Monomial ideals of minimal depth

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Abstract

Let $S$ be a polynomial algebra over a field. We study classes of monomial ideals (as for example lexsegment ideals) of $S$ having minimal depth. In particular, Stanley’s conjecture holds for these ideals. Also we show that if $I$ is a monomial ideal with $\text{Ass}(S/I) = \{P_1, P_2, \ldots, P_s\}$ and $P_i \not\subset \sum_{j \neq i} P_j$ for all $i \in [s]$, then Stanley’s conjecture holds for $S/I$.

Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over $K$. Let $I \subseteq S$ be a monomial ideal and $I = \cap_{i=1}^{s} Q_i$ an irredundant primary decomposition of $I$, where the $Q_i$ are monomial ideals. Let $Q_i$ be $P_i$-primary. Then each $P_i$ is a monomial prime ideal and $\text{Ass}(S/I) = \{P_1, \ldots, P_s\}$.

According to Lyubeznik [9] the size of $I$, denoted $\text{size}(I)$, is the number $a + (n - b) - 1$, where $a$ is the minimum number $t$ such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{i=1}^{t} Q_{j_i}} = \left( \sum_{j=1}^{s} Q_j \right),$$

and where $b = \text{ht}(\sum_{j=1}^{s} Q_j)$. It is clear from the definition that $\text{size}(I)$ depends only on the associated prime ideals of $S/I$. In the above definition if we replace “there exists $j_1 < \cdots < j_t$” by “for all $j_1 < \cdots < j_t$”, we obtain the definition of $\text{bigsiee}(I)$, introduced by Popescu [11]. Clearly $\text{bigsiee}(I) \geq \text{size}(I)$.

Key Words: Monomial ideal, Stanley decomposition, Stanley depth, Lexsegment ideal, Minimal depth.


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Theorem 0.1. (Lyubeznik [9]) Let $I \subset S$ be a monomial ideal then $\text{depth}(I) \geq 1 + \text{size}(I)$.

Herzog, Popescu and Vladoiu say in [5] that a monomial ideal $I$ has minimal depth, if $\text{depth}(I) = \text{size}(I) + 1$. Suppose above that $P_i \not\subset \sum_{j \neq i} P_j$ for all $i \in [s]$. Then $I$ has minimal depth as shows our Corollary 1.3 which extends [11, Theorem 2.3]. It is easy to see that if $I$ has bigsize 1 then it must have minimal depth (see our Corollary 1.5).

Next we consider the lexicographical order on the monomials of $S$ induced by $x_1 > x_2 > \cdots > x_n$. Let $d \geq 2$ be an integer and $M_d$ the set of monomials of degree $d$ of $S$. For two monomials $u, v \in M_d$, with $u \geq_{\text{lex}} v$, the set

$$L(u, v) = \{ w \in M_d | u \geq_{\text{lex}} w \geq_{\text{lex}} v \}$$

is called a lexsegment set. A lexsegment ideal in $S$ is a monomial ideal of $S$ which is generated by a lexsegment set. We show that a lexsegment ideal has minimal depth (see our Theorem 1.6).

Now, let $M$ be a finitely generated multigraded $S$-module, $z \in M$ be a homogeneous element in $M$ and $zK[Z], Z \subseteq \{x_1, \ldots, x_n\}$ the linear $K$-subspace of $M$ of all elements $zf, f \in K[Z]$. Such a linear $K$-subspace $zK[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where $|Z|$ denotes the number of indeterminates in $Z$. A presentation of $M$ as a finite direct sum of spaces $D : M = \bigoplus_{i=1}^{r} z_i K[Z_i]$ is called a Stanley decomposition.

Stanley depth of a decomposition $D$ is the number

$$sdepth(D) = \min\{|Z_i| : i = 1, \ldots, r\}.$$ 

The number

$$sdepth(M) := \max\{sdepth(D) : \text{Stanley decomposition of } M\}$$

is called Stanley depth of $M$. In [14] R. P. Stanley conjectured that

$$sdepth(M) \geq \text{depth}(M).$$

Theorem 0.2 ([5]). Let $I \subset S$ be a monomial ideal then $sdepth(I) \geq 1 + \text{size}(I)$. In particular, Stanley’s conjecture holds for the monomial ideals of minimal depth.

As a consequence, Stanley’s depth conjecture holds for all ideals considered above since they have minimal depth. It is still not known a relation between $sdepth(I)$ and $sdepth(S/I)$, but our Theorem 2.3 shows that Stanley’s conjecture holds also for $S/I$ if $P_i \not\subset \sum_{j \neq i} P_j$ for all $i \in [s]$. Some of the recent development about the Stanley’s conjecture is given in [12].
1 Minimal depth

We start this section extending some results of Popescu in [11]. Lemma 1.1, Proposition 1.2, Lemma 1.4 and Corollary 1.5 were proved by Popescu when I is a squarefree monomial ideal. We show that with some small changes the same proofs work even in the non-squarefree case.

Lemma 1.1. Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_s \not\subseteq \sum_{i=1}^{s-1} P_i$, then

$$\text{depth}(S/I) = \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1+\text{depth}(S/\cap_{i=1}^{s-1} (Q_i+Q_s))\}.$$ 

Proof. We have the following exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{s-1} Q_i \oplus S/Q_s \longrightarrow S/\cap_{i=1}^{s-1} (Q_i+Q_s) \longrightarrow 0.$$ 

Clearly $\text{depth}(S/I) \leq \text{depth}(S/Q_s)$ by [1, Proposition 1.2.13]. Choosing $x_a^a$ where $x_j^a \in P_s \not\subseteq \sum_{i=1}^{s-1} P_i$ and $a$ is minimum such that $x_j^a \in Q_s$ we see that $I : x_j^a = \cap_{i=1}^{s-1} Q_i$ and by [13, Corollary 1.3] we have

$$\text{depth}(S/I) \leq \text{depth}(S/(I : x_j^a)) = \text{depth}(S/(\cap_{i=1}^{s-1} Q_i)).$$

Now by using Depth Lemma (see [15, Lemma 1.3.9]) we have

$$\text{depth}(S/I) = \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1+\text{depth}(S/\cap_{i=1}^{s-1} (Q_i+Q_s))\},$$

which is enough. \qed

Proposition 1.2. Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of I as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subseteq \sum_{1=i\neq j}^{s-1} P_j$ for all $i \in [s]$. Then $\text{depth}(S/I) = s - 1$.

Proof. It is enough to consider the case when $\sum_{i=1}^{s} P_j = \mathfrak{m}$. We use induction on s. If $s = 1$ the result is trivial. Suppose that $s > 1$. By Lemma 1.1 we get

$$\text{depth}(S/I) \leq \min\{\text{depth}(S/\cap_{i=1}^{s-1} Q_i), \text{depth}(S/Q_s), 1+\text{depth}(S/\cap_{i=1}^{s-1} (Q_i+Q_s))\}.$$ 

Then by induction hypothesis we have

$$\text{depth}(S/\cap_{i=1}^{s-1} Q_i) = s - 2 + \dim(S/(\sum_{i=1}^{s-1} Q_i)) \geq s - 1.$$
We see that $\cap_{i=1}^{s-1} (Q_i + Q_s)$ satisfies also our assumption, the induction hypothesis gives $\text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s)) = s - 2$. Since $Q_i \not\subset Q_s$, $i < s$ by our assumption we get $\text{depth}(S/Q_s) > \text{depth}(S/(Q_i + Q_s))$ for all $i < s$. It follows

$$\text{depth}(S/Q_s) \geq 1 + \text{depth}(S/\cap_{i=1}^{s-1} (Q_i + Q_s))$$

which is enough.

**Corollary 1.3.** Let $I \subset S$ be a monomial ideal such that $\text{Ass}(S/I) = \{P_1, \ldots, P_s\}$ where $P_i \not\subset \sum_{i \neq j} P_j$ for all $i \in [s]$. Then $I$ has minimal depth.

**Proof.** Clearly $\text{size}(I) = s - 1$ and by Proposition 1.2 we have $\text{depth}(I) = s$, thus we have $\text{depth}(I) = \text{size}(I) + 1$, i.e. $I$ has minimal depth.

**Lemma 1.4.** Let $I = \cap_{i=1}^{s} Q_i$ be the irredundant primary decomposition of $I$ and $\sqrt{Q_i} \neq m$ for all $i$. Suppose that there exists $1 \leq r < s$ such that $\sqrt{Q_i + Q_1} = m$ for each $r < j \leq s$ and $1 \leq i \leq r$. Then $\text{depth}(I) = 2$.

**Proof.** The proof follows by using Depth Lemma on the following exact sequence.

$$0 \rightarrow S/I \rightarrow S/\cap_{i=1}^{s} Q_i \oplus S/\cap_{j=r}^{s} Q_j \rightarrow S/\cap_{i=1}^{s} \cap_{j=r}^{s} (Q_i + Q_j) \rightarrow 0.$$

**Corollary 1.5.** Let $I \subset S$ be a monomial ideal. If $\text{bigs}ize(I)$ is one then $I$ has minimal depth.

**Proof.** We know that $\text{size}(I) \leq \text{bigs}ize(I)$. If $\text{size}(I) = 0$ the $\text{depth}(I) = 1$ and the result follows in this case. Now let us suppose that $\text{size}(I) = 1$. By Lemma 1.4 we have $\text{depth}(I) = 2$. Hence the result follows.

Let $d \geq 2$ be an integer and $M_d$ the set of monomials of degree $d$ of $S$. For two monomials $u, v \in M_d$, with $u \geq_{\text{lex}} v$, we consider the lexsegment set

$$\mathcal{L}(u, v) = \{w \in M_d | u \geq_{\text{lex}} w \geq_{\text{lex}} v\}.$$

**Theorem 1.6.** Let $I = (\mathcal{L}(u, v)) \subset S$ be a lexsegment ideal. Then $\text{depth}(I) = \text{size}(I) + 1$, that is $I$ has minimal depth.

**Proof.** For the trivial cases $u = v$ the result is obvious. Suppose that $u = x_1^{a_1} \ldots x_n^{a_n}$, $v = x_1^{b_1} \ldots x_n^{b_n} \in S$. First assume that $b_1 = 0$. If there exist $r$ such that $a_1 = \cdots = a_r = 0$ and $a_{r+1} \neq 0$, then $I$ is a lexsegment ideal in $S' := K[x_{r+1}, \ldots, x_n]$. We get $\text{depth}(IS) = \text{depth}(IS') + r$ and by definition of size we have $\text{size}(IS) = \text{size}(IS') + r$. This means that without loss of generality we can assume that $a_1 > 0$. If $x_n u/x_1 \geq_{\text{lex}} v$, then by [3, Proposition 3.2] $\text{depth}(I) = 1$ which implies that $m \in \text{Ass}(S/I)$, thus $\text{size}(I) = 0$ and the result
follows in this case. Now consider the complementary case \( x_nu/x_1 <_{lex} v \), then \( u \) is of the form \( u = x_1x_1^n \cdots x_n^a \) where \( l \geq 2 \). Let \( I = \cap_{i=1}^s Q_i \) be an irredundant primary decomposition of \( I \), where \( Q_i \)'s are monomial primary ideals. If \( l \geq 4 \) and \( v = x_2^1 \) then by [3, Proposition 3.4] we have \( \text{depth}(I) = l-1 \). After [6, Proposition 2.5(ii)] we know that

\[
\sqrt{\sum_{i=1}^s Q_i} = (x_1, x_2, x_1, \ldots, x_n) \notin \text{Ass}(S/I),
\]

but \( (x_1, x_2), (x_2, x_1, \ldots, x_n) \in \text{Ass}(S/I) \). Therefore, \( \text{size}(I) = l - 2 \) and we have \( \text{depth}(I) = \text{size}(I) + 1 \), so we are done in this case. Now consider the case \( v = x_2^{d-1}x_j \) for some \( 3 \leq j \leq n - 2 \) and \( l \geq j + 2 \), then again by [3, Proposition 3.4] we have \( \text{depth}(I) = l - j + 1 \) and by [6, Proposition 2.5(ii)] we have

\[
\sqrt{\sum_{i=1}^s Q_i} = (x_1, \ldots, x_j, x_1, \ldots, x_n) \notin \text{Ass}(S/I),
\]

and \( (x_1, \ldots, x_j), (x_2, \ldots, x_j, x_1, \ldots, x_n) \in \text{Ass}(S/I) \). Therefore, \( \text{size}(I) = l - j \) and again we have \( \text{depth}(I) = \text{size}(I) + 1 \). Now for all the remaining cases by [3, Proposition 3.4] we have \( \text{depth}(I) = 2 \), and by [6, Proposition 2.5(ii)]

\[
\sqrt{\sum_{i=1}^s Q_i} = (x_1, \ldots, x_n) \notin \text{Ass}(S/I),
\]

but \( (x_1, \ldots, x_j), (x_2, \ldots, x_n) \in \text{Ass}(S/I) \), for some \( j \geq 2 \). Therefore \( \text{size}(I) = 1 \). Thus the equality \( \text{depth}(I) = \text{size}(I) + 1 \) follows in all cases when \( b_1 = 0 \).

Now let us consider that \( b_1 > 0 \), then \( I = x_1^{b_1}I' \) where \( I' = (I : x_1^{b_1}) \). Clearly \( I' \) is a lexsegment ideal generated by the lexsegment set \( L(u', v') \) where \( u' = u/x_1^{b_1} \) and \( v' = v/x_1^{b_1} \). The ideals \( I, I' \) are isomorphic, therefore \( \text{depth}(I') = \text{depth}(I) \). It is enough to show that \( \text{size}(I') = \text{size}(I) \). We have the exact sequence

\[
0 \rightarrow S/I' \xrightarrow{x_1^{b_1}} S/I \rightarrow S/(I, x_1^{b_1}) = S/(x_1^{b_1}) \rightarrow 0,
\]

and therefore

\[
\text{Ass}(S/I') \subset \text{Ass}(S/I) \subset \text{Ass}(S/I') \cup \{(x_1)\}.
\]

As \( \{(x_1)\} \in \text{Ass}(S/I) \) since it is a minimal prime over \( I \), we get \( \text{Ass}(S/I) = \text{Ass}(S/I') \cup \{(x_1)\} \). Let \( s' \) be the minimum number such that there exist \( P_1, \ldots, P_s \in \text{Ass} S/I' \) such that \( \sum_{i=1}^s P_i = a := \sum_{P \in \text{Ass}(S/I')} P \). Then
size(I') = s' + \dim(S/a) - 1. Let s be the minimum number t such that there exist t prime ideals in \text{Ass}(S/I') whose sum is (a, x_1). By [6, Lemma 2.1] we have that at least one prime ideal from \text{Ass}(S/I') contains necessarily x_1, we have x_1 \in a. It follows s \leq s' because any \( \sum_{i=1}^{s'} P_i = a = \sum_{P \in \text{Ass}(S/I')} P \).

If we have \( P_1', \ldots, P_{s-1}' \in \text{Ass}(S/I') \) such that \( \sum_{i=1}^{s-1} P_i' + (x_1) = a \) then we have also \( \sum_{i=1}^{s-1} P_i' + P_1 = a \) for some \( P_1 \in \text{Ass}(S/I') \) which contains \( x_1 \). Thus \( s = s' \) and so size(I) = size(I').

2 Stanley depth of cyclic modules defined by ideals of minimal depth

Using Corollaries 1.3, 1.5 and Theorems 1.6, 0.2 we get the following theorem.

**Theorem 2.1.** Stanley’s conjecture holds for \( I \), if it satisfies one of the following statements:

1. \( P_i \not\subset \sum_{i=j\neq i}^{s} P_j \) for all \( i \in [s] \).
2. the bigsize of \( I \) is one,
3. \( I \) is a lexsegment ideal.

**Remark 2.2.** Usually, if Stanley’s conjecture holds for an ideal \( I \) then we may show that it holds for the module \( S/I \) too. There exist no general explanation for this fact. If \( I \) is a monomial ideal of bigsize one then Stanley’s conjecture holds for \( S/I \). Indeed, case depth(S/I) = 0 is trivial. Suppose depth(S/I) \neq 0, then by Lemma 1.4 depth(S/I) = 1, therefore by [2, Proposition 2.13] sdepth(S/I) \geq 1. If \( I \) is a lexsegment ideal then Stanley’s conjecture holds for \( S/I \) [6]. Below we show this fact in the first case of the above theorem.

**Theorem 2.3.** Let \( I = \bigcap_{i=1}^{s} Q_i \) be the irredundant presentation of \( I \) as an intersection of primary monomial ideals. Let \( P_i := \sqrt{Q_i} \). If \( P_i \not\subset \sum_{i=\neq j}^{s} P_j \) for all \( i \in [s] \) then sdepth(S/I) \geq depth(S/I), that is the Stanley’s conjecture holds for \( S/I \).

**Proof.** Using [4, Lemma 3.6] it is enough to consider the case \( \sum_{i=1}^{s} P_i = m \).

By Proposition 1.2 we have depth(S/I) = s − 1. We show that sdepth(S/I) \geq s − 1. Apply induction on \( s \), case \( s = 1 \) being clear. Fix \( s > 1 \) and apply induction on \( n \). If \( n \leq 5 \) then the result follows by [10]. Let \( A := \bigcup_{i=1}^{s}(G(P_i) \setminus \sum_{i=j\neq i}^{s} G(P_j)) \). If \( A = m \) then note that \( G(P_i) \cap G(P_j) = \emptyset \) for all \( i \neq j \).

By [7, Theorem 2.1] and [8, Theorem 3.1] we have sdepth(S/I) \geq s − 1. Now suppose that \( (A) \neq m \). By renumbering the primes and variables we can
assume that $x_n \not\in A$. There exists a number $r$, $2 \leq r \leq s$ such that $x_n \in G(P_j)$, $1 \leq j \leq r$ and $x_n \notin G(P_j)$, $r + 1 \leq j \leq s$. Let $S' := K[x_1, \ldots, x_{n-1}]$. First assume that $r < s$. Let $Q_j' = Q_j \cap S'$, $P_j' = P_j \cap S'$ and $J = \bigcap_{i=r+1}^{s} Q_i' \subset S'$, $L = \bigcap_{i=1}^{s} Q_i' \subset S'$. We have $(I, x_n) = ((J \cap L), x_n)$ because $(Q_j, x_n) = (Q_j', x_n)$ using the structure of monomial primary ideals given in [15]. In the exact sequence

$$0 \to S/(I : x_n) \to S/I \to S/(I, x_n) \to 0,$$

the sdepth of the right end is $\geq s-1$ by induction hypothesis on $n$ for $J \cap L \subset S'$ (note that we have $P_i' \not\subseteq \sum_{i \neq j} P_j'$ for all $i \in [s]$ since $x_n \notin A$). Let $e_I$ be the maximum degree in $x_n$ of a monomial from $G(I)$. Apply induction on $e_I$. If $e_I = 1$ then $(I : x_n) = JS$ and the sdepth of the left end in the above exact sequence is equal with $\text{sdepth}(S/JS) \geq (s-r-1)+r = s-1$ since there are at least $r$ variables which do not divide the minimal monomial generators of ideal $(I : x_n)$ and we may apply induction hypothesis on $s$ for $J$. By [13, Theorem 3.1] we have $\text{sdepth}(S/I) = \min\{\text{sdepth}(S/(I : x_n)), \text{sdepth}(S/(I, x_n))\} \geq s-1$. If $e_I > 1$ then note that $e_{(I : x_n)} < e_I$ and by induction hypothesis on $e_I$ or $s$ we get $\text{sdepth}(S/(I : x_n)) \geq s-1$. As above we obtain by [13, Theorem 3.1] $\text{sdepth}(S/I) \geq s-1$.

Now let $r = s$. If $e_I = 1$ then $I = (L, x_n)$ and by induction on $n$ we have $\text{sdepth}(S/I) = \text{sdepth}(S'/L) \geq s-1$. If $e_I > 1$ then by induction hypothesis on $e_I$ and $s$ we get $\text{sdepth}(S/(I : x_n)) \geq s-1$. As above we are done using [13, Theorem 3.1].

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References


