follows that of the previous section, except that here involutions are products of $k$ signed transpositions for $k \equiv 0$ or $1$ modulo $4$. So in particular $\pm \langle 1, 2 \rangle$ is an involution in this double cover.

Hence in $\hat{S}_5$, $\hat{S}_6$ and $\hat{S}_7$ the only elements of order two are $-1$ and signed transpositions of the form $\pm \langle a, b \rangle$. Thus we have immediately that $i(\hat{S}_5) = 4$, $i(\hat{S}_6) = 5$ (which, as $\hat{S}_6 \cong \hat{S}_5$, implies $i(\hat{S}_6) = 5$) and $i(\hat{S}_7) = 6$, generators being $\langle 1, 2 \rangle, \ldots, \langle 1, n \rangle$. Also, as the only involutions in $\hat{S}_8$ are $-1$ and signed cycles of type $2^1$ and $2^4$, it is readily verified (via GAP [4]) that $i(\hat{S}_8) > 3$. But as
$$\langle 1, 2 \rangle, \langle 1, 6 \rangle, \langle 1, 2 \rangle \langle 3, 8 \rangle \langle 4, 7 \rangle \langle 5, 6 \rangle \quad \text{and} \quad \langle 1, 6 \rangle \langle 2, 3 \rangle \langle 4, 8 \rangle \langle 5, 7 \rangle$$
generate $\hat{S}_8$ we have $i(\hat{S}_8) = 4$.

**Proposition 3.4.** For $n \geq 9$, when $n \equiv 1, 2$ or $3$ modulo $8$, $\hat{S}_n$ may be generated by three involutions.

*Proof.* We only need apply the decomposition referred to in the introduction to see that we may express $\pm \langle 1, 2, \ldots, n \rangle$ as the product $ab$ where
$$a = \langle 1, n - 1 \rangle \langle 2, n - 2 \rangle \ldots \langle r, n - r \rangle, \quad b = \langle 1, n \rangle \langle 2, n - 1 \rangle \ldots \langle t, n + 1 - t \rangle.$$
Hence if we take $a$, $b$ along with $\langle 1, 2 \rangle$ the result follows. \hfill $\square$

**Proposition 3.5.** For $12 \leq n \leq 16$, the following involutions generate $\hat{S}_n$.

*When $n = 16$*

\[
\begin{align*}
    a &= \langle 1, 15 \rangle \langle 2, 14 \rangle \langle 3, 13 \rangle \langle 6, 10 \rangle \langle 7, 9 \rangle \\
    b &= \langle 4, 12 \rangle \langle 5, 11 \rangle \langle 7, 10 \rangle \langle 2, 15 \rangle \langle 3, 14 \rangle \\
    c &= \langle 4, 13 \rangle \langle 5, 12 \rangle \langle 6, 11 \rangle \langle 1, 16 \rangle \langle 8, 9 \rangle.
\end{align*}
\]

*For $12 \leq n \leq 15$*

\[
\begin{align*}
    a &= \langle 1, 2 \rangle \langle 3, 11 \rangle \langle 4, 10 \rangle \langle 5, 9 \rangle \langle 6, 8 \rangle, \quad b = \langle 1, 3 \rangle \langle 4, 11 \rangle \langle 5, 10 \rangle \langle 6, 9 \rangle \langle 7, 8 \rangle \\
    \text{and} \quad c &= \begin{cases} 
        \langle 3, 12 \rangle \langle 4, 10 \rangle \langle 5, 9 \rangle \langle 6, 8 \rangle & \text{when } n = 12, \\
        \langle 3, 12 \rangle \langle 4, 13 \rangle \langle 5, 9 \rangle \langle 6, 8 \rangle & \text{when } n = 13, \\
        \langle 3, 12 \rangle \langle 4, 13 \rangle \langle 5, 14 \rangle \langle 6, 8 \rangle & \text{when } n = 14, \\
        \langle 3, 12 \rangle \langle 4, 13 \rangle \langle 5, 14 \rangle \langle 6, 15 \rangle & \text{when } n = 15.
    \end{cases}
\end{align*}
\]
Proof. For \( n = 16 \) we have \( abc = \pm \langle 1, 2, \ldots, 16 \rangle \) so we need only show that \( \langle 1, 2 \rangle \) is also generated. But

\[
(abc)^6 ((c(ab)^4b)^3(b(abcb)^3b)^3)^3(abcb)^{-6} = \pm \langle 1, 4 \rangle \quad \text{and} \\
((abc)^3b(abcb)^3)^2 \pm \langle 1, 4 \rangle ((abc)^2b(abcb)^3)^{-2} = \pm \langle 1, 2 \rangle .
\]

When \( n \in \{12, 13, 14, 15\} \), \( (ac)^3 = \pm \langle 1, 2 \rangle \) and \( ab = \pm \langle 1, 2, \ldots, 11 \rangle \) so for each \( n \) we may generate the subgroup \( \hat{S}_{11} \), in particular the signed cycles \( d = \langle 1, 3 \rangle, e = \langle 1, 4 \rangle, f = \langle 1, 5 \rangle \) and \( g = \langle 1, 6 \rangle \).

Hence we need only show that \( \langle 1, 2, \ldots, n \rangle \) can also be generated for each \( n \). But when \( n = 12 \) we have \( abcd = \pm \langle 1, 2, \ldots, 12 \rangle \). When \( n = 13 \), \( abcdec = \pm \langle 1, 2, \ldots, 13 \rangle \). When \( n = 14 \), \( abcdedec = \pm \langle 1, 2, \ldots, 14 \rangle \) and when \( n = 15 \), \( abcdedecf = \pm \langle 1, 2, \ldots, 15 \rangle \). \( \square \)

We again make use of the notation \( T(\alpha, \alpha + \delta, \alpha + \beta) \), as in the previous section, but here this represents the \( \delta + 1 \) signed transpositions

\[ \langle \alpha, \beta \rangle \langle \alpha + 1, \beta - 1 \rangle \langle \alpha + 2, \beta - 2 \rangle \ldots \langle \alpha + \delta, \beta - \delta \rangle . \]

Proposition 3.6. For \( n \geq 17 \), the following involutions, which are dependent on the value of \( n \) modulo 8, generate \( \hat{S}_n \).

\[
\begin{align*}
\text{n \equiv 0} & \quad a = \langle 1, 2 \rangle T(3, (n - 4)/2, n - 2), \\
& \quad b = \langle 1, 3 \rangle T(4, (n - 2)/2, n - 1), \\
& \quad c = \langle 3, n \rangle T(4, n/2, n + 2).
\end{align*}
\]

\[
\begin{align*}
\text{n \equiv 4} & \quad a = \langle 1, 2 \rangle T(3, n/2, n + 2), \\
& \quad b = \langle 1, 3 \rangle T(4, n/2, n + 3), \\
& \quad c = \langle 3, n \rangle T(4, n/2, n + 2).
\end{align*}
\]

\[
\begin{align*}
\text{n \equiv 5} & \quad a = \langle 1, 2 \rangle T(3, (n - 1)/2, n + 1), \\
& \quad b = \langle 1, 3 \rangle T(4, (n + 1)/2, n + 2), \\
& \quad c = \langle 3, n \rangle T(4, (n + 1)/2, n + 2).
\end{align*}
\]

\[
\begin{align*}
\text{n \equiv 6} & \quad a = \langle 1, 2 \rangle T(3, (n - 1)/2, n), \\
& \quad b = \langle 1, 3 \rangle T(4, n/2, n + 1), \\
& \quad c = \langle 3, n \rangle T(3, 5, n + 3) T(6, (n - 2)/2, n).
\end{align*}
\]

\[
\begin{align*}
\text{n \equiv 7} & \quad a = \langle 1, 2 \rangle T(3, (n - 3)/2, n - 1), \\
& \quad b = \langle 1, 3 \rangle T(4, (n - 1)/2, n), \\
& \quad c = \langle 3, n \rangle T(3, 6, n + 3) T(7, (n - 3)/2, n - 1).
\end{align*}
\]

Proof. For each \( n \) here we have \( (ac)^3 = \pm \langle 1, 2 \rangle \), so we only need show that \( \pm \langle 1, 2, \ldots, n \rangle \) is also generated in each case. Now for

\[
\begin{align*}
\text{n \equiv 4} & \quad ab = \pm \langle 1, 2, \ldots, n - 1 \rangle \quad \text{so } \hat{S}_{n-1} \text{ is generated,} \\
\text{n \equiv 5} & \quad ab = \pm \langle 1, 2, \ldots, n - 2 \rangle \quad \text{so } \hat{S}_{n-2} \text{ is generated,} \\
\text{n \equiv 6} & \quad ab = \pm \langle 1, 2, \ldots, n - 3 \rangle \quad \text{so } \hat{S}_{n-3} \text{ is generated,} \\
\text{n \equiv 7} & \quad ab = \pm \langle 1, 2, \ldots, n - 4 \rangle \quad \text{so } \hat{S}_{n-4} \text{ is generated,} \\
\text{n \equiv 0} & \quad ab = \pm \langle 1, 2, \ldots, n - 5 \rangle \quad \text{so } \hat{S}_{n-5} \text{ is generated.}
\end{align*}
\]
In particular, as $n \geq 17$, we may generate for each $n$ the signed cycles $d = \langle 1, 7 \rangle$, $e = \langle 1, 6 \rangle$, $f = \langle 1, 5 \rangle$, $g = \langle 1, 4 \rangle$ and $h = \langle 1, 3 \rangle$. Thus we may obtain $\pm \langle 1, 2, \ldots, n \rangle$ from $abchc$ when $n \equiv 4$, $abchc$ when $n \equiv 5$, $abchgc$ when $n \equiv 6$, $abchgc$ when $n \equiv 7$ and $abchgc$ when $n \equiv 0$. 

These last three propositions, along with the fact that $(1, 2)$, $(1, 3)$ and $(1, 4)$ generate $S_4$, give directly the following theorem.

**Theorem 3.7.** For $n = 4$ and $n \geq 9$, $i(\hat{S}_n) = 3$.

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**References**


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