Some Combinatorial Properties of the $k$-Fibonacci and the $k$-Lucas Quaternions

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Abstract
In this paper, we define the $k$-Fibonacci and the $k$-Lucas quaternions. We investigate the generating functions and Binet formulas for these quaternions. In addition, we derive some sums formulas and identities such as Cassini’s identity.

1 Introduction
The Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (see, e.g., [20]). The Fibonacci numbers $F_n$ are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$ 

The first few terms are 0, 1, 1, 2, 3, 5, 8, 13, ... (sequence A000045). Another important sequence is the Lucas sequence. This sequence is defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1}, \quad n \geq 1.$$ 

The first few terms are 2, 1, 3, 7, 11, 18, 29, 37, ... (sequence A000032).

Many kinds of generalizations of the Fibonacci sequence have been presented...
In the literature (see, e.g., [20, 21]). In particular, there exist a generalization called the \( k \)-Fibonacci and the \( k \)-Lucas numbers. For any positive real number \( k \), the \( k \)-Fibonacci sequence, say \( \{F_{k,n}\}_{n \in \mathbb{N}} \), is defined by

\[
F_{k,0} = 0, \quad F_{k,1} = 1, \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1,
\]

and the \( k \)-Lucas sequence, say \( \{L_{k,n}\}_{n \in \mathbb{N}} \), is defined by

\[
L_{k,0} = 2, \quad L_{k,1} = k, \quad \text{and} \quad L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad n \geq 1.
\]

These sequences were studied by Horadam in [12]. Recently, Falcón and Plaza [6] found the \( k \)-Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. The interested reader is also referred to [2, 3, 4, 5, 6, 7, 22, 23, 24, 25], for further information about these sequences.

On the other hand, Horadam [13] introduced the \( n \)-th Fibonacci and the \( n \)-th Lucas quaternion as follow:

\[
Q_n = F_n + iF_{n+1} + jF_{n+2} + \kappa F_{n+3}, \quad (2)
\]
\[
K_n = L_n + iL_{n+1} + jL_{n+2} + \kappa L_{n+3}, \quad (3)
\]

respectively. Here the basis \( i, j, \kappa \) satisfy the following rules:

\[
i^2 = j^2 = \kappa^2 = ij\kappa = -1. \quad (4)
\]

Note that the rules (4) imply

\[
i j = \kappa = -ji, \quad j\kappa = i = -\kappa j, \quad \kappa i = j = -i\kappa.
\]

In general, a quaternion is a hyper-complex number and is defined by the following equation:

\[
q = q_0 + iq_1 + jq_2 + \kappa q_3,
\]

where \( i, j, \kappa \) are as in (4). Note that we can write \( q = q_0 + u \) where \( u = iq_1 + jq_2 + \kappa q_3 \). The conjugate of the quaternion \( q \) is denoted by \( q^* \) and \( q^* = q_0 - u \).

The Fibonacci and Lucas quaternions have been studied in several papers. For example, Swamy [26] gave some relations for the \( n \)-th Fibonacci quaternion. Horadam [14] studied some recurrence relations associated with the Fibonacci quaternions. Iyer [18, 19] derived relations connecting the Fibonacci and Lucas quaternions. Iakin [15, 16, 17] introduced the higher order

In analogy with (2) and (3), we introduce the $k$-Fibonacci and $k$-Lucas quaternions. We give some properties, the generating functions and Binet formulas for $k$-Fibonacci and $k$-Lucas quaternions. Moreover, we obtain some sums formulas for these quaternions and some identities such as Cassini’s identity to $k$-Fibonacci quaternions.

2 Some properties of the $k$-Fibonacci and $k$-Lucas Numbers

The characteristic equation associated with the recurrence relation (1) is $z^2 - kz - 1 = 0$. The roots of this equation are

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2}, \quad \beta = \frac{k - \sqrt{k^2 + 4}}{2}.$$ 

Then we have the following basic identities:

$$\alpha + \beta = k, \quad \alpha - \beta = \sqrt{k^2 + 4}, \quad \alpha\beta = -1.$$ 

Some of the properties that the $k$-Fibonacci numbers verify are summarized bellow (see [6, 7] for the proofs).

Binet formula:

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 0.$$ 

(5)

$$F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}, \quad n \geq 0.$$ 

(6)

Generating function:

$$f_k(z) = \frac{z}{1 - kz - z^2}.$$ 

(7)

$$\alpha^n = \alpha F_{k,n} + F_{k,n-1}.$$ 

(8)

Some properties that the $k$-Lucas numbers verify are summarized bellow
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(see [3] for the proofs).

**Binet formula:**

$$L_{k,n} = \alpha^n + \beta^n, \ n \geq 0.$$  
$$L_{k,n} = F_{k,n-1} + F_{k,n+1}, \ n \geq 1.$$  
$$L_{k,n}^2 + L_{k,n+1}^2 = (k^2 + 4)F_{k,2n+1}.$$  

**Generating function:**

$$l_k(z) = \frac{2 - kz}{1 - kz - z^2}.$$  

3 Some properties of the $k$-Fibonacci and $k$-Lucas Quaternions

**Definition 1.** The $k$-Fibonacci quaternion $D_{k,n}$ is defined by

$$D_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3}, \ n \geq 0,$$

where $F_{k,n}$ is the $n$-th $k$-Fibonacci number.  

The $k$-Lucas quaternion $P_{k,n}$ is defined by

$$P_{k,n} = L_{k,n} + iL_{k,n+1} + jL_{k,n+2} + \kappa L_{k,n+3}, \ n \geq 0,$$

where $L_{k,n}$ is the $n$-th $k$-Lucas number.

**Proposition 2.** The following identities hold:

(i) $D_{k,n}D_{k,n}^* = (k^2 + 2)F_{k,2n+3}.$  
(ii) $P_{k,n}P_{k,n}^* = (k^2 + 2)(k^2 + 4)F_{k,2n+3}.$  
(iii) $D_{k,n}^2 = 2F_{k,n}D_{k,n} - D_{k,n}D_{k,n}^*.$  
(iv) $P_{k,n}^2 = 2L_{k,n}P_{k,n} - P_{k,n}P_{k,n}^*.$  
(v) $D_{k,n} + D_{k,n}^* = 2F_{k,n}.$  
(vi) $P_{k,n} + P_{k,n}^* = 2L_{k,n}.$  
(vii) $D_{k,n+2} = kD_{k,n+1} + D_{k,n}.$  
(viii) $P_{k,n+2} = kP_{k,n+1} + P_{k,n}.$
Proof. (i) From Equations (6) and (1)
\[
D_{k,n}D_{k,n}^* = F_{k,n}^2 + F_{k,n+1}^2 + F_{k,n+2}^2 + F_{k,n+3}^2
\]
\[
= F_{k,2n+1} + F_{k,2n+5}
\]
\[
= F_{k,2n+1} + k(F_{k,2n+3} + F_{k,2n+2}) + F_{k,2n+3}
\]
\[
= (k^2 + 1)F_{k,2n+3} + kF_{k,2n+2} + F_{k,2n+1}
\]
\[
= (k^2 + 1)F_{k,2n+3} + F_{k,2n+3}
\]
\[
= (k^2 + 2)F_{k,2n+3}.
\]

(ii) The proof is similar to (i).

(iii) From Proposition 2(i)
\[
D_{k,n}^2 = (F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3})^2
\]
\[
= F_{k,n}^2 - F_{k,n+1}^2 - F_{k,n+2}^2 - F_{k,n+3}^2 + i(2F_{k,n}F_{k,n+1})
\]
\[
+ j(2F_{k,n}F_{k,n+2}) + \kappa(2F_{k,n}F_{k,n+3})
\]
\[
= 2F_{k,n}(F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3})
\]
\[
- F_{k,n}^2 - F_{k,n+1}^2 - F_{k,n+2}^2 - F_{k,n+3}^2
\]
\[
= 2F_{k,n}D_{k,n} - D_{k,n}D_{k,n}^*.
\]

(iv) The proof is similar to (iii).

The other identities are clear from definition.

4 Main Results

**Theorem 3** (Binet’s Formula). For \( n \geq 0 \), the Binet formulas for the \( k \)-Fibonacci and \( k \)-Lucas quaternions are as follow:
\[
D_{k,n} = \frac{1}{\sqrt{k^2 + 4}}(\delta\alpha^n - \beta\beta^n) = \frac{\delta\alpha^n - \beta\beta^n}{\alpha - \beta}, \quad (9)
\]
and
\[
P_{k,n} = \delta\alpha^n - \beta\beta^n, \quad (10)
\]
respectively, where \( \delta = 1 + i\alpha + j\alpha^2 + \kappa\alpha^3 \) and \( \beta = 1 + i\beta + j\beta^2 + \kappa\beta^3 \).
Proof. The characteristic equation of recurrence relation in Proposition 2(vii) is $z^2 - k z - 1 = 0$. Moreover, the initial values are $D_{k,0} = (0, 1, k, k^2 + 1)$ and $D_{k,1} = (1, k, k^2 + 1, k^3 + 2k)$. Hence,

$D_{k,n} = A_\alpha^n + B_\beta^n$.

Then, $D_{k,0} = A + B$ and $D_{k,1} = A\alpha + B\beta$, and from Equation (8) we obtain that

$$A = \frac{1}{\alpha - \beta} (D_{k,1} - \beta D_{k,0}) = \frac{1}{\sqrt{k^2 - 4}} (1 + i\alpha + j\alpha^2 + \kappa\alpha^3).$$

Analogously, $B = \frac{1}{\sqrt{k^2 - 4}} (1 + i\beta + j\beta^2 + \kappa\beta^3)$. Therefore,

$$D_{k,n} = \frac{1}{\sqrt{k^2 + 4}} (\hat{\alpha}_\alpha^n - \hat{\beta}_\beta^n) = \frac{\hat{\alpha}_\alpha^n - \hat{\beta}_\beta^n}{\alpha - \beta}.$$

Similarly, we can get Equation (10).

Note that if $k = 1$ (see Equations (3.1) and (3.2) in [11]), then

$$D_{1,n} = \frac{1}{\sqrt{5}} (\hat{\alpha}_\alpha^n - \hat{\beta}_\beta^n),$$

and

$$P_{1,n} = \hat{\alpha}_\alpha^n - \hat{\beta}_\beta^n.$$

**Theorem 4** (Cassini’s identity). For $n \geq 1$, we have the following formula:

$$D_{k,n} - D_{k,n+1} - D_{k,n}^2 = (-1)^n (2D_{k,1} - (k^2 + 2k)\kappa).$$

Proof. We proceed by induction on $n$. If $n = 1$,

$$D_{k,0}D_{k,2} - D_{k,1}^2 = (F_{k,0} + F_{k,1}i + F_{k,2}j + F_{k,3}k)(F_{k,2} + F_{k,3}i + F_{k,4}j + F_{k,5}k)$$

$$- (F_{k,1} + F_{k,2}i + F_{k,3}j + F_{k,4}k)(F_{k,1} + F_{k,2}i + F_{k,3}j + F_{k,4}k)$$

$$= -(2 + 2(k + 1)i + (k^2 + 1)j + (k^3 + 2k)k)$$

$$= (-1)^1(2D_{k,1} - (k^2 + 2k)\kappa).$$

It is not difficult to show that the proposition is true for $n + 1$.

Note that if $k = 1$ (see Equation (3.9) in [11]), then

$$D_{1,n} - D_{1,n+1} - D_{1,n}^2 = (-1)^n (2D_{1,1} - 3\kappa).$$

From a numerical test in Mathematica we obtained the following conjecture:
Conjecture 5. For \( n \geq r \geq 1 \), we conjecture the following formula:

\[
D_{k,n-r}D_{k,n+r} - D^2_{k,n} = (-1)^{n-r}(2F_{k,r}D_{k,r} - G_{k,r} \kappa),
\]

where \( G_{k,r} \) is a sequence defined by

\[
G_{k,0} = 0, G_{k,1} = k^2 + 2k, \text{ and } G_{k,n} = (k^2 + 2)G_{k,n-1} - G_{k,n-2}, \quad n \geq 2.
\]

Example 6. If \( n = 10 \) and \( r = 3 \) in (11), then

\[
D_{k,n-r}D_{k,n+r} - D^2_{k,n} = (2+4k^2+2k^4) + (4k+6k^3+2k^5)i + (2+8k^2+8k^4+2k^6)j + (3k^3+4k^5+k^7)\kappa
\]

\[
2F_{k,r}D_{k,r} = (2+4k^2+2k^4) + (4k+6k^3+2k^5)i + (2+8k^2+8k^4+2k^6)j + (6k+14k^3+10k^5+2k^7)\kappa.
\]

Then,

\[
D_{k,7}D_{k,13} - D^2_{k,10} = 2F_{k,3}D_{k,3} - G_{k,3}\kappa.
\]

Note that, if this conjecture is true, then Cassini’s identity is a particular case, \( r = 1 \).

Theorem 7. For the \( k \)-Fibonacci quaternions \( D_{k,n} \), we have

\[
\sum_{i=0}^{n} D_{k,mi+j} = \begin{cases} (-1)^m D_{k,n^m+j} - D_{k,n^m+m+j} + (-1)^{m+1} D_{k,n^m+1} & \text{if } j < m; \\ (-1)^m D_{k,n^m+j} - D_{k,n^m+m+j} - (-1)^{m+1} D_{k,n^m+1} & \text{otherwise.} \end{cases}
\]

Proof.

\[
\sum_{i=0}^{n} D_{k,mi+j} = \sum_{i=0}^{n} \frac{\hat{\alpha}^{mi+j} - \hat{\beta}^{mi+j}}{\sqrt{k^2+4}^{i}} = \frac{1}{\sqrt{k^2+4}} \left( \hat{\alpha} \sum_{i=0}^{n} \alpha^{mi} - \hat{\beta} \sum_{i=0}^{n} \beta^{mi} \right)
\]

\[
= \frac{1}{\sqrt{k^2+4}} \left( \frac{\hat{\alpha}^{nm+m+j} - \hat{\beta}^{nm+m+j}}{\alpha^m - 1} - \frac{\hat{\beta}^{nm+m+1} - \beta^m}{\beta^m - 1} \right)
\]

\[
= \frac{1}{\sqrt{k^2+4}} \left( (\alpha^m - \hat{\beta}^{nm+m+1}) + (\hat{\alpha}^{nm+m+j} - \hat{\beta}^{nm+m+1}) \right)
\]

\[
= \frac{1}{\sqrt{k^2+4}} \left( (\hat{\alpha}^{nm+m+j} - \hat{\beta}^{nm+m+1} + \hat{\alpha}^{nm+m+j} - \hat{\beta}^{nm+m+1}) - (\hat{\alpha}^{nm+m+j} - \hat{\beta}^{nm+m+1}) \right)
\]

\[
= \frac{(-1)^m D_{k,n^m+j} - D_{k,n^m+m+j} - \frac{\hat{\alpha}^{jm} - \hat{\beta}^{jm}}{\sqrt{k^2+4}} + D_{k,j}}{(-1)^m - L_{k,m} + 1}.
\]
But

\[ \hat{\alpha}^j \beta^m - \hat{\beta}^j \alpha^m = \begin{cases} (-1)^{j+1} \sqrt{k^2 + 4D_{k,m-j}}, & \text{if } j < m; \\ (-1)^m \sqrt{k^2 + 4D_{k,j-m}}, & \text{otherwise}. \end{cases} \]

Therefore, Equation (12) is clear. \( \square \)

From Theorem 7 we obtain the following corollary.

**Corollary 8.** For the \( k \)-Fibonacci quaternions \( D_{k,n} \), we have

\[ \sum_{i=0}^{n} D_{k,mi} = \frac{(-1)^m D_{k, nm} - D_{k, nm+m} + D_{k,m} + D_{k,0}}{(-1)^m - L_{k,m} + 1}, \]

\[ \sum_{i=0}^{n} D_{k,i} = \frac{1}{k} (D_{k,n} + D_{k,n+1} - D_{k,1} - D_{k,0}). \]

**Theorem 9.** For \( n \geq 0 \), we have the following summation formulas:

\[ \sum_{i=0}^{n} \binom{n}{i} D_{k,i} k^i = D_{k,2n}, \]

\[ \sum_{i=0}^{n} \binom{n}{i} P_{k,i} k^i = P_{k,2n}. \]

**Proof.**

\[ \sum_{i=0}^{n} \binom{n}{i} D_{k,i} k^i = \frac{\hat{\alpha}^2 - \hat{\beta}^2}{\alpha - \beta} = D_{k,2n}. \]

The proof of the second sum is analogously. \( \square \)
Theorem 10. The generating function for the $k$-Fibonacci and $k$-Lucas quaternions are

\[ G_k(z) = \frac{z + i + j(k + z) + \kappa(k^2 + 1 + k z)}{1 - k z - z^2}, \quad (13) \]

and

\[ J_k(z) = \frac{2 - k z + i(k + 2 z) + j(k^2 + 2 + k z) + \kappa(k^3 + 3k + (k^2 + 2)z)}{1 - k z - z^2}, \quad (14) \]

respectively.

Proof. We begin with the formal power series representation of the generating function for \{\(D_{k,n}\)\}_{n=0}^\infty,

\[ G_k(z) = D_{k,0} + D_{k,1}z + D_{k,2}z^2 + \cdots + D_{k,l}z^k + \cdots. \]

Then

\[ k z G_k(z) = k D_{k,0} z + k D_{k,1} z^2 + k D_{k,2} z^3 + \cdots + k D_{k,l} z^{k+1} + \cdots \]

\[ z^2 G_k(z) = D_{k,0} z^2 + D_{k,1} z^3 + D_{k,2} z^4 + \cdots + D_{k,l} z^{k+2} + \cdots. \]

Therefore

\[ (1 - k z - z^2) G_k(z) = D_{k,0} + (D_{k,1} - k D_{k,0}) z. \]

So

\[ G_k(z) = \frac{D_{k,0} + (D_{k,1} - k D_{k,0}) z}{1 - k z - z^2}. \]

The proof of Equation (14) runs like this. \(\square\)

Theorem 11. For \(m,n \in \mathbb{Z}\) the generating function of the $k$-Fibonacci quaternion \(D_{k,m+n}\) and $k$-Lucas quaternion \(P_{k,m+n}\) are

\[ \sum_{n=0}^\infty D_{k,n+m}z^n = \frac{D_{k,m} + D_{k,m-1} z}{1 - k z - z^2}, \]

and

\[ \sum_{n=0}^\infty P_{k,n+m}z^n = \frac{P_{k,m} + P_{k,m-1} z}{1 - k z - z^2}. \]
Proof.

\[
\sum_{n=0}^{\infty} D_{k,n+m} z^n = \sum_{n=0}^{\infty} \left( \frac{\hat{\alpha}^{n+m} - \hat{\beta}^{n+m}}{\alpha - \beta} \right) z^n \\
= \frac{1}{\alpha - \beta} \left( \hat{\alpha}^m \sum_{n=0}^{\infty} \alpha^n z^n - \hat{\beta}^m \sum_{n=0}^{\infty} \beta^n z^n \right) \\
= \frac{1}{\sqrt{k^2 - 4}} \left( \frac{\hat{\alpha}^m}{1 - \alpha z} - \frac{\hat{\beta}^m}{1 - \beta z} \right) \\
= \frac{1}{\sqrt{k^2 - 4}} \left( \frac{(\hat{\alpha}^m - \hat{\beta}^m)}{1 - k z - z^2} \right) \\
= \frac{D_{k,m} + D_{k,m-1} z}{1 - k z - z^2}.
\]

\[ \square \]

5 Conclusion

In this paper, we study a generalization of the Fibonacci and Lucas quaternions. Particularly, we define the $k$-Fibonacci and $k$-Lucas quaternions, and we find some combinatorial identities.

The $k$-Fibonacci sequence is a special case of a sequence called $s$-bonacci sequence which is defined recursively as a linear combination of the preceding $s$ terms:

\[
a_{n+s} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{s-1} a_{n+s-1},
\]

where $c_0, c_1, \ldots, c_{s-1}$ are real constants. It would be interesting to introduce a $s$-bonacci quaternions.

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References


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