On the Classification of Simple Singularities in Positive Characteristic

Deeba Afzal, Muhammad Ahsan Binyamin and Faira Kanwal Janjua

Abstract

The aim of the article is to describe the classification of simple isolated hypersurface singularities over a field of positive characteristic by certain invariants without computing the normal form. We also give a description of the algorithms to compute the classification which we have implemented in the Singular libraries classifyCeq.lib and classifyReq.lib.

1 Introduction

Simple hypersurface singularities in characteristic $p > 0$ were classified by Greuel and Kröning [4] with respect to contact equivalence. Greuel and Nguyen Hong Duc [6] classified the simple hypersurface singularities in characteristic $p > 2$ with respect to right equivalence and the simple plane curve singularities in characteristic 2 with respect to right equivalence. We complete the classification of hypersurface singularities in characteristic 2.

Also we describe our implementation of a classifier for simple singularities with respect to contact equivalence respectively right equivalence in SINGULAR [5],[2]. We use for distinguishing the different cases the blowing up as a new tool.

Key Words: Simple Singularity, Normal form, Contact equivalent, Right equivalent.

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2 Basic Definitions

Let $K[[x]] = K[[x_1, \ldots, x_n]]$ be the local ring of formal power series and $m$ is its maximal ideal, $K$ an algebraically closed field of characteristic $p > 0$. The following definitions can be found in [3].

**Definition 2.1.** Let $f \in m - \{0\}$ be a formal power series, then the ideal $j(f) = \langle f_{x_1}, \ldots, f_{x_n} \rangle$ is called Jacobian ideal and the $K$-algebra $M_f = K[[x]]/j(f)$ is called Milnor algebra, and $\mu(f) = \dim(M_f)$ is called the Milnor number.

**Definition 2.2.** Let $f \in m - \{0\}$ is a formal power series, then the ideal $tj(f) = \langle f, f_{x_1}, \ldots, f_{x_n} \rangle$ is called Tjurina ideal and the $K$-algebra $N_f = K[[x]]/tj(f)$ is called Tjurina algebra, and $\tau(f) = \dim(N_f)$ is called the Tjurina number.

**Definition 2.3.** Let $f$ and $g \in m \subset K[[x]]$ $f$ is called to be right equivalent to $g$, $f \sim_R g$ if there exists an automorphism $\phi$ of $K[[x]]$ such that $\phi(f) = g$.

$f$ is called to be contact equivalent to $g$, $f \sim_c g$ if there exists an automorphism $\phi$ of $K[[x]]$ such that $< \phi(f) > = < g >$, that is, there exists a unit $u$ such that $\phi(f) = ug$.

**Definition 2.4.** Let $f \in m \subset K[[x]]$. Then $f$ is called $k$-determined if all $g \in K[[x]]$ with $f - g \in m^{k+1}$ are right equivalent to $f$.

**Definition 2.5.** Let $f \in K[[x]]$ then $k$-jet of $f$ is the Taylor expansion of $f$ up to degree $k$ terms.

**Definition 2.6.** Let $f \in m^2 \subset K[[x]]$. Then Hesse matrix of $f$ is defined by

$$H(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right)_{1 \leq i, j \leq n}$$

The corank of $f$ is defined as $\text{corank}(f) = n - \text{rank}(H(f))$.

3 Arnold’s Classification Of Simple Singularities

Arnold gave a classification of simple singularities with respect to right equivalence over the field of complex numbers [1]. These are also the simple singularities with respect to contact equivalence.

Normal forms of simple singularities with respect to the Arnold’s classification are given as follows
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<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_k$</td>
<td>$x_1^{k+1} + x_2^2 + \cdots + x_n^k$, $k \geq 1$</td>
</tr>
<tr>
<td>$D_k$</td>
<td>$x_1 x_2^k + x_2^{k+1} + x_3^2 + \cdots + x_n^2$, $k \geq 4$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$x_1^3 + x_2^3 + x_3^3 + \cdots + x_n^3$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$x_1^4 + x_1 x_2^2 + x_3^3 + \cdots + x_n^4$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$x_1^3 + x_2^3 + x_3^3 + \cdots + x_n^3$</td>
</tr>
</tbody>
</table>

Note that for these singularities the Milnor number is equal to the Tjurina number and given by the index in the notation.

4  Greuel and Kröning’s Classification of Simple Singularities

Greuel and Kröning gave the classification of simple singularities with respect to contact equivalence in characteristic $p > 0$ [4].

Proposition 4.1. Let $p = \text{char}(K)$. A plane curve singularity is contact simple if and only if it is contact equivalent to one of the following forms:

(i) $p \neq 2$

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form for $f \in K[[x, y]]$</th>
<th>$\tau$</th>
<th>$r$, $p = 5$</th>
<th>$r$, $p = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_k$</td>
<td>$x_1^{k+1} + y^2$, $k \geq 1$</td>
<td>$k$</td>
<td>$k$ if $5 \mid k + 1$</td>
<td>$k$ if $3 \mid k + 1$</td>
</tr>
<tr>
<td>$D_k$</td>
<td>$x y^k$, $k \geq 4$</td>
<td>$k$</td>
<td>$k$ if $5 \mid k + 1$</td>
<td>$k$ if $3 \mid k + 1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$x^3 + y^3$, in char $= 3$</td>
<td>$6$</td>
<td>$7$</td>
<td>$9$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$x^3 + y^3 + x^2 y^2$, in char $= 3$</td>
<td>$7$</td>
<td>$7$</td>
<td>$9$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$x^3 + y^3$, in char $= 5$</td>
<td>$8$</td>
<td>$10$</td>
<td>$12$</td>
</tr>
</tbody>
</table>

(ii) $p = 2$

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form for $f \in K[[x, y]]$</th>
<th>$\tau$</th>
<th>$r$, $p = 5$</th>
<th>$r$, $p = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{m-1}$</td>
<td>$x^m + y^m$, $m \geq 1$</td>
<td>$2m$</td>
<td>$m$ even</td>
<td></td>
</tr>
</tbody>
</table>
Proposition 4.2. Let $p = \text{char}(K) > 2$. A hypersurface singularity $f \in m^2 \subset K[[x]]$ is contact simple if and only if it is contact equivalent to one of the following form:

$$f(x_1, \ldots, x_n) = g(x_1, x_2) + x_3^2 + \cdots + x_n^2,$$

where $g \in K[[x_1, x_2]]$ is one of the list in proposition 4.1 part (i).

Proposition 4.3. Let $p = \text{char}(K) = 2$. A surface singularity is contact simple if and only if it is contact equivalent to one of the following forms:

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form for $f \in K[[x, y, z]]$</th>
<th>$k$</th>
<th>$k+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{2m}$</td>
<td>$x^2 + x^m y^m$</td>
<td>$m \geq 2$</td>
<td>$4m$</td>
</tr>
<tr>
<td>$D_{2m+1}$</td>
<td>$x^2 + x^m y^m + x^m z^m$</td>
<td>$m \geq 2$</td>
<td>$4m$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$z^2 + x^3 + y^3 + xyz$</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$E_7$</td>
<td>$z^2 + x^3 + y^3 + x^2 y y$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$E_8$</td>
<td>$z^2 + x^3 + y^3 + x^2 y y$</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Proposition 4.4. Let $p = \text{char}(K) = 2$. A hypersurface singularity $f \in m^2 \subset K[[x]]$ is contact simple if and only if it is contact equivalent to the following form:

$$f(x_1, \ldots, x_n) = g(x_1, x_2, x_3) + x_4 x_5 + \cdots + x_{2k} x_{2k+1}, n = 2k + 1,$$

where $g \in K[[x_1, x_2, x_3]]$ is one of the list in proposition 4.3., or

$$f(x_1, \ldots, x_n) = g(x_1, x_2) + x_3 x_4 + \cdots + x_{2k+1} x_{2k}, n = 2k,$$

where $g \in K[[x_1, x_2]]$ is one of the list in proposition 4.1 part (ii).

To distinguish the different singularities in the classification we need detailed information on the resolution of the singularities. In the next proposition we give the blowing up sequences and the corresponding sequences of the Tjurina numbers. Here the notation $A_3 \leftarrow A_1$ means that the blowing up of $A_3$ gives $A_1$.

Proposition 4.5. Assume $f \in K[[x, y]]$ then
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(1) $A_{2m-1} \leftarrow A_{2(m-1)-1} \leftarrow \ldots A_3 \leftarrow A_1$
   is the sequence of blowing ups in the resolution of the $A_{2m-1}$-singularities.
   The corresponding sequence of Tjurina numbers is
   $(\ldots, 2k+5, 2k+4, 2k+1, 2k, 2k-3, \ldots, 8, 5, 4, 1)$

(2) $A_{2m}^0 \leftarrow A_{2(m-1)}^0 \leftarrow \ldots A_2^0$
   $(4m, 4m-4, \ldots, 12, 8, 4)$

(3) $A_{2m}^r \leftarrow A_{2(m-1)}^{r-1} \leftarrow \ldots \leftarrow A_{2(m-(r-1))}^2 \leftarrow A_{2(m-r)}^0 \leftarrow \ldots \leftarrow A_2^0$
   $(4m-2r, \ldots, 4(m-r)+5, 4(m-r)+4, 4(m-r)+1, 4(m-r), \ldots, 8, 4)$
   if $r$ is even.
   $(4m-2r-1, \ldots, 4(m-r)+5, 4(m-r)+4, 4(m-r)+1, 4(m-r), \ldots, 8, 4)$
   if $r$ is odd.

(4) $D_{2m} \leftarrow A_{2(m-2)-1} \leftarrow \ldots \leftarrow A_1$
   $(2m, 2m-4, 2m-7, 2m-8, 2m-11, \ldots, 1)$ if $m$ is even.
   $(2m, 2m-5, 2m-6, 2m-9, 2m-10, \ldots, 1)$ if $m$ is odd

(5) $D_{2m+1}^0 \leftarrow A_{2(m-2)}^0 \leftarrow \ldots \leftarrow A_2^0$
   $(4m, 4m-8, 4m-12, \ldots, 4)$

(6) $D_{2m+1}^r \leftarrow A_{2(m-2)}^{r-2} \leftarrow A_{2(m-3)}^{r-3} \leftarrow \ldots \leftarrow A_{2(m-(r-1))}^2 \leftarrow A_{2(m-r)}^0 \leftarrow \ldots \leftarrow A_2^0$
   $(4m-2r, 4m-2r-4, 4m-2r-7, \ldots, 4(m-r)+1, 4(m-r), \ldots, 8, 4)$
   if $r$ is even.
   $(4m-2r, 4m-2r-5, 4m-2r-6, \ldots, 4(m-r)+1, 4(m-r), \ldots, 8, 4)$
   if $r$ is odd.

(7) $E_6^0$ and $E_6^1$, the sequence of Tjurina numbers is (8) respectively (6).

(8) $E_7 \leftarrow A_1$, the sequence of Tjurina numbers is (7, 1).

(9) $E_8 \leftarrow A_0^2$, the sequence of Tjurina numbers is (8, 4).

Proposition 4.6. Assume $f \in K[[x, y, z]]$ then
(1) $A_k \leftarrow A_{k-2} \leftarrow \ldots \leftarrow A_1$ $k$ is odd
$(k + 1, k - 1, \ldots, 2)$
$A_k \leftarrow A_{k-2} \leftarrow \ldots \leftarrow A_2$ $k$ is even
$(k, k - 2, \ldots, 2)$

(2) $D_{2m}^0 \leftarrow D_{2(m-1)}^0 \leftarrow \ldots \leftarrow D_4^0 \leftarrow A_1$
$(4m, 4m - 4, \ldots, 8, 2)$

(3) $D_{2m}^r \leftarrow D_{2(m-1)}^{r-1} \leftarrow \ldots \leftarrow D_{2(m-(r-1))}^1 \leftarrow D_{2(m-r)}^0 \leftarrow \ldots \leftarrow D_4^0 \leftarrow A_1$
$(4m - 2r, 4m - 2r - 2, \ldots, 4m - 4r + 2, 4(m - r), \ldots, 8, 2)$

(4) $D_{2m+1}^0 \leftarrow D_{2m-1}^0 \leftarrow \ldots \leftarrow D_5^0 \leftarrow A_3 \leftarrow A_1$
$(4m, 4(m - 1), \ldots, 8, 4, 2)$

(5) $D_{2m+1}^r \leftarrow D_{2(m-1)+1}^{r-1} \leftarrow \ldots \leftarrow D_{2(m-(r-1))+1}^1 \leftarrow D_{2(m-r)+1}^0 \leftarrow \ldots \leftarrow D_5^0 \leftarrow A_3 \leftarrow A_1$
$(4m - 2r, 4m - 2r - 2, \ldots, 4m - 4r + 2, 4(m - r), \ldots, 8, 4, 2)$

(6) $E_6^0 \leftarrow A_5 \leftarrow A_3 \leftarrow A_1$
$E_6^1 \leftarrow A_5 \leftarrow A_3 \leftarrow A_1$ $(8, 6, 4, 2)$

(7) $E_7^0 \leftarrow D_6^0 \leftarrow D_4^0 \leftarrow A_1$
$E_7^1 \leftarrow D_6^0 \leftarrow D_4^0 \leftarrow A_1$ $(14, 12, 8, 2)$

(8) $E_7^2 \leftarrow D_6^1 \leftarrow D_4^0 \leftarrow A_1$
$E_7^3 \leftarrow D_6^1 \leftarrow D_4^1 \leftarrow A_1$ $(10, 10, 8, 2)$

(9) $E_8^0 \leftarrow E_7^0 \leftarrow D_6^0 \leftarrow D_4^0 \leftarrow A_1$
$(16, 14, 12, 8, 2)$

(10) $E_8^1 \leftarrow E_7^0 \leftarrow D_6^0 \leftarrow D_4^1 \leftarrow A_1$
$(14, 14, 12, 8, 2)$

(11) $E_8^2 \leftarrow E_7^1 \leftarrow D_6^1 \leftarrow D_4^1 \leftarrow A_1$
$(12, 12, 12, 8, 2)$
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(12) \( E_8^3 \leftarrow E_6^2 \leftarrow D_6^0 \leftarrow A_1 \)  
(10, 10, 10, 8, 2)

(13) \( E_8^4 \leftarrow E_6^2 \leftarrow D_6^0 \leftarrow A_1 \)  
(8, 8, 6, 2)

**Proposition 4.7.** Let \( f \in (x, y^2)^3 \subseteq K[[x, y]], \text{char}(K) = 2 \) then \( \tau(f) \geq 10 \).

**Proof.** Let \( f = x^3 + ax^2y^2 + bxy^4 + cy^6 + h \) of weighted order at least 7 with respect to the weights 2,1 for \( x, y \). Then \( \frac{\partial f}{\partial x} = x^2 + by^4 + \frac{\partial h}{\partial x} \) and \( \frac{\partial f}{\partial y} = \frac{\partial h}{\partial y} \).

Therefore \( f \) can be reduced using \( \frac{\partial f}{\partial x} \) to \( (c - ab)y^6 + k \) of weighted degree at least 7. This implies that \( 1, y, \ldots, y^6, x, xy, xy^2, xy^3 \) are linearly independent modulo \( <f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> \). \( \square \)

**Proposition 4.8.** Let \( f \in (x, y^2)^3 \subseteq K[[x, y]], \text{char}(K) > 2 \) then \( \tau(f) \geq 9 \).

**Proof.** Let \( f = g + h \), \( g \) weighted homogeneous of degree 6 with respect to the weights 2 resp. 1 for \( x \) resp. \( y \), or zero and \( h \) of weighted order at least 7. Then \( 6g = 2x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} \). This implies if \( \text{char}(K) \neq 3 \) then \( <f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> = <h, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> \).

But the contribution of \( h \) are of weighted degree at least 7. Analyzing the proof of the proposition 5.4 we obtain \( \tau(f) \geq 9 \). If \( \text{char}(K) = 3 \) then let \( g = x^3 + ax^2y^2 + bxy^4 + cy^6 \). We have \( \frac{\partial f}{\partial x} = 2axy^2 + by^4 + \frac{\partial h}{\partial x} \) and \( \frac{\partial f}{\partial y} = 2ax^2y + bxy^3 + \frac{\partial h}{\partial y} \).

In case \( a \neq 0 \) all s-polys of \( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) have weighted order at least 7 (since \( 2x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = 0 \)). This implies that in case \( a \neq 0 \), \( 1, y, \ldots, y^6, x, xy, x^2 \) are linearly independent modulo \( <f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> \).

In case \( a = 0 \) we obtain that \( 1, y, y^2, x, xy, xy^2, x^2, x^2y, x^2y^2 \) are linearly independent modulo \( <f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> \). This implies that \( \tau(f) \geq 10 \) in this case. \( \square \)

5 Greuel and Nguyen Hong Duc’s Classification of Simple Singularities in positive Characteristic w.r.t. Right Equivalence

The normal forms of simple singularities with respect to right equivalence [6] are given as follows

**Proposition 5.1.** Let \( p = \text{char}(K) \). A plane curve singularity is right simple if and only if it is right equivalent to one of the following forms:
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\( p \neq 2 \)

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form for ( f \in K[[x, y]] )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_k</td>
<td>( x^2 + y^{k+1} ) ( 1 \leq k \leq p - 2 )</td>
<td>( k )</td>
</tr>
<tr>
<td>D_k</td>
<td>( x^2y + y^{k-1} ) ( 4 \leq k \leq p )</td>
<td>( k )</td>
</tr>
<tr>
<td>E_6</td>
<td>( x^3 + y^4 ) ( p \neq 3 )</td>
<td>6</td>
</tr>
<tr>
<td>E_7</td>
<td>( x^3 + xy^3 ) ( p \neq 3 )</td>
<td>7</td>
</tr>
<tr>
<td>E_8</td>
<td>( x^3 + y^6 ) ( p \neq 3, 5 )</td>
<td>8</td>
</tr>
</tbody>
</table>

\( p = 2 \)

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form for ( f \in K[[x, y]] )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>( xy )</td>
<td>1</td>
</tr>
</tbody>
</table>

**Proposition 5.2.** Let \( p = \text{char}(K) \). A hypersurface singularity \( f \in m^2 \subset K[[x]] \) is right simple if and only if it is right equivalent to one of the following forms:

(i) \( p \neq 2 \)

\[
 f(x_1, \ldots, x_n) = g(x_1, x_2) + x_3^2 + \cdots + x_n^2, 
\]

where \( g \in K[[x_1, x_2]] \) is one of the list in proposition 5.1 part (i).

(ii) \( p = 2 \)

\[
 f(x_1, \ldots, x_n) = x_1x_2 + x_3x_4 + \cdots + x_{2k+1}x_{2k}, n = 2k. 
\]

**Proof.** (i) is proved in [6].

To prove (ii) we use the splitting lemma (proposition 3 of [4]).

Let \( f \in m \subset K[[x]] \) and \( \text{mult}(f) = 2 \) then either

\[
 f \sim x_1^2 + x_2x_3 + \cdots, + x_{2l-1}x_{2l} + g(x_{2l+2}, \ldots, x_n)x_1 + h(x_{2l+2}, \ldots, x_n) 
\]

with \( g \in m^2 \) and \( h \in m^3 \) or

\[
 f \sim x_1x_2 + \cdots, + x_{2l-1}x_{2l} + h(x_{2l+1}, \ldots, x_n) 
\]

with \( h \in m^3 \).

In the first case it is easy to see that the Milnor number is not finite i.e. \( f \) is not simple.

In the second case we use induction to prove that \( f \) is simple if and only if \( n = 2l \). This is clear since \( f \) is simple if and only if \( h \) is simple. \( \square \)
Proposition 5.3. Let \( f \in K[[x]] \) and \( \mu(f) = 1 \) then \( f \) defines \( A_1 \)-singularity.

Proof. If \( \text{char}(K) > 2 \) then the splitting lemma implies that \( f \sim x_1^2 + \cdots + x_l^2 + h(x_{l+1}, \ldots, x_n) \) with \( h \in < x_{l+1}, \ldots, x_n >^3 \) then \( \mu(f) = \mu(h) \). But \( h \in < x_{l+1}, \ldots, x_n >^3 \) implies \( f_{x_1}, \ldots, f_{x_n} \subseteq < x_{l+1}, \ldots, x_n > \). This implies \( \mu(h) > 1 \). We obtain \( l = 1 \) and \( f \) defines \( A_1 \)-singularity.

If \( \text{char}(K) = 2 \) then either

\[ f \sim x_1^2 + x_2 x_3 + \cdots, + x_{2l} x_{2l+1} + g(x_{2l+2}, \ldots, x_n) x_1 + h(x_{2l+2}, \ldots, x_n) \]

with \( g \in m^2 \) and \( h \in m^3 \) or

\[ f \sim x_1 x_2 + \cdots, + x_{2l-1} x_{2l} + h(x_{2l+1}, \ldots, x_n) \]

with \( h \in m^3 \).

In the first case it is easy to see that the Milnor number is not finite. The second case can be settled as above. \( \square \)

Proposition 5.4. Let \( f \in (x, y^2)^3 \subseteq K[[x, y]] \), \( \text{char}(K) > 2 \) then \( \mu(f) > 8 \).

Proof. Let \( f = g + h \), \( g \) weighted homogeneous of degree 6 with respect to the weights 2 respectively 1 for \( x \) respectively \( y \) or zero and \( h \) of weighted order at least 7. If \( g = 0 \) then the leading monomials of a standard basis of \( < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \) with respect to the local weighted ordering have degree at least 5. This implies that \( 1, y, y^2, y^3, x, xy, xy^2, x^2 \) are linearly independent modulo \( < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \). This implies that \( \mu(f) \geq 9 \) in this case.

If \( g \neq 0 \) then 3 cases are possible. We may assume that \( g = x^3 + ax^2 y^2 + bxy^4 + cy^6 \). Over the algebraic closure of \( K \) \( g \) splits, \( g = (x + \alpha_1 y^2)(x + \alpha_2 y^2)(x + \alpha_3 y^2) \). If \( \alpha_1 = \alpha_2 = \alpha_3 \), we may assume (after the transformation \( x \to x - \alpha_1 y^2 \)) that \( g = x^3 \). Then \( \frac{\partial f}{\partial x} = 3x^2 + \frac{\partial h}{\partial x} \) and \( \frac{\partial f}{\partial y} = \frac{\partial h}{\partial y} \). If \( \text{char}(K) = 3 \) then we are in the same situation as the previous case and obtain \( \mu(f) > 9 \). If \( \text{char}(K) \neq 3 \) then a minimal standard basis of \( < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \) contains \( \frac{\partial f}{\partial y} \) with leading monomial \( x^2 \). \( \frac{\partial f}{\partial y} \) is of the weighted order at least 6. This implies that \( 1, y, y^5, x, xy, xy^2, xy^3 \) are linearly independent modulo \( < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \) and therefore \( \mu(f) \geq 10 \) in this case.

If two of the \( \alpha_i \) are equal, we may assume \( f = x^3 + ax^2 y^2 \). Then \( \frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + \frac{\partial h}{\partial x} \) and \( \frac{\partial f}{\partial y} = 2x^2 y + \frac{\partial h}{\partial y} \). If \( \text{char}(K) = 3 \) then \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) are in the standard basis and all the other elements have weighted order at least 6. This implies that \( 1, y, y^5, x, xy, x^2 \) are linearly independent modulo \( < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \) and \( \mu(f) \geq 9 \).

If \( \text{char}(K) \neq 3 \) then we can reduce \( \frac{\partial f}{\partial y} \) using \( \frac{\partial f}{\partial x} \) to \( a x y^3 + \varphi \), where \( \text{deg}(\varphi) \geq 6 \). If \( a = 0 \) then with the same argument as in the case \( g = x^3 \) we obtain
\[ \mu(f) \geq 10. \] If \( a \neq 0 \) then we have \( x^2 \) and \( xy^3 \) in the leading ideal of \( \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \). The next element in the standard basis has weighted order at least 6 i.e. \( 1, y, \ldots, y^5, x, xy, xy^2 \) are linearly independent, i.e. \( \mu(f) \geq 9 \).

Now assume that all the roots are different. Then (after some transformation) we may assume \( g = x(x^2 + axy^2 + y^4) \) and \( a \neq \pm 2 \). Then \( \frac{\partial f}{\partial x} = 3x^2 + 2axy^2 + \frac{\partial h}{\partial x} \), and \( \frac{\partial f}{\partial y} = 2ax^2y + 4xy^3 + \frac{\partial h}{\partial y} \). If \( \text{char}(K) \neq 3 \) we use \( f_1 := \frac{\partial f}{\partial x} \) to reduce \( \frac{\partial f}{\partial y} \) to \( f_2 := (a^2 - 3)xy^3 + \frac{a}{2}y^5 + h_1, h_1 \) of weighted order at least 6. If \( a^2 = 3 \) then \( \frac{\partial f}{\partial x}, \frac{a}{2}y^5 + h_1 \) is a standard basis of \( \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \) and \( \mu(f) = 10 \). If \( a^2 \neq 3 \) then we obtain as standard basis \( \{ \frac{\partial f}{\partial x}, f_2, y^7 \} \) and also \( \mu(f) = 10 \). If \( \text{char}(K) = 3 \) we conclude similar to the case \( g = x^3 + x^2y^2 \) that \( \mu(f) \geq 9 \).

### 6 Description of the Classifier

In this section we give the description of our classifier and its implementation in SINGULAR. This classifier is used for computing the type and normal forms of the simple singularities in characteristic \( p > 0 \) with respect to contact and right equivalence.

Our classifier is based on two main algorithms

**Algorithm 1:** classifyCeq

**Algorithm 2:** classifyReq

First algorithm of our classifier is classifyCeq, which classifies the simple singularities in characteristic \( p > 0 \) with respect to the contact equivalence. This algorithm further consist on two algorithms classifyC1eq and classifyC2eq. classifyC1eq, classifies the simple singularities with respect to contact equivalence when characteristic \( p \neq 2 \) and classifyC2eq, classifies the simple singularities with respect to the contact equivalence when characteristic \( p = 2 \).

The second algorithm of our classifier classifyReq, classifies the simple singularities in characteristic \( p > 0 \) with respect to the right equivalence.

Now we explain our classifier in detail.

#### 6.1 The Algorithm(classifyCeq)

In the first case we consider \( \text{char}(K) \neq 2 \), \( n=\)number of variables= 2. If Tjurina number of \( f \) is not finite then \( f \) is not an isolated singularity and therefore not simple.

For this case our classifier computes the \( \text{corank}(f) \). If \( \text{corank}(f) = 0 \) then it gives \( A_1 \)- singularity and if \( \text{corank}(f) = 1 \) then it gives \( A_k \)-singularities, \( k > 1 \). It makes a single blow-up of \( f \) at the origin and computes the difference of the
Now if \( \text{corank}(f) = 2 \) then it computes \( j^3(f) \). If \( j^3(f) = 0 \) then it gives \( f \) is not simple. If \( j^3(f) \) has only one factor then it is transformed to \( x \), it gives either \( f \) is not simple or \( E \)-singularities. If \( f \in (x, y^2)^3 \), then \( f \) is not simple. This can be detected by \( \tau(f) \geq 9 \), cf. Proposition 4.8. When \( \text{char}(K) \neq 3, 5 \) and \( \tau(f) \leq 8 \) then for \( \tau(f) = 6, 7, 8 \) it gives \( E_6^0, E_7^0 \) and \( E_8^0 \) singularities respectively. When \( \text{char}(K) = 5 \) then for \( \tau(f) = 6, 7, 10, 8 \) it gives \( E_6^5, E_7^5 \), \( E_8^5 \) and \( E_9^5 \) singularities respectively. When \( \text{char}(K) = 3 \) then for \( \tau(f) = 12, 10, 8 \) it gives \( E_6^3, E_7^3 \) and \( E_8^3 \) singularities respectively. If \( \text{char}(K) = 3 \) and \( \tau(f) = 7 \) then it makes a single blow-up of \( f \) at the origin. If \( f \) is smooth after this blow up then it gives \( E_6^3 \), otherwise it gives \( E_7^3 \). If \( \text{char}(K) = 3 \) and \( \tau(f) = 9 \) then again it makes a single blow-up of \( f \) at the origin. If \( f \) is smooth after this blow up then it gives \( E_6^3 \), otherwise it gives \( E_7^3 \). And if \( j^3(f) \) has two or three factors then for \( \tau(f) = k \) it gives \( D_k \)-singularities.

If \( n \)-number of variables \( > 2 \) then our classifier splits \( f \) as explained is section 4 and reduce the case into two variables.

Now we consider \( \text{char}(K) = 2 \), \( n \)-number of variables \( = 2 \). If Tjurina number of \( f \) is not finite then \( f \) is not an isolated singularity and therefore not simple.

In this case our classifier checks the order of \( f \), if \( \text{ord}(f) = 2 \) then it gives \( A_k \) singularities. To determine \( k \), it performs a blow-up of \( f \) at the origin and computes the difference of the tjurina number of \( f \) before and after the blowing-up. If the difference is equal to 4, then it gives the \( A_{2k} \) singularities, where \( k = \tau(f)/4 \) if \( \tau(f) \) is even and \( k = (\tau(f) + 1)/4 \) if \( \tau(f) \) is odd. And for the other case it give \( A_{2k-1} \) singularities, where \( k = \tau(f)/2 \) if \( \tau(f) \) is even and \( k = (\tau(f) + 1)/2 \) if \( \tau(f) \) is odd. To find value of \( r \) it makes successive blow ups and find the difference of tjurina numbers for every two consecutive blow ups. Let \( t - 1 \) and \( t \) are consecutive blow ups such that difference of tjurina numbers of these blow ups becomes 4 then it gives \( r = t - 1 \).

Now if \( \text{ord}(f) = 3 \) then it computes \( j^3(f) \). If \( j^3(f) \) has only one factor then it is transformed to \( x \). If \( f \in (x, y^2)^3 \) then \( f \) is not simple. This can be detected by \( \tau(f) \geq 10 \), cf. Proposition 4.7. If \( \tau(f) \leq 9 \) then it gives \( E \)-singularities. For \( \tau(f) = 6, 7 \) it gives \( E_6^1 \) and \( E_7 \) respectively. If \( \tau(f) = 8 \) then it makes a single
blow-up of $f$ at the origin. If $f$ is smooth after this blow up then it gives $E_6^0$, otherwise it gives $E_8$. If $j^3(f)$ has 3 factors then it gives the $D_4$-singularity, and if $j^3(f)$ has two factors then it makes a blow-up and computes the difference of Tjurina numbers before and after blow-up, if the difference is 8 then it gives $D_{2k+1}^0$. Otherwise it makes successive blow-ups and computes the Tjurina number after each blow-up until it gets Tjurina number 0. If Tjurina number of the last singularity in the resolution is 4 then it gives $D_{2k+1}^r$-singularities with $k = (\tau(f) + 2r)/4$ and for the other case it gives $D_{2k}$-singularities with $k = \tau(f)/2$. To find value of $r$ it makes successive blow ups and find the difference of Tjurina numbers for every two consecutive blow ups. Let $t − 1$ and $t$ are consecutive blow ups such that difference of Tjurina numbers of these blow ups becomes 4 then it gives $r = t$. If $ord(f) \geq 4$ then the singularity is not simple.

In case of $char(K) = 2$ and $n=$number of variables$= 3$. Our classifier checks the multiplicity of $f$. If the multiplicity of $f$ is 2 then it factorizes $j^2(f)$. If $j^2(f)$ has only one factor of multiplicity 1 then it gives the $A_1$-singularity. If $j^2(f)$ has two different factors then we have $A_k$-singularities. In order to obtain the $k$ we resolve the singularity. If the last singularity before becoming smooth is $A_2$ then we have $k = \tau(f)$ otherwise $k = \tau(f) − 1$.

At this step our classifier makes some coordinate change and transform $j^3(f)$ to $z^2$. Then we have $f = z^2 + h(x, y) + h_1(x, y)z + z^2\phi + \psi$, where $\phi \in m, \psi \in m^4$ and $h, h_1$ are homogeneous polynomials of degree 3 and 2. Now it factorizes $g = h(x, y) + h_1(x, y)z$. If it is irreducible then it gives $E_6^0$ or $E_8^0$ corresponding to the $\tau(f) = 8$ or 6 respectively. If $g$ has one linear and one quadratic factor of multiplicity 1 then it gives $D_{3}^1$ corresponding to $\tau(f) = 6$ and $D_{0}^0, E_{1}^3$ or $E_{3}^4$ corresponding to $\tau(f) = 8$. To distinguish the $D_{0}^0, E_{3}^3$ and $E_{4}^4$ singularities, it performs blow ups of $f$ at origin to compute the sequence of the Tjurina number: $(8, 4, 2)$ for $D_{0}^0$, $(8, 8, 6, 2)$ for $E_{7}^3$ and $(8, 8, 8, 6, 2)$ for $E_{4}^4$. Now if $g$ has three different factors then it gives $D_{4}^0$ and $D_{1}^1$ corresponding to the $\tau(f) = 8, 6$ respectively. If $g$ has one factor of multiplicity 1 and one factor of multiplicity 2 then it gives $D_k$ singularities. To determine $k$, it performs a blow-up and computes the difference of Tjurina numbers before an after blow-up. If the difference is 4 then it gives $D_{2k+1}^0$ or $D_{2k}^0$ singularities with $k = \tau(f)/4$ and if the difference is 2 then it gives $D_{r_{2k+1}}^r$ or $D_{r_{2k}}^r$ singularities with $k = (\tau(f) + 2r)/4$, where $r$ is the number of that blow up after which either the difference of the Tjurina numbers of blow ups becomes 4 (i.e $r \neq k−2$) or if the difference is not 4 then $r = k−2$. The resolution sequence separates the cases: $D_{2m+1}^r$, $D_{2m+1}^r$ resolution have $... \leftarrow A_3 \leftarrow A_1$ at the end. And $D_{2m}^r$, $D_{2m+1}^r$ resolution have $... \leftarrow D_4^1 \leftarrow A_1$ respectively. $D_4 \leftarrow A_1$ at the end cf. Proposition 4.6.
Now if \( g \) has only one factor of multiplicity 3 then it gives \( E_k \) singularities. If \( \tau(f) = 10, 12, 14 \) then it gives \( E_k \) singularities. If \( \tau(f) = 16 \) then it gives \( E_0^8 \).

If \( n \) = number of variables > 3 then our classifier splits \( f \) [7] and reduces the case into two or three variables.

6.2 The AlgorithmclassifyReq

We consider \( \text{char}(K) \neq 2 \), \( n \) = number of variables = 2. If the Milnor number of \( f \) is not finite then \( f \) is not simple. And if the Milnor number of \( f \) is finite then our classifier computes \( \text{corank}(f) \). If \( \text{corank}(f) \leq 1 \) then it gives \( A_k \)-singularities, where \( 1 \leq k = \mu(f) \leq \text{char}(K) - 2 \). And if \( \text{corank}(f) = 2 \) then it computes \( j^3(f) \). If \( j^3(f) \) has only one factor and \( \mu(f) \leq 8 \) then it gives \( E \)-singularities. If the factor is transformed to \( x \) and \( f \in (x,y)^3 \) then \( f \) is not simple. This can be detected by \( \mu(f) \geq 9 \), cf. proposition 5.4. If \( \text{char}(K) \neq 3 \) and \( \mu(f) = 6, 7 \) then it gives \( E_8 \), \( E_7 \) respectively. If \( \text{char}(K) \neq 2, \text{char}(K) \neq 3, 5 \) and \( \mu(f) = 8 \) then it gives \( E_8 \). And if \( j^3(f) \) has two or three factors then it gives \( D_k \)-singularities, where \( 4 \leq k = \mu(f) \leq \text{char}(K) \).

If \( \text{char}(K) \neq 2 \) and \( n \) = number of variables > 2 then our classifier splits \( f \) as described in proposition-5.2 part (i) and reduces the case into two variables.

For the case when \( \text{char}(K) = 2 \), our classifier gives \( A_2 \)-singularity for \( \mu(f) = 1 \). This is the only normal form in this case.

7 Singular Examples

LIB"classifyCeq.lib";

\begin{verbatim}
ring R=2,(x,y,z),ds;
poly f1 = x2+y2+z2+x3+x2y+xy2+y3+y5;
classifyCeq(f1);
> E_8^0 = z2+x3+y5

ring R=2,(x,y),ds;
poly f2 = x2+y2+x12+x11y+x21;
classifyCeq(f2);
> A_2k^r
> A_18^9 = x2+y21+xy11
\end{verbatim}
ring R = 5, (x, y), ds;
poly f3 = x^3 - 2x^2y - 2xy^2 + y^3 + x^4 + x^3y;
classifyCeq(f3);
> E_7^0 = x^3 + xy^3

LIB "classifyReq.lib";

ring R = 7, (x, y), ds;
poly f4 = x^3 + 2x^2y - xy^2 - y^3 + xy^3 + 3x^3y^4 + 3x^3y^2 + 2x^2y^3 + 3x^5y + 2x^4y^2 + x^7 + 3x^6y + 3x^2y^5 + 2xy^6 - x^4y^4 - 3x^3y^5 + 3x^6y^3 + 2x^5y^4 + 3x^3y^7 + 2x^2y^8 + 3x^5y^6 + 2x^4y^7 + 4x^9y + 3x^3y^10;
classifyReq(f4);
> E7 = x^3 + xy^3

ring R = 2, (x, y), ds;
poly f5 = x^2 + xy;
classifyReq(f5);
> A1 = xy

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References


Deeba Afzal,
Abdus Salam School of Mathematical Sciences GC University,
68-B New Muslim Town
Lahore 54600, Pakistan.
Email: deebafzal@gmail.com

Muhammad Ahsan Binyamin,
Department of Mathematics
GC University,Faisalabad Pakistan.
Email: ahsanbanyamin@gmail.com

Faira Kanwal Janjua,
Abdus Salam School of Mathematical Sciences GC University,
68-B New Muslim Town
Lahore 54600, Pakistan.
Email: fairakanwaljanjua@gmail.com