A new characterization of $A_7$ and $A_8$

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Abstract

Let $G$ be a finite group and $\pi_e(G)$ be the set of element orders of $G$. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. Set $\text{nse}(G):=\{m_k|k \in \pi_e(G)\}$. It is proved that $A_n$ are uniquely determined by $\text{nse}(A_n)$, where $n \in \{4,5,6\}$. In this paper, we prove that if $G$ is a group such that $\text{nse}(G)=\text{nse}(A_n)$ where $n \in \{7,8\}$, then $G \cong A_n$.

1 Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. Also the set of element orders of $G$ is denoted by $\pi_e(G)$. A finite group $G$ is called a simple $K_n-$group, if $G$ is a simple group with $|\pi(G)|=n$.

Set $m_i=\text{m}_i(G)=|\{g \in G|\text{ the order of } g \text{ is } i\}|$. In fact, $m_i$ is the number of elements of order $i$ in $G$, and $\text{nse}(G):=\{m_i|i \in \pi_e(G)\}$, the set of sizes of elements with the same order. For the set $\text{nse}(G)$, the most important problem is related to Thompson’s problem. In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows. For each finite group $G$ and each integer $d \geq 1$, let $G(d) = \{x \in G| x^d = 1\}$. Defining $G_1$ and $G_2$ is of the same order type if, and only if, $|G_1(d)| = |G_2(d)|$, $d = 1$, $d = 2$, $d = 3$, $d = 4$. We are interested in the possible values of $d$.

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Suppose $G_1$ and $G_2$ are of the same order type. If $G_1$ is solvable, is $G_2$ necessarily solvable? ([10, Problem 12.37])

We know that if groups $G_1$ and $G_2$ are of the same order type, then $|G_1| = |G_2|$ and nse($G_1$) = nse($G_2$). So it is natural to investigate the Thompson’s Problem by $|G|$ and nse($G$).

In [4], [2], [3] and [1], it is proved that all simple $K_4$—groups, symmetric groups $S_r$ where $r$ is prime, sporadic simple groups and $L_2(p)$ where $p$ is prime, can be uniquely determined by nse($G$) and the order of $G$. In [9] and [8], it is proved that the groups $A_4$, $A_5$ and $A_6$, $L_2(q)$ for $q \in \{7, 8, 11, 13\}$ are uniquely determined by only nse($G$). In [9], the authors gave the following problem:

**Problem:** Is a group $G$ isomorphic to $A_n$ ($n \geq 4$) if and only if nse($G$) = nse($A_n$)?

In this paper, we give a positive answer to this problem and show that the alternating group $A_n$ is characterizable by only nse($G$) for $n \in \{7, 8\}$. In fact the main theorem of our paper is as follows:

**Main Theorem:** Let $G$ be a group such that nse($G$) = nse($A_n$), where $n \in \{7, 8\}$. Then $G \cong A_n$.

We note that there are finite groups which are not characterizable even by nse($G$) and $|G|$. In 1987, Thompson gave an example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of $M_{23}$. Then nse($G_1$) = nse($G_2$) and $|G_1| = |G_2|$, but $G_1 \not\cong G_2$. Throughout this paper, we denote by $\phi$ the Euler totient function. If $G$ is a finite group, then we denote by $P_q$ a Sylow $q$—subgroup of $G$ and $n_q(G)$ is the number of Sylow $q$—subgroup of $G$, that is, $n_q(G) = |\text{Syl}_q(G)|$. All other notations are standard and we refer to [7], for example.

## 2 Main Results

In this section, for the proof of main theorem, we need the following Lemmas:

**Lemma 2.1.** [5] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.

**Lemma 2.2.** [6] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$—subgroup of $G$ and $n = p^s m$, where $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 2.3.** [9] Let $G$ be a group containing more than two elements.
Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup \{m_k | k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(A_n)$, where $n \in \{7, 8\}$. By Lemma 2.3, we can assume that $G$ is finite. Let $m_n$ be the number of elements of order $n$ in $G$. We note that $m_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$ in $G$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \mid |G|$, then by Lemma 2.1 and the above notation we have

$$\begin{align*}
\phi(n) & \mid m_n \\
n & \mid \sum_{d | n} m_d
\end{align*}$$

(\ast)

In the proof of the main theorem, we often apply (\ast) and the above comments.

3 Proof of the Main Theorem

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(A_7) = \{1, 105, 210, 350, 504, 630, 720\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Since 105 \in $\text{nse}(G)$, it follows that by (\ast), 2 \in $\pi(G)$ and $m_2 = 105$. Let 2 \neq p \in $\pi(G)$. By (\ast), $p \mid (1 + m_p)$ and $(p - 1) \mid m_p$, which implies that $p \in \{3, 5, 7, 211, 631\}$. If 211 \in $\pi(G)$, then by (\ast), $m_{211} = 210$. On the other hand, by (\ast) we conclude that if 422 \in $\pi_e(G)$, then $m_{422} = 210 \text{ or } 630$ and 422 \mid $(1 + m_2 + m_{211} + m_{422})$, and hence 422 \mid 526 or 422 \mid 946, which is a contradiction. Thus 422 \notin $\pi_e(G)$. Since 422 \notin $\pi_e(G)$, the group $P_{211}$ acts fixed point freely on the set of elements of order 2, and so $|P_{211}| \mid m_2$, which is a contradiction. Hence 211 \notin $\pi_e(G)$. Similar to the above discussion 631 \notin $\pi_e(G)$.

If 3, 5 and 7 \in $\pi(G)$, then $m_3 = 350$, $m_5 = 504$ and $m_7 = 720$, by (\ast). Also we can see easily that $G$ does not contain any elements of order 35, 81, 64, 125 and 343. Similarly, we can see that if 10, 14, 15, 21 \in $\pi_e(G)$, then $m_{10} = 720$, $m_{14} \in \{210, 504\}$, $m_{15} = 720$ and $m_{21} = 504$.

If $2^a \times 3^b \in $\pi_e(G)$, then $2^a \times 3^{b-1} \mid m_{2^a \times 3^b}$. Hence $1 \leq a \leq 3$ and $1 \leq b \leq 3$.

If $2^c \times 5^d \in $\pi_e(G)$, then $2^{c+1} \times 5^{d-1} \mid m_{2^{c+1} \times 5^d}$. Hence $1 \leq c \leq 3$ and $1 \leq d \leq 2$.

If $2^e \times 7^f \in $\pi_e(G)$, then $2^e \times 3 \times 7^{f-1} \mid m_{2^e \times 7}$. Hence $1 \leq e \leq 3$ and $1 \leq f \leq 2$.

If $3^k \times 5^h \in $\pi_e(G)$, then $2^3 \times 3^{k-1} \times 5^{h-1} \mid m_{3^k \times 5^h}$. Hence $1 \leq k \leq 3$ and $1 \leq h \leq 2$.

If $3^l \times 7^m \in $\pi_e(G)$, then $2^2 \times 3^l \times 7^{m-1} \mid m_{3^l \times 7^m}$. Hence $1 \leq l \leq 2$ and $1 \leq m \leq 2$. 


In follow, we show that \( \pi(G) \) could not be the sets \( \{2\}, \{2, 3\}, \{2, 3, 7\} \) and \( \{2, 3, 5\} \), and \( \pi(G) \) must be equal to \( \{2, 3, 5, 7\} \).

**Case a.** If \( \pi(G) = \{2\} \), then \( \pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5\} \). Since \( nse(G) \) has seven elements, this case impossible.

**Case b.** We know that \( 2 \in \pi(G) \). We claim that \( 3 \in \pi(G) \). Suppose that \( 3 \not\in \pi(G) \). If \( 5, 7 \notin \pi(G) \), then by Case a, we get a contradiction. Hence \( 5 \) or \( 7 \in \pi(G) \).

Let \( 5 \in \pi(G) \). Since \( 125 \not\in \pi_e(G) \), \( \exp(P_5) = 5 \) or \( 25 \). If \( \exp(P_5) = 5 \), then by Lemma 2.1, \( |P_5| \mid |1 + m_5| = 505 \). Hence \( |P_5| = 5 \). Then \( n_5 = m_5/\phi(5) = 504/4 \mid |G| \), a contradiction. If \( \exp(P_5) = 25 \), then \( |P_5| \mid |1 + m_5 + m_{25}| \). Hence \( |P_5| = 25 \) and \( n_5 = m_{25}/20 \mid |G| \). Since \( m_{25} = 720 \), we get a contradiction. Thus \( 5 \not\in \pi(G) \).

Let \( 7 \in \pi(G) \). Since \( 7^3 \not\in \pi_e(G) \), \( \exp(P_7) = 7 \) or \( 49 \). If \( \exp(P_7) = 7 \), then by Lemma 2.1, \( |P_7| \mid |1 + m_7| = 721 \). Hence \( |P_7| = 7 \) and \( n_7 = m_7/\phi(7) \mid |G| \), which is a contradiction.

If \( \exp(P_7) = 49 \), then \( |P_7| \mid |1 + m_7 + m_{49}| \). Hence \( |P_7| = 49 \). Since \( m_{49} \in \{210, 504\} \), \( n_7 = m_{49}/\phi(49) = 5 \) or \( 12 \). By Sylow’s theorem \( n_7 = 7k + 1 \) for some \( k \), since \( n_7 = 5 \) or \( 12 \), we get a contradiction. Thus \( 7 \not\in \pi(G) \).

**Case c.** Let \( \pi(G) = \{2, 3\} \). Since \( 3^4 \not\in \pi_e(G) \), \( \exp(P_3) = 3, 3^2 \) or \( 3^3 \). If \( \exp(P_3) = 3 \), \( |P_3| \mid |1 + m_3| = 351 \), by Lemma 2.1. Thus \( |P_3| \mid 3^3 \). If \( |P_3| = 3 \), then \( n_3 = m_3/2 \mid |G| \), a contradiction. If \( |P_3| > 3 \), then since \( \exp(P_3) = 3 \), \( |\pi_e(G)| \leq 11 \). Therefore \( |G| = 2^m \times 3^n = 2520 + 350k_1 + 504k_2 + 720k_3 + 630k_4 + 210k_5 \), where \( m, n, k_1, k_2, k_3, k_4 \) and \( k_5 \) are non-negative integers and \( 0 \leq k_1 + k_2 + k_3 + k_4 + k_5 \leq 4 \). It is clear that \( |G| \leq 5400 \). If \( n = 2 \), then \( m = 9 \). It easy to check that the above equation has no solution. If \( n = 3 \), then \( m = 7 \), arquing as above, the equation has no solution. Therefore \( \exp(P_3) \neq 3 \).

If \( \exp(P_3) = 9 \), by Lemma 2.1, \( |P_3| \mid |1 + m_3 + m_{9}| \). Since \( m_9 \in \{504, 630, 720\} \), \( |P_3| = 9 \). Hence \( n_3 = m_9/6 \mid |G| \), a contradiction.

If \( \exp(P_3) = 27 \), then since \( m_{27} \in \{504, 630, 720\} \), \( |P_3| \mid 3^5 \). If \( |P_3| = 27 \), then \( n_9 = m_{27}/18 \mid |G| \), a contradiction. If \( |P_3| = 81 \) or \( 243 \), then by Lemma 2.2, \( 3^3 \mid m_{27} \), a contradiction.

**Case d.** Let \( \pi(G) = \{2, 3, 7\} \). Since \( 7^3 \not\in \pi_e(G) \), \( \exp(P_7) = 7 \) or \( 7^2 \). If \( \exp(P_7) = 7 \), then \( |P_7| \mid |1 + m_7| = 721 \). Hence \( |P_7| = 7 \) and \( n_7 = m_7/6 = 120 \mid |G| \), a contradiction.

If \( \exp(P_7) = 49 \), then \( |P_7| \mid |1 + m_7 + m_{49}| \). Thus \( |P_7| = 49 \) and \( n_7 = m_{49}/42 = 5 \) or \( 12 \). By Sylow’s theorem, we get a contradiction.
Case e. Let $\pi(G) = \{2, 3, 5\}$. Since $125 \not\in \pi_e(G)$, $\exp(G) = 5$ or $25$. If $\exp(G) = 5$, then $|P_5| \mid (1 + n_5) = 505$. Hence $|P_5| = 5$ and $n_5 = m_5/4 = 126 \mid |G|$, which is a contradiction.

If $\exp(P_3) = 25$, then $|P_5| \mid (1 + m_5 + m_{25}) = 1225$. Hence $|P_5| = 25$ and then the group $P_5$ is cyclic. Thus $n_5 = m_{25}/20 = 36$. Since a cyclic group of order 25 has 4 elements of order 5, $m_5 \leq 4 \times n_5 = 144$, which is a contradiction.

Case f. Let $\pi(G) = \{2, 3, 5, 7\}$. Since $35 \not\in \pi_e(G)$, the group $P_7$ acts fixed point freely on the set of elements of order 5, and so $|P_7| \mid m_5 = 504$, which implies that $|P_7| = 7$. Similarly, $|P_3| = 5$.

We know that if $P$ and $Q$ are Sylow 7–subgroups of $G$, then $P$ and $Q$ are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in $G$. Therefore $m_{21} = \phi(21) \cdot n_7 \cdot k$, where $k$ is the number of cyclic subgroups of order 3 in $C_G(P_7)$. Since $n_7 = m_7/\phi(7) = 120$, $2 \times 720 \mid m_{21}$, which is a contradiction. Hence $21 \not\in \pi_e(G)$. Similarly, $10 \not\in \pi_e(G)$.

Since $21 \not\in \pi_e(G)$, the group $P_3$ acts fixed point freely on the set of elements of order 7, and $|P_3| \mid m_7$. Thus $|P_3| \mid 9$. Also since $10 \not\in \pi_e(G)$, $|P_2| \mid m_5 = 504$, and so $|P_2| \mid 2^3$.

If $|P_3| = 3$, then $|G| = 2^3 \times 105$ and $m \leq 3$. On the other hand, $2520 \leq |G|$, a contradiction. Therefore $|P_3| = 9$ and $|G| = 2^m \times 315$ where $m \leq 3$. Since $2520 \leq |G|$, $m = 3$ and then $|G| = 2520 = |A_7|$. By [4], since $A_7$ is a simple $K_4$–group, $G \cong A_7$.

Now suppose that $G$ be a group such that $\text{nse}(G) = \text{nse}(A_8) = \{1, 315, 1232, 1344, 2688, 3780, 5040, 5760\}$. First we prove that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Since $315 \in \text{nse}(G)$, it follows that by (*) $2 \in \pi(G)$ and $m_2 = 315$. Let $2 \not\mid p \in \pi(G)$, by (*), $p \mid (1 + m_p)$ and $(p - 1) \mid m_p$, which implies that $p \in \{3, 5, 7, 19, 2689, 5041\}$.

If $19 \in \pi(G)$, then $m_{19} = 3780$. On the other hand, by (*) we conclude that if $38 \in \pi_e(G)$, then $m_{38} \in \{5760, 3780, 5040\}$ and $38 \mid (1 + m_2 + m_{19} + m_{38})$, a contradiction. Therefore $38 \not\in \pi_e(G)$. Thus the group $P_{19}$ acts fixed point freely on the set of elements of order 2, and $|P_{19}| \mid m_2$, which is a contradiction. Hence $19 \not\in \pi(G)$. Similar to the above discussion $2689, 5041 \not\in \pi(G)$, and so $\pi(G) \subseteq \{2, 3, 5\}$.

If $3, 5$ and $7 \in \pi(G)$, then $m_3 = 1232, m_5 = 1344$ and $m_7 = 5760$, by (*). Also we can see easily that $G$ does not contain any elements of order 35, 512, 81, 125, 343 and 768.

If $15, 25, 49 \in \pi_e(G)$, then $m_{15} = 2688, m_{25} = 3780$ and $m_{49} = 1344$.

If $2^a \times 3^b \in \pi_e(G)$, then $1 \leq a \leq 6$ and $1 \leq b \leq 4$. 

If $2^c \times 5^d \in \pi_e(G)$, then $1 \leq c \leq 6$ and $1 \leq d \leq 2$.
If $2^e \times 7^f \in \pi_e(G)$, then $1 \leq e \leq 7$ and $1 \leq f \leq 2$. If $3^h \times 5^h \in \pi_e(G)$, then $1 \leq k \leq 3$ and $1 \leq h \leq 2$.
If $3^l \times 7^m \in \pi_e(G)$, then $1 \leq l \leq 3$ and $1 \leq m \leq 2$.
We show that $\pi(G)$ could not be the sets $\{2\}, \{2, 3\}$ and $\{2, 3, 5\}$ and $\{2, 3, 7\}$, and $\pi(G)$ must be equal to $\{2, 3, 5, 7\}$.

Case a. If $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8\}$ and $|G| = 2^m = 20160 + 1232k_1 + 1344k_2 + 2688k_3 + 3780k_4 + 5040k_5 + 5760k_6$, where $m, k_1, k_2, k_3, k_4, k_5$ and $k_6$ are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 1$. It easy to check that this equation has no solution.

Case b. We claim that $3 \in \pi(G)$. Suppose contrary, i.e., $3 \notin \pi(G)$. If $5$, $7 \notin \pi(G)$, then by Case a, we get a contradiction. Hence $5$ or $7 \in \pi(G)$.
Let $5 \in \pi(G)$. Since $125 \notin \pi_e(G)$, $\exp(P_5) = 5$ or $25$.
If $\exp(P_5) = 5$, then $|P_5| \mid (1 + m_5) = 1345$. Hence $|P_5| = 5$, and so $n_5 = 1344/4 \mid |G|$. Because $3 \notin \pi(G)$, we get a contradiction.
If $\exp(P_5) = 25$, then $|P_5| \mid 125$. Suppose that $|P_5| = 25$, then $n_5 = 3780/20 \mid |G|$, a contradiction. If $|P_5| = 125$, then by Lemma 2.2, $25 \mid m_{25}$, a contradiction. Thus $5 \notin \pi(G)$.
Let $7 \in \pi(G)$. Since $7^3 \notin \pi_e(G)$, $\exp(P_7) = 7$ or $49$.
If $\exp(P_7) = 7$, then $|P_7| = 7$ and $n_7 = m_7/6 = 5760/6 \mid |G|$, a contradiction.
If $\exp(P_7) = 49$, then $|P_7| = 49$. Hence $n_7 = m_{49}/42 = 32$ and by Sylow’s theorem, we get a contradiction.

Case c. Let $\pi(G) = \{2, 3\}$. Since $3^4 \notin \pi_e(G)$, $\exp(P_3) = 3$, $3^2$ or $3^3$. If $\exp(P_3) = 3$, then $|P_3| \mid (1 + m_3) = 1233$. Hence $|P_3| \mid 9$. Thus $|P_3| = 3$ or $9$. First suppose that $|P_3| = 3$. Then $n_3 = m_3/2 = 1232/2 \mid |G|$, a contradiction. Suppose that $|P_3| = 9$. Thus $|G| = 2^m \times 9 = 20160 + 1232k_1 + 1344k_2 + 2688k_3 + 3780k_4 + 5040k_5 + 5760k_6$, where $m, k_1, k_2, k_3, k_4, k_5$ and $k_6$ are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leq 9$. It is clear that $|G| \leq 72000$. Since $20160 \leq |G| \leq 72000$, $m = 12$. Therefore $16704 = 1232k_1 + 1344k_2 + 2688k_3 + 3780k_4 + 5040k_5 + 5760k_6$. By using an easy computer calculation, we can see that this equation has no solution.
Let $\exp(P_3) = 9$. Since $m_9 \in \{3780, 5760\}$ and $|P_3| \mid (1 + m_3 + m_9)$, $|P_3| = 9$. Hence $n_3 = m_9/6 \mid |G|$, which is a contradiction.
Let $\exp(P_3) = 27$. Since $m_{27} \in \{3780, 5760\}$ and $|P_3| \mid (1 + m_3 + m_9 + m_{27})$, $|P_3| \mid 81$. If $|P_3| = 81$, then by Lemma 2.2, $27 \mid m_{27}$, which is a contradiction. If $|P_3| = 27$, then $n_3 = m_{27}/18 \mid |G|$, a contradiction.
Case d. Let $\pi(G) = \{2, 3, 5\}$. Since $5^3 \notin \pi_e(G)$, $\exp(P_5) = 5$ or 25. If $\exp(P_5) = 5$, then $|P_5| | (1 + m_5) = 1345$. Hence $|P_5| = 5$, and $n_5 = 1344/4 | G|$, a contradiction. If $\exp(P_5) = 25$, then $|P_5| | (1 + m_5 + m_{25})$. Thus $|P_5| | 125$ and $|P_5| = 25$ or 125. If $|P_5| = 25$, then $n_5 = 3780/20 | G|$, a contradiction. If $|P_5| = 125$, then $25 | m_{25}$, a contradiction.

Case e. Let $\pi(G) = \{2, 3, 7\}$. Since $7^3 \notin \pi_e(G)$, $\exp(P_7) = 7$ or 49. If $\exp(P_7) = 7$, then $|P_7| | (1 + m_7)$. Thus $|P_7| = 7$ and $n_7 = 960$. Since $n_7 | G|$ and $5 \notin \pi(G)$, we get a contradiction.

If $\exp(P_7) = 49$, then $|P_7| | (1 + m_7 + m_{49})$. Thus $|P_7| = 49$ and $n_7 = 32$. By Sylow’s theorem, we get a contradiction.

Case f. Let $\pi(G) = \{2, 3, 5, 7\}$. Since $35 \notin \pi_e(G)$, the group $P_7$ acts fixed point freely on the set of elements of order 5, and so $|P_5| | m_5 = 1344$, which implies that $|P_5| = 7$. We have $m_{21} = \phi(21) \cdot n_7 \cdot k$, where $k$ is the number of cyclic subgroups of order 3 in $C_G(P_2)$. Since $n_7 = m_7/\phi(7) = 960$, $2 \times 5760 | m_{21}$, a contradiction. Hence $21 \notin \pi_e(G)$. Similarly, $10 \notin \pi_e(G)$.

Since $21 \notin \pi_e(G)$, the group $P_3$ acts fixed point freely on the set of elements of order 7. Then $|P_3| | m_7 = 5760$. Thus $|P_3| | 9$. Also since $10 \notin \pi_e(G)$, $|P_3| | m_5 = 1344$, and so $|P_3| | 2^6$. If $|P_3| = 3$, then $|G| = 2^m \times 105$. On the other hand, since $|P_2| | 2^6$, $m \leq 6$. Since $|G| \leq 20160$, $2^m \times 105 \leq 20160$, a contradiction. Therefore $|P_3| = 9$ and then $|G| = |A_8|$. By [4], since $A_8$ is a simple $K_4$–group, $G \cong A_8$, and the proof is complete.

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