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Existence and data dependence for multivalued weakly Ćirić-contractive operators

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Abstract. In this paper we define the concept of weakly Ćirić-contractive operator and give a fixed point result for this type of operators. Then we study the data dependence for the fixed point set.

1 Introduction

Let (X, d) be a metric space. A singlevalued operator T from X into itself is called contractive if there exists a real number $r \in [0, 1)$ such that $d(T(x), T(y)) \leq rd(x, y)$ for every $x, y \in X$. It is well known that if X is a complete metric space, then a contractive operator from x into itself has a unique fixed point in X.

In 1996, Japanese mathematicians O. Kada, T. Suzuki and W. Takahashi introduced the w-distance (see [4]) and discussed some properties of this new distance. Later, T. Suzuki and W. Takahashi, starting by the definition above, gave some fixed points result for a new class of operators, weakly contractive operators (see [8]).

The purpose of this paper is to give a fixed point theorem for a new class of operators, namely the so-called weakly Ćirić-contractive operators. Then, we present a data dependence result for the fixed point set.

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2 Preliminaries

Let (X, d) be a complete metric space. We will use the following notations:
$P(X)$ - *the set of all nonempty subsets of X*;
$\mathcal{P}(X) = P(X) \bigcup \emptyset$
$P_{cl}(X)$ - *the set of all nonempty closed subsets of X*;
$P_b(X)$ - *the set of all nonempty bounded subsets of X*;
$P_{b,cl}(X)$ - *the set of all nonempty bounded and closed, subsets of X*;
For two subsets $A, B \in P_b(X)$, we recall the following functionals:
$D : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, D(Z, Y) = \inf\{d(x, y) : x \in Z\ , y \in Y\}$, $Z \subset X$ – *the gap functional.*
$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \delta(A, B) := \sup\{d(a, b)|x \in A, b \in B\}$ – *the diameter functional*;
$\rho : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, \rho(A, B) := \sup\{D(a, B)|a \in A\}$ – *the excess functional*;
$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+, H(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$ – *the Pompeiu-Hausdorff functional*;
$Fix\ F := \{x \in X \mid x \in F(x)\}$ – *the set of the fixed points of* F;

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [4]) as follows:
Let (X,d) be a metric space, $w : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

1. $w(x, z) \le w(x, y) + w(y, z)$, for any $x, y, z \in X$;
2. for any $x \in X : w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
3. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ implies $d(x, y) \le \varepsilon$.

Let us give some examples of w-distances (see [4]).

Example 1 *Let* (X, d) *be a metric space . Then the metric "d" is a w-distance on X.*

Example 2 *Let X be a normed liniar space with norm* $||\cdot||$. *Then the function* $w : X \times X \to [0, \infty)$ *defined by* $w(x, y) = ||x|| + ||y||$ *for every* $x, y \in X$ *is a w-distance.*

Example 3 *Let (X,d) be a metric space and let* $g : X \to X$ *a continuous mapping. Then the function* $w : X \times Y \to [0, \infty)$ *defined by:*

$$w(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$$

for every $x, y \in X$ *is a w-distance.*

For the proof of the main results we need the following crucial result for w-distance (see [8]).

Lemma 1 *Let* (X, d) *be a metric space, and let* w *be a* w*-distance on* X*. Let* (x_n) *and* (y_n) *be two sequences in* X*, let* (α_n)*,* (β_n) *be sequences in* $[0, +\infty[$ *converging to zero and let* $x, y, z \in X$*. Then the following holds:*

1. *If* $w(x_n, y) \le \alpha_n$ *and* $w(x_n, z) \le \beta_n$ *for any* $n \in \mathbb{N}$*, then* $y = z$*.*
2. *If* $w(x_n, y_n) \le \alpha_n$ *and* $w(x_n, z) \le \beta_n$ *for any* $n \in \mathbb{N}$*, then* (y_n) *converges to* z*.*
3. *If* $w(x_n, x_m) \le \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$*, then* (x_n) *is a Cauchy sequence.*
4. *If* $w(y, x_n) \le \alpha_n$ *for any* $n \in \mathbb{N}$*, then* (x_n) *is a Cauchy sequence.*

3 Existence of fixed points for multivalued weakly Ćirić-contractive operators

At the beginning of this section let us define the notion of multivalued weakly Ćirić-contractive operators.

Definition 1 *Let* (X, d) *be a metric space and* $T : X \to P(X)$ *a multivalued operator. Then* T *is called weakly Ćirić-contractive if there exists a* w*-distance on* X *such that for every* $x, y \in X$ *and* $u \in T(x)$ *there is* $v \in T(y)$ *with* $w(u, v) \le q \; max\{w(x, y), D_w(x, T(x), D_w(y, T(y)), \frac{1}{2}D_w(x, T(y))\}$*,*
for every $q \in [0, 1)$*.*

Let (X, d) be a metric space, w be a w-distance on X $x_0 \in X$ and $r > 0$. Let us define:

$B_w(x_0; r) := \{x \in X | w(x_0, x) < r\}$ the open ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}(x_0; r) := \{x \in X | w(x_0, x) \leq r\}$ the closed ball centered at x_0 with radius r with respect to w;

$\widetilde{B_w}^d(x_0; r)$- the closure in (X, d) of the set $B_w(x_0; r)$.

One of the main results is the following fixed point theorem for weakly Ćirić-contractive operators.

Theorem 1 *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \widetilde{B_w}(x_0; r) \to P_{cl}(X)$ a multivalued operator such that:*

(i) T is weakly Ćirić-contractive operator;

(ii) $D_w(x_0, T(x_0)) \leq (1 - q)r$.

Then there exists $x^* \in X$ such that $x^* \in T(x^*)$.

Proof. Since $D_w(x_0, T(x_0)) \leq (1 - q)r$, then for every $x_0 \in X$ there exists $x_1 \in T(x_0)$ such that $D_w(x_0, T(x_0)) \leq w(x_0, x_1) \leq (1 - q)r < r$.
Hence $x_1 \in \widetilde{B_w}(x_0; r)$.
For $x_1 \in \widetilde{B_w}(x_0; r)$, there exists $x_2 \in T(x_1)$ such that:

i. $w(x_1, x_2) \leq qw(x_0, x_1)$

ii. $w(x_1, x_2) \leq qD_w(x_0, T(x_0)) \leq qw(x_0, x_1)$

iii. $w(x_1, x_2) \leq qD_w(x_1, T(x_1)) \leq qw(x_1, x_2)$

iv. $w(x_1, x_2) \leq \frac{q}{2} D_w(x_0, T(x_1)) \leq \frac{q}{2} w(x_0, x_2)$

$w(x_1, x_2) \leq \frac{q}{2}[w(x_0, x_1) + w(x_1, x_2)]$

$(1 - \frac{q}{2})w(x_1, x_2) \leq \frac{q}{2} w(x_0, x_1)$

$w(x_1, x_2) \leq \frac{q}{2-q} w(x_0, x_1)$.

Then $w(x_1, x_2) \leq \max \{q, \frac{q}{2-q}\} w(x_0, x_1)$
Since $q > \frac{q}{2-q}$ for every $q \in [0, 1)$, then $w(x_1, x_2) \leq qw(x_0, x_1) \leq q(1 - q)r$.
Then $w(x_0, x_2) \leq w(x_0, x_1) + w(x_1, x_2) < (1 - q)r + q(1 - q)r = (1 - q^2)r < r$.
Hence $x_2 \in \widetilde{B_w}(x_0; r)$.
For $x_1 \in \widetilde{B_w}(x_0; r)$ and $x_2 \in T(x_1)$, there exists $x_3 \in T(x_2)$ such that

i. $w(x_2, x_3) \leq qw(x_1, x_2)$

ii. $w(x_2, x_3) \leq qD_w(x_1, T(x_1)) \leq qw(x_1, x_2)$

iii. $w(x_2, x_3) \leq qD_w(x_2, T(x_2)) \leq qw(x_2, x_3)$

iv. $w(x_2, x_3) \leq \frac{q}{2} D_w(x_1, T(x_2)) \leq \frac{q}{2} w(x_1, x_3)$

$w(x_2, x_3) \leq \frac{q}{2}[w(x_1, x_2) + w(x_2, x_3)]$

$(1 - \frac{q}{2})w(x_2, x_3) \leq \frac{q}{2} w(x_1, x_2)$

$w(x_2, x_3) \leq \frac{q}{2-q} w(x_1, x_2)$.

Then $w(x_2, x_3) \leq \max \{q, \frac{q}{2-q}\} w(x_1, x_2)$.
Since $q > \frac{q}{2-q}$ for every $q \in [0, 1)$, then $w(x_2, x_3) \leq qw(x_1, x_2) \leq q^2(x_0, x_1) \leq q^2(1 - q)r$.

Then $w(x_0, x_3) \leq w(x_0, x_2) + w(x_2, x_3) \leq (1-q^2)r + q^2(1-q)r =$
$= (1-q)(1+q+q^2)r = (1-q^3)r < r$. Hence $x_3 \in \widetilde{B_w}(x_0; r)$.
By this procedure we get a sequence $(x_n)_{n\in\mathbb{N}} \in X$ of successive applications for T starting from arbitrary $x_0 \in X$ and $x_1 \in T(x_0)$, such that

(1) $x_{n+1} \in T(x_n)$, for every $n \in \mathbb{N}$;

(2) $w(x_n, x_{n+1}) \leq q^n w(x_0, x_1) \leq q^n(1-q)r$, for every $n \in \mathbb{N}$.

For every $m, n \in \mathbb{N}$, with $m > n$, we have

$$\begin{aligned} w(x_n, x_m) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + ... + w(x_{m-1}, x_m) \leq \\ &\leq q^n w(x_0, x_1) + q^{n+1} w(x_0, x_1) + ... + q^{m-1} w(x_0, x_1) \leq \\ &\leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r. \end{aligned}$$

By Lemma 1(3) we have that the sequence $(x_n)_{n\in\mathbb{N}} \in \widetilde{B_w}(x_0; r)$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, then there exists $x^* \in \widetilde{B^d_w}(x_0; r)$ such that $x_n \xrightarrow{d} x^*$.
Fix $n \in \mathbb{N}$. Since $(x_m)_{m\in\mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous, we have

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r.$$

For $x^* \in \widetilde{B^d_w}(x_0; r)$ and $x_n \in T(x_{n-1})$, there exists $u_n \in T(x^*)$ such that

i. $w(x_n, u_n) \leq q w(x_{n-1}, x^*) \leq \frac{q^n}{1-q} w(x_0, x_1)$

ii. $w(x_n, u_n) \leq q D_w(x_{n-1}, T(x_{n-1})) \leq q w(x_{n-1}, x_n) \leq ... \leq q^n w(x_0, x_1)$

iii. $w(x_n, u_n) \leq q D_w(x^*, T(x^*)) \leq q w(x^*, u_n) \leq \frac{q^n}{1-q} w(x_0, x_1)$

iv. $w(x_n, u_n) \leq \frac{q}{2} D_w(x_{n-1}, T(x^*)) \leq \frac{q}{2} w(x_{n-1}, u_n) \leq \frac{q}{2} \cdot \frac{q^{n-1}}{1-q} w(x_0, x_1)$
$= \frac{q^n}{2(1-q)} w(x_0, x_1)$.

Then $w(x_n, u_n) \leq max\{\frac{q^n}{1-q}, q^n, \frac{q^n}{2(1-q)}\} w(x_0, x_1)$.
Since for $q \in [0, 1)$ we have true $\frac{q^n}{1-q} > q^n$ and $\frac{q^n}{1-q} > \frac{q^n}{2(1-q)}$ we get that $w(x_n, u_n) \leq \frac{q^n}{1-q} w(x_0, x_1) \leq q^n r$.
So, for every $n \in \mathbb{N}$ we have:
$w(x_n, x^*) \leq q^n r$
$w(x_n, u_n) \leq q^n r$.

Then, from 1(2), we obtain that $u_n \xrightarrow{d} x^*$. As $u_n \in T(x^*)$ and using the closure of T result that $x^* \in T(x^*)$. ■

A global result for previous theorem is the following fixed point result for multivalued weakly Ćirić-contractive operators.

Theorem 2 *Let* (X, d) *be a complete metric space,* $T : X \to P_{cl}(X)$ *a multivalued weakly Ćirić-contractive operator. Then there exists* $x^* \in X$ *such that* $x^* \in T(x^*)$.

4 Data dependence for weakly Ćirić-contractive multivalued operators

The main result of this section is the following data dependence theorem with respect to the above global theorem 2.

Theorem 3 *Let* (X, d) *be a complete metric space,* $T_1, T_2 : X \to P_{cl}(X)$ *be two multivalued weakly Ćirić-contractive operators with* $q_i \in [0, 1)$ *with* $i = \{1, 2\}$. *Then the following are true:*

1. $\mathrm{Fix}T_1 \neq \emptyset \neq \mathrm{Fix}T_2$*;*

2. *We suppose that there exists* $\eta > 0$ *such that for every* $u \in T_1(x)$ *there exists* $v \in T_2(x)$ *such that* $w(u, v) \leq \eta$*, (respectively for every* $v \in T_2(x)$ *there exists* $u \in T_1(x)$ *such that* $w(v, u) \leq \eta$*).*

Then for every $u^* \in \mathrm{Fix}T_1$, *there exists* $v^* \in \mathrm{Fix}T_2$ *such that*

$$w(u^*, v^*) \leq \frac{\eta}{1-q}, \text{ where } q = q_i \text{ for } i = \{1, 2\};$$

(respectively for every $v^* \in \mathrm{Fix}T_2$ *there exists* $u^* \in \mathrm{Fix}T_1$ *such that*

$$w(v^*, u^*) \leq \frac{\eta}{1-q}, \text{ where } q = q_i \text{ for } i = \{1, 2\}).$$

Proof. From the above theorem we have that $\mathrm{Fix}T_1 \neq \emptyset \neq \mathrm{Fix}T_2$.
Let $u_0 \in \mathrm{Fix}T_1$, then $u_0 \in T_1(u_0)$. Using the hypothesis (2) we have that there exists $u_1 \in T_2(u_0)$ such that $w(u_0, u_1) \leq \eta$.
Since T_1, T_2 are weakly Ćririć-contractive with $q_i \in [0, 1)$ and $i = \{1, 2\}$ we have that for every $u_0, u_1 \in X$ with $u_1 \in T_2(u_0)$ there exists $u_2 \in T_2(u_1)$ such that

i. $w(u_1, u_2) \leq q w(u_0, u_1)$

ii. $w(u_1, u_2) \leq D_w(u_0, T_2(u_0)) \leq q w(u_0, u_1)$

iii. $w(u_1, u_2) \leq D_w(u_1, T_2(u_1)) \leq q w(u_1, u_2)$

iv. $w(u_1, u_2) \leq \frac{q}{2} D_w(u_0, T_2(u_1)) \leq \frac{q}{2} w(u_0, u_2)$

$w(u_1, u_2) \leq \frac{q}{2}[w(u_0, u_1) + w(u_1, u_2)]$

$w(u_1, u_2) \leq \frac{q}{2-q} w(u_0, u_1)$.

Then $w(u_1, u_2) \leq \max\{q, \frac{q}{2-q}\}w(u_0, u_1)$.
Since for $q \in [0, 1)$ we have true $q > \frac{q}{2-q}$, then we have

$$w(u_1, u_2) \leq qw(u_0, u_1).$$

For $u_1 \in X$ and $u_2 \in T_2(u_1)$, there exists $u_3 \in T_2(u_2)$ such that

i. $w(u_2, u_3) \leq qw(u_1, u_2)$

ii. $w(u_2, u_3) \leq D_w(u_1, T_2(u_1)) \leq qw(u_1, u_2)$

iii. $w(u_2, u_3) \leq D_w(u_2, T_2(u_2)) \leq qw(u_2, u_3)$

iv. $w(u_2, u_3) \leq \frac{q}{2}D_w(u_1, T_2(u_2)) \leq \frac{q}{2}w(u_1, u_3)$
$w(u_2, u_3) \leq \frac{q}{2}[w(u_1, u_2) + w(u_2, u_3)]$
$w(u_2, u_3) \leq \frac{q}{2-q}w(u_1, u_2)$

Then $w(u_2, u_3) \leq \max\{q, \frac{q}{2-q}\}w(u_1, u_2)$.
Since for $q \in [0, 1)$ we have true $q > \frac{q}{2-q}$, then we have

$$w(u_2, u_3) \leq qw(u_1, u_2) \leq q^2w(u_0, u_1).$$

By induction we obtain a sequence $(u_n)_{n\in\mathbb{N}} \in X$ such that

(1) $u_{n+1} \in T_2(u_n)$, for every $n \in \mathbb{N}$;

(2) $w(u_n, u_{n+1}) \leq q^n w(u_0, u_1)$.

For $n, m \in \mathbb{N}$, with $m > n$ we have the inequality

$$\begin{aligned} w(u_n, u_m) \leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + \cdots + w(u_{m-1}, u_m) \leq \\ < q^n w(u_0, u_1) + q^{n+1}w(u_0, u_1) + \cdots + q^{m-1}w(u_0, u_1) \leq \\ \leq \frac{q^n}{1-q}w(u_0, u_1) \end{aligned}$$

By Lemma 1(3) we have that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is a complete metric space, we have that there exists $v^* \in X$ such that $u_n \xrightarrow{d} v^*$.
By the lower semicontinuity of $w(x, \cdot) : X \to [0, \infty)$ we have
$w(u_n, v^*) \leq \liminf_{m\to\infty} w(u_n, u_m) \leq \dfrac{q^n}{1-q}w(u_0, u_1)$.
For $u_{n-1}, v^* \in X$ and $u_n \in T_2(u_{n-1})$ there exists $z_n \in T_2(v^*)$ such that we have

i. $w(u_n, z_n) \leq qw(u_{n-1}, v^*) \leq \frac{q^n}{1-q}w(u_0, u_1)$

ii. $w(u_n, z_n) \leq qD_w(u_{n-1}, T_2(u_{n-1})) \leq qw(u_{n-1}, u_n) \leq ... \leq q^n w(u_0, u_1)$

iii. $w(u_n, z_n) \leq qD_w(v^*, T_2(v^*)) \leq w(v^*, z_n) \leq \frac{q^n}{1-q}w(u_0, u_1)$

iv. $w(u_n, z_n) \leq \frac{q}{2}D_w(u_{n-1}, T_2(v^*)) \leq \frac{q}{2}w(u_{n-1}, z_n) \leq \frac{q^n}{2(1-q)}w(u_0, u_1)$.

Then $w(u_n, z_n) \leq \max\{\frac{q^n}{1-q}, q^n, \frac{q^n}{2(1-q)}\}w(u_0, u_1)$.
Since $\frac{q^n}{1-q} > q^n$ and $\frac{q^n}{1-q} > \frac{q^n}{2(1-q)}$ for every $q \in [0, 1)$ we have that

$$w(u_n, z_n) \leq \frac{q^n}{1-q} w(u_0, u_1).$$

So, we have:
$w(u_n, v^*) \leq \frac{q^n}{1-q} w(u_0, u_1)$
$w(u_n, z_n) \leq \frac{q^n}{1-q} w(u_0, u_1)$.
Applying Lemma 1(2), from the above relations we have that $z_n \stackrel{d}{\to} v^*$.
Then, we know that $z_n \in T_2(v^*)$ and $z_n \stackrel{d}{\to} v^*$. In this case, by the closure of T_2, it results that $v^* \in T_2(v^*)$. Then, by $w(u_n, v^*) \leq \frac{q^n}{1-q} w(u_0, u_1)$, with $n \in \mathbb{N}$, for $n = 0$, we obtain

$$w(u_0, v^*) \leq \frac{1}{1-q} w(u_0, u_1) \leq \frac{\eta}{1-q},$$

which completes the proof. ■

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