Correction to
Connecting invariant manifolds
and the solution of the $C^1$ stability
and $\Omega$-stability conjectures for flows

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There is a gap in the proof of Lemma VII.4 in [1]. We present an alternative proof of Theorem B ($C^1$ $\Omega$-stable vector fields satisfy Axiom A) in [1]. The novel and essential part in the proof of the stability and $\Omega$-stability conjectures for flows is the connecting lemma introduced in [1]. A mistake in the proof of the last conjecture was pointed out to me by Toyoshiba [5], who later also provided an independent proof of it, again based on the connecting lemma and previous arguments by Mañé and Palis.

The crucial step to the proof of Theorem B is the separation of singularities from periodic orbits ([1, Corollary III]) by the $C^1$ connecting lemma ([1, Theorem A]). After the separation, the proof proceeds based on Mañé’s theorems used in [3] and we still rely on Palis’s argument in [4], proving first the density of Axiom A diffeomorphisms in the set of $C^1$ $\Omega$-stable ones to then show that every $C^1$ $\Omega$-stable diffeomorphism satisfies Axiom A.

Let $\mathcal{G}_{\Omega}^1(M)$ be the set of $C^1$ $\Omega$-stable vector fields on a compact smooth boundaryless manifold $M$ with the $C^1$ topology and $X \in \mathcal{G}_{\Omega}^1(M)$. As in [1], we prove the hyperbolicity of $\overline{\text{Per}(X)} (= \Omega(X) - \text{Sing}(X))$ by induction. In fact, we prove that $\overline{\mathcal{P}_j}(X)$ is hyperbolic assuming that $\bigcup_{i=0}^{j-1} \overline{\mathcal{P}_i}(X)$ is hyperbolic for some $1 \leq j \leq \dim M - 1$, where $\overline{\mathcal{P}_i}(X)$ is the closure of the set of periodic points with index $i$ (dimension of the stable subspace), which is enough to conclude that $X$ satisfies Axiom A. For a dense subset of $\mathcal{G}_{\Omega}^1(M)$, we can use the statement of [1, Lemma VII.4] by an already classic argument on set-valued functions of $C^1$ vector fields. In fact, there is a residual subset of the set of $C^1$ vector fields (therefore of $\mathcal{G}_{\Omega}^1(M)$) in which the closure of the set of hyperbolic periodic points of saddle type moves continuously with respect to vector fields (see for instance the proof of [1, Corollary II] for this kind of argument). Therefore, as proved in [1], we get the density of Axiom A vector fields in $\mathcal{G}_{\Omega}^1(M)$. Then, by $\Omega$-conjugacy, we see that $\Omega(X)$ can be decomposed into a finite union of disjoint compact invariant sets which are isolated and transitive. Moreover, Palis’s argument ([4]) for flows shows that each component is homogeneous in the sense that the index of every periodic...
point in it is the same. Thus, the proof of Theorem B is reduced to proving the following claim:

Claim. Every homogeneous component of $\mathcal{P}_j(X)$ is hyperbolic.

Let $\mathcal{G}^1(M)$ be the interior of the set of $C^1$ vector fields on $M$, with the $C^1$ topology, such that all periodic orbits and singularities are hyperbolic. Then $\mathcal{G}^1_{\Omega}(M) \subset \mathcal{G}^1(M)$. Denote by $L_t^X$, $t \in \mathbb{R}$ the linear Poincaré flow of $X \in \mathcal{G}^1(M)$ on $N^*$ (see [1, p. 126] for the definition). As in [1, p. 131], let $N^*|\mathcal{P}_j(X) = E_j \oplus F_j$ be the dominated splitting such that

\begin{equation}
\|L^X_{-m}|E_j(y)\| \cdot \|L^X_{-m}|F_j(X_{m}(y))\| \leq \lambda
\end{equation}

for all $y \in \mathcal{P}_j(X)$ with $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$ given by [1, Lemma VII.1], which is the continuous extension of hyperbolic splittings of periodic orbits of index $j$ with respect to $L^X_t$. To prove the Claim, it is enough to show that $E_j$ is contracting by the following lemma proved in [3, Theorem II.1], which is Lemma VII.5 in [1] and written for this setting:

**Lemma 1** (Mañé). Let $\Lambda$ be a compact invariant set of $X \in \mathcal{G}^1(M)$ such that $\Lambda \cap \text{Sing}(X) = \emptyset$ and $\Omega(X|\Lambda) = \Lambda$. Suppose that $N^*|\Lambda = E \oplus F$ is a dominated splitting such that the dimension of the subspaces $E(y)$, $y \in \Lambda$ is constant,

\[ \|L^X_{-m}|E(y)\| \cdot \|L^X_{-m}|F(X_{m}(y))\| \leq \lambda \]

for all $y \in \Lambda$, and

\[ \liminf_{n \to +\infty} \frac{1}{n} \sum_{t=1}^{n} \log \|L^X_{-m}|F(X_{mt}(x))\| \leq \log \lambda \]

holds for a dense set of points $x \in \Lambda$, where $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$ are given in (1). Then, if $E$ is contracting, $F$ is expanding (and therefore $\Lambda$ is hyperbolic).

Let $\Sigma(X)$ be the set of “strongly closable points” given in [1, Lemma VII.6 (Ergodic Closing Lemma for time-one maps)] and originally introduced by Mañé in [2]. We shall need the following lemma:

**Lemma 2.** Let $X \in \mathcal{G}^1(M)$. If $x \in \Sigma(X)$ and $\overline{O_X(x)} \cap \text{Sing}(X) = \emptyset$, then $O_X(x)$ contains a hyperbolic set, where $O_X(x) = \{X_t(x) : t \in \mathbb{R}\}$.

**Proof.** We can suppose that $x \in \Sigma(X) - \text{Per}(X)$. Let $U_n$, $n \geq 0$, be a basis of neighborhoods of $X$. Then, by the definition of $\Sigma(X)$ ([1, p. 132]; see also [2, p. 506]), there exists $\{t_n > 0 : n \geq 0\}$ with $\lim_{n \to +\infty} t_n = +\infty$, $X^n \in U_n$ and $y_n \in \text{Per}(X^n)$ having period $T_n$ such that $\{X_t(x) : 0 \leq t \leq t_n\}$ can be approximated by $\{X^n_t(y_n) : 0 \leq t \leq T_n\}$ for large $n$. Without loss of
generality we may assume that the index of the $X^n$-periodic point $y_n$ is the same for all $n \geq 0$ (by taking a subsequence if necessary). Then, by [1, Lemma VII.1], the following properties hold for all $n \geq 0$ with $T_n \geq m$:

$$\|L_{X^m} E_n^s(y)\| \cdot \|L_{-X^m} E_n^u(X^m_n(y))\| \leq \lambda$$

for all $y \in \{X^t_n(y_n) : 0 \leq t \leq T_n\}$, and

$$\prod_{\ell=1}^{[T_n/m]} \|L_{-X^m} E_n^u(X^m_{m\ell}(y_n))\| \leq K\lambda^{[T_n/m]}$$

where $E_n^s \oplus E_n^u$ is the hyperbolic splitting with respect to $L_{X^n}$ over the $X^n$-periodic orbit with $y_n$. Note that the dimension of $E_n^s(y)$ is constant, and the angle between $E_n^s(y)$ and $E_n^u(y)$ is uniformly bounded away from 0 by [2, Lemma II.9]. Then, defining the splitting $E \oplus F$ over $O_X(x)$ by accumulation of $\{E_n^s \oplus E_n^u : n \geq 0\}$ (see the definition of $\sum(X)$ again), we have, by continuity, the following properties:

$$\|L_{X^m} E(y)\| \cdot \|L_{-X^m} F(X_m(\ell))\| \leq \lambda$$

for all $y \in O_X(x)$, and

$$\liminf_{n \to +\infty} -\frac{1}{n} \sum_{\ell=1}^n \log \|L_{X^m} F(X_m(\ell))\| \leq \log \lambda.$$

It is well known that the dominated splitting $E \oplus F$ over $O_X(x)$ can be continuously extended to $\tilde{E} \oplus \tilde{F}$ over $\tilde{O}_X(x)$ with the same $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$. Hence, we can apply Lemma 1 to $\Lambda = \overline{O}_X(x)$. If $\overline{O}_X(x)$ is not hyperbolic; that is, $\tilde{E}$ is not contracting, then, as in [1, p. 132], there exists $p \in \overline{O}_X(x) \cap \sum(X)$ such that

$$\lim_{n \to +\infty} -\frac{1}{n} \sum_{\ell=0}^{n-1} \log \|L_{X^m} \tilde{E}(X_m(\ell))\| \geq 0.$$ (2)

When $p \in \text{Per}(X)$, $O_X(p)$ is a hyperbolic set, we may assume that $p \notin \text{Per}(X)$. Then, we can continue this argument for $\overline{O}_X(p)$ instead of $\overline{O}_X(x)$. As observed in [1, pp. 132–133], the index $i_0$ of periodic point created by the Ergodic Closing Lemma from $O_X(p)$ is less than $\dim \tilde{E}$. If $i_0 = 0$, then, by the same argument as in the proof of the finiteness of periodic orbits in $\overline{P}_0(M)$, $p$ cannot be recurrent. Therefore, we may suppose that $\overline{O}_X(p)$ has a contracting subbundle (by continuing this further if necessary) and therefore, by Lemma 1, $\overline{O}_X(p)$ is hyperbolic, proving Lemma 2.

Now let us prove the Claim. Assume that a homogeneous component $\tilde{\Lambda}$ of $\overline{\Lambda}_j(X)$ is not hyperbolic. Then, we can find $p \in \sum(X) \cap \tilde{\Lambda}$ satisfying property
with $\tilde{E}$ replaced by $E_j$ in (1). Lemma 2 implies that $\overline{O_X(p)}$ contains a hyperbolic set $\Lambda$, and, as in the proof of Lemma 2, the index of any periodic point in $\Lambda$ is less than $j$. This contradicts the homogeneity of $\Lambda$.

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References


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