On the binomial edge ideals of block graphs

Faryal Chaudhry, Ahmet Dokuyucu, Rida Irfan

Abstract

We find a class of block graphs whose binomial edge ideals have minimal regularity. As a consequence, we characterize the trees whose binomial edge ideals have minimal regularity. Also, we show that the binomial edge ideal of a block graph has the same depth as its initial ideal.

1 Introduction

In this paper we study homological properties of some classes of binomial edge ideals.

Let $G$ be a simple graph on the vertex set $[n]$ and let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables over a field $K$. For $1 \leq i < j \leq n$, we set $f_{ij} = x_i y_j - x_j y_i$. The binomial edge ideal of $G$ is defined as $J_G = (f_{ij} : \{i,j\} \in E(G))$. Binomial edge ideals were introduced in [8] and [12]. Algebraic and homological properties of binomial edge ideals have been studied in several papers. In [5], it was conjectured that $J_G$ and $\text{in}_<(J_G)$ have the same extremal Betti numbers. Here $<$ denotes the lexicographic order in $S$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. This conjecture was proved in [3] for cycles and complete bipartite graphs. In [6], it was shown that, for a closed graph $G$, $J_G$ and $\text{in}_<(J_G)$ have the same regularity which can be expressed in the combinatorial data of the graph. We recall that a graph $G$ is closed if and only if it has a quadratic Gröbner basis with respect to the lexicographic order.

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In support of the conjecture given in [5], we show, in Section 3, that if \( G \) is a block graph, then \( \text{depth}(S/J_G) = \text{depth}(S/\text{in}_<(J_G)) \); see Theorem 3.2. By a block graph we mean a chordal graph \( G \) with the property that any two maximal cliques of \( G \) intersect in at most one vertex.

Also, in the same section, we show a similar equality for regularity. More precisely, in Theorem 3.4 we show that \( \text{reg}(S/J_G) = \text{reg}(S/\text{in}_<(J_G)) = \ell \) if \( G \) a \( C_\ell \)-graph. \( C_\ell \)-graphs constitute a subclass of the block graphs; see Section 3 for definition and Figure 1 for an example.

In [10] it was shown that, for any connected graph \( G \) on the vertex set \([n]\), we have
\[
\ell \leq \text{reg}(S/J_G) \leq n - 1,
\]
where \( \ell \) is the length of the longest induced path of \( G \).

The main motivation of our work was to answer the following question. May we characterize the connected graphs \( G \) whose longest induced path has length \( \ell \) and \( \text{reg}(S/J_G) = \ell \)? We succeeded to answer this question for trees. In Theorem 4.1, we show that if \( T \) is a tree whose longest induced path has length \( \ell \), then \( \text{reg}(S/J_T) = \ell \) if and only if \( T \) is caterpillar. A caterpillar tree is a tree \( T \) with the property that it contains a path \( P \) such that any vertex of \( T \) is either a vertex of \( P \) or it is adjacent to a vertex of \( P \).

In [11], the so-called weakly closed graphs were introduced. This is a class of graphs which includes closed graphs. In the same paper, it was shown that a tree is caterpillar if and only if it is a weakly closed graph. Having in mind our Theorem 4.1 and Theorem 3.2 in [6] which states that \( \text{reg}(S/J_G) = \ell \) if \( G \) is a connected closed graph whose longest induced path has length \( \ell \), and by some computer experiments, we are tempted to formulate the following.

**Conjecture 1.1.** If \( G \) is a connected weakly closed graph whose longest induced path has length \( \ell \), then \( \text{reg}(S/J_G) = \ell \).

## 2 Preliminaries

In this section we introduce the notation used in this paper and summarize a few results on binomial edge ideals.

Let \( G \) be a simple graph on the vertex set \([n] = \{1, \ldots, n\}\), that is, \( G \) has no loops and no multiple edges. Furthermore, let \( K \) be a field and \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) be the polynomial ring in \( 2n \) variables. For \( 1 \leq i < j \leq n \), we set \( f_{ij} = x_i y_j - x_j y_i \). The **binomial edge ideal** \( J_G \subset S \) associated with \( G \) is generated by all the quadratic binomials \( f_{ij} = x_i y_j - x_j y_i \) such that \( \{i, j\} \in E(G) \). Binomial edge ideals were introduced in the papers [8] and [12].

We first recall some basic definitions from graph theory. A vertex \( i \) of \( G \) whose deletion from the graph gives a graph with more connected components
than G is called a cut point of G. A chordal graph is a graph without cycles of length greater than or equal to 4. A clique of a graph G is a complete subgraph of G. The cliques of a graph G form a simplicial complex, Δ(G), which is called the clique complex of G. Its facets are the maximal cliques of G. A graph G is a block graph if and only if it is chordal and every two maximal cliques have at most one vertex in common. This class was considered in [5, Theorem 1.1].

The clique complex Δ(G) of a chordal graph G has the property that there exists a leaf order on its facets. This means that the facets of Δ(G) may be ordered as F_1, ..., F_r such that, for every i > 1, F_i is a leaf of the simplicial complex generated by F_1, ..., F_i. A leaf F of a simplicial complex Δ is a facet of Δ with the property that there exists another facet of Δ, say G, such that, for every facet H ≠ F of Δ, H ∩ F ⊆ G ∩ F.

Let < be the lexicographic order on S induced by the natural order of the variables. As it was shown in [8, Theorem 2.1], the Gröbner basis of J_G with respect to this order may be given in terms of the admissible paths of G. We recall the definition of admissible paths from [8].

Definition 2.1. [8] Let i < j be two vertices of G. A path i = i_0, i_1, ..., i_r = j from i to j is called admissible if the following conditions are fulfilled:

1. i_k ≠ i_l for k ≠ l;
2. for each k = 1, ..., r − 1 on has either i_k < i or i_k > j;
3. for any proper subset \{j_1, ..., j_s\} of \{i_1, ..., i_r−1\}, the sequence i, j_1, ..., j_s, j is not a path in G.

Given an admissible path π in G from i to j, we set u_π = (\prod_{i_k > j} x_{i_k})(\prod_{i_k < i} y_{i_k}).

By [8, Theorem 2.1], it follows that

\[ \text{in}_<(J_G) = \langle u_\pi x_i y_j : i < j, \pi \text{ is an admissible path from } i \text{ to } j \rangle. \]

In particular, \text{in}_<(J_G) is a radical monomial ideal which implies that the binomial edge ideal J_G is radical as well. Hence J_G is equal to the intersection of all its minimal prime ideals. The minimal prime ideals were determined in [8, Section 3] in terms of the combinatorial data of the graph.

3 Initial ideals of binomial edge ideals of block graphs

In this section, we first show that, for a block graph G on [n] with c connected components, we have depth(S/J_G) = depth(S/\text{in}_<(J_G)) = n + c, where <
denotes the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ in the ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

We begin with the following lemma.

**Lemma 3.1.** Let $G$ be a graph on the vertex set $[n]$ and let $i \in [n]$. Then

$$\text{in}_{<}(J_G, x_i, y_i) = (\text{in}_{<}(J_G), x_i, y_i).$$

**Proof.** We have $\text{in}_{<}(J_G, x_i, y_i) = \text{in}_{<}(J_{G \setminus \{i\}}, x_i, y_i) = (\text{in}_{<}(J_{G \setminus \{i\}}), x_i, y_i).$ Therefore, we have to show that $(\text{in}_{<}(J_G), x_i, y_i) = (\text{in}_{<}(J_{G \setminus \{i\}}), x_i, y_i).$ The inclusion $\supseteq$ is obvious since $J_{G \setminus \{i\}} \subseteq J_G$. For the other inclusion, let us take $u$ to be a minimal generator of $\text{in}_{<}(J_G)$. If $x_i \nmid u$ or $y_i \nmid u$, obviously $u \in (\text{in}_{<}(J_{G \setminus \{i\}}), x_i, y_i)$. Let now $x_i \nmid u$ and $y_i \nmid u$. This means that $u = u_x x_i y_i$ for some admissible path $\pi$ from $k$ to $l$ which does not contain the vertex $i$. Then it follows that $\pi$ is a path from $k$ to $l$ in $G \setminus \{i\}$, hence $u \in \text{in}_{<}(J_{G \setminus \{i\}})$. □

**Theorem 3.2.** Let $G$ be a block graph. Then

$$\text{depth}(S/J_G) = \text{depth}(S/\text{in}_{<}(J_G)) = n + c,$$

where $c$ is the number of connected components of $G$.

**Proof.** Let $G_1, \ldots, G_c$ be the connected components of $G$ and $S_i = K[x_j, y_j]_{j \in G_i}$. Then $S/J_G \cong S_1/J_{G_1} \otimes \cdots \otimes S_c/J_{G_c}$, so that

$$\text{depth} S/J_G = \text{depth} S_1/J_{G_1} + \cdots + \text{depth} S_c/J_{G_c}.$$ Moreover, we have $S/\text{in}_{<}(J_G) \cong S/\text{in}_{<}(J_{G_1}) \otimes \cdots \otimes S/\text{in}_{<}(J_{G_c})$, thus

$$\text{depth} S/\text{in}_{<}(J_G) = \text{depth} S_1/\text{in}_{<}(J_{G_1}) + \cdots + \text{depth} S_c/\text{in}_{<}(J_{G_c}).$$

Hence, without loss of generality, we may assume that $G$ is connected. By [5, Theorem 1.1] we know that $\text{depth}(S/J_G) = n + 1$. In order to show that $\text{depth}(S/\text{in}_{<}(J_G)) = n + 1$, we proceed by induction on the number of maximal cliques of $G$. Let $\Delta(G)$ be the clique complex of $G$ and let $F_1, \ldots, F_r$ be a leaf order on the facets of $\Delta(G)$. If $r = 1$, then $G$ is a simplex and the statement is well known. Let $r > 1$; since $F_r$ is a leaf, there exists a unique vertex, say $i \in F_r$, such that $F_r \cap F_j = \{i\}$ where $F_j$ is a branch of $F_r$. Let $F_{t_1}, \ldots, F_{t_q}$ be the facets of $\Delta(G)$ which intersect the leaf $F_r$ in the vertex $\{i\}$. Following the proof of [5, Theorem 1.1] we may write $J_G = J_1 \cap J_2$ where $J_1 = \bigcap_{l \notin S} P_l(G)$ and $J_2 = \bigcap_{l \in S} P_l(G)$. Then, as it was shown in the proof of [5, Theorem 1.1], it follows that $J_1 = J_{G'}$ where $G'$ is obtained from $G$ by replacing the cliques $F_{t_1}, \ldots, F_{t_q}$ and $F_r$ by the clique on the vertex set $F_r \cup \bigcup_{l \in S} F_{t_l}$. Also, $J_2 = (x_i, y_i) + J_{G''}$ where $G''$ is the restriction of $G$ to the vertex set $[n] \setminus \{i\}$.
We have \( \in_{<}(J_G) = \in_{<}(J_1 \cap J_2) \). By [1, Lemma 1.3], we have \( \in_{<}(J_1 \cap J_2) = \in_{<}(J_1) \cap \in_{<}(J_2) \) if and only if \( \in_{<}(J_1 + J_2) = \in_{<}(J_1) + \in_{<}(J_2) \). But \( \in_{<}(J_1 + J_2) = \in_{<}(J_G) + (x_i, y_i) \), hence, by Lemma 3.1, we get \( \in_{<}(J_1 + J_2) = \in_{<}(J_G') + (x_i, y_i) = \in_{<}(J_1) + \in_{<}(J_2) \). Therefore, we get \( \in_{<}(J_G) = \in_{<}(J_1) \cap \in_{<}(J_2) \) and, consequently, we have the following exact sequence of \( S \)-modules

\[
0 \to \frac{S}{\in_{<}(J_G)} \to \frac{S}{\in_{<}(J_1)} \oplus \frac{S}{\in_{<}(J_2)} \to \frac{S}{\in_{<}(J_1 + J_2)} \to 0.
\]

By using again Lemma 3.1, we have \( \in_{<}(J_G) = \in_{<}(J_G') + (x_i, y_i) + \in_{<}(J_G') \). Thus, we have actually the following exact sequence

\[
0 \to \frac{S}{\in_{<}(J_G)} \to \frac{S}{\in_{<}(J_G')} \oplus (x_i, y_i) + \in_{<}(J_G') \to \frac{S}{(x_i, y_i) + \in_{<}(J_G')} \to 0.
\]

(1)

Since \( G' \) inherits the properties of \( G \) and has a smaller number of maximal cliques than \( G \), it follows, by the inductive hypothesis, that

\[
dePTH(S/J_G') = \dePTH(S/\in_{<}(J_G')) = n + 1.
\]

Let \( S_i \) be the polynomial ring \( S/(x_i, y_i) \). Then \( S/((x_i, y_i) + \in_{<}(J_G')) \cong S_i/\in_{<}(J_G') \). Since \( G'' \) is a graph on \( n - 1 \) vertices with \( q + 1 \) connected components and satisfies our conditions, the inductive hypothesis implies that \( \dePTH(S/((x_i, y_i) + \in_{<}(J_G'))) = n + q \geq n + 1 \). Hence,

\[
dePTH(S/\in_{<}(J_G') + S/((x_i, y_i) + \in_{<}(J_G')) = n + 1.
\]

Next, we observe that \( S/((x_i, y_i) + \in_{<}(J_G')) \cong S_i/\in_{<}(J_H) \), where \( H \) is obtained from \( G' \) by replacing the clique on the vertex set \( F' \cup (\bigcup_{i=1}^p F_i) \) by the clique on the vertex set \( F' \cup (\bigcup_{i=1}^p F_i) \setminus \{i\} \). Hence, by the inductive hypothesis, \( \dePTH(S/((x_i, y_i) + \in_{<}(J_G')) = n \) since \( H \) is connected and its vertex set has cardinality \( n - 1 \). Hence, by applying the Depth lemma to exact sequence (1), we get

\[
dePTH S/J_G = \dePTH S/\in_{<}(J_G) = n + 1.
\]

\[\square\]

**Definition 3.3.** Let \( \ell \geq 2 \) be an integer. A \( C_{\ell} \)-graph is a connected graph \( G \) on the vertex set \([n]\) which consists of

(i) a sequence of maximal cliques \( F_1, \ldots, F_{\ell} \) with \( \dim F_i \geq 1 \) for all \( i \) such that \( |F_i \cap F_{i+1}| = 1 \) for \( 1 \leq i \leq \ell - 1 \) and \( F_i \cap F_j = \emptyset \) for any \( i < j \) such that \( j \neq i + 1 \), together with
some additional edges of the form $F = \{j, k\}$ where $j$ is an intersection point of two consecutive cliques $F_i, F_{i+1}$ for some $1 \leq i \leq \ell - 1$, and $k$ is a vertex of degree 1.

In other words, $G$ is obtained from a graph $H$ with $\Delta(H) = \langle F_1, \ldots, F_\ell \rangle$ whose binomial edge ideal is Cohen-Macaulay (see [5, Theorem 3.1]) by attaching edges in the intersection points of the facets of $\Delta(H)$. Obviously, such a graph has the property that its longest induced path has length equal to $\ell$. In the case that $\dim F_i = 1$ for $1 \leq i \leq \ell$, then $G$ is called a caterpillar graph. Figure 1 displays a $C_\ell$-graph with $\ell = 5$.

![Figure 1: $C_\ell$-graph](image)

We should also note that any $C_\ell$-graph is chordal and has the property that any two distinct maximal cliques intersect in at most one vertex. So that any $C_\ell$-graph is a connected block graph. But, obviously, there are block graphs which are not $C_\ell$-graphs. Such an example is displayed in Figure 2.

![Figure 2: A block graph which is not a $C_\ell$-graph](image)

**Theorem 3.4.** Let $G$ be a $C_\ell$-graph on the vertex set $[n]$. Then

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\in_{<}(J_G)) = \ell.$$  

**Proof.** Let $G$ consists of the sequence of maximal cliques $F_1, \ldots, F_\ell$ as in condition (i) in Definition 3.3 to which we add some edges as in condition (ii). So the maximal cliques of $G$ are $F_1, \ldots, F_\ell$ and all the additional whiskers. We proceed by induction on the number $r$ of maximal cliques of $G$. If $r = \ell$, then $G$ is a closed graph whose binomial edge ideal is Cohen-Macaulay, hence the statement holds by [6, Theorem 3.2]. Let $r > \ell$ and let $F'_1, \ldots, F'_r$ be a leaf order on the facets of $\Delta(G)$. Obviously, we may choose a leaf order on $\Delta(G)$
such that $F'_r = F_r$. With the same arguments and notation as in the proof of Theorem 3.2, we get the sequence (1).

We now observe that $G'$ is a $C_{\ell-1}$-graph, hence, by the inductive hypothesis,
\[ \text{reg} \frac{S}{J_{G'}} = \text{reg} \frac{S}{\text{in}_<(J_{G'})} = \ell - 1. \tag{2} \]

The graph $G''$ has at most two non-trivial connected components. One of them, say $H_1$, is a $C_{\ell'}$-graph with $\ell' \in \{\ell - 2, \ell - 1\}$. The other possible non-trivial component, say $H_2$, occurs if $|F_\ell| \geq 3$ and, in this case, $H_2$ is a clique of dimension $|F_\ell| - 2 \geq 1$. By the inductive hypothesis, we obtain
\[ \text{reg} \frac{S}{J_{G''}} = \text{reg} \frac{S}{\text{in}_<(J_{G''})} = \text{reg} \frac{S}{J_{H_1}} + \text{reg} \frac{S}{J_{H_2}} \leq \ell - 1 + 1 = \ell. \tag{3} \]

Relations (2) and (3) yield $\text{reg}(S/\text{in}_<(J_{G'}) \oplus S/(x_i, y_i) + \text{in}_<(J_{G''})) \leq \ell$. From the exact sequence (1) we get
\[ \text{reg} \left( \frac{S}{\text{in}_<(J_G)} \right) \leq \max \{\text{reg} \left( \frac{S}{\text{in}_<(J_{G'})} \oplus \frac{S}{(x_i, y_i) + \text{in}_<(J_{G''})} \right), \text{reg} \frac{S}{\text{in}_<(J_G)} + 1\} \leq \ell. \tag{4} \]

By [7, Theorem 3.3.4], we know that $\text{reg}(S/J_G) \leq \text{reg}(S/\text{in}_<(J_G))$, and by [10, Theorem 1.1], we have $\text{reg}(S/J_G) \geq \ell$. By using all these inequalities, we get the desired conclusion.

4 Binomial edge ideals of caterpillar trees

Matsuda and Murai showed in [10] that, for any connected graph $G$ on the vertex set $[n]$, we have $\ell \leq \text{reg}(S/J_G) \leq n - 1$, where $\ell$ denotes the length of the longest induced path of $G$, and conjectured that $\text{reg}(S/J_G) = n - 1$ if and only if $T$ is a line graph. Several recent papers are concerned with this conjecture; see, for example, [6], [13], and [14]. One may ask as well to characterize connected graphs $G$ whose longest induced path has length $\ell$ and $\text{reg}(S/J_G) = \ell$. In this section, we answer this question for trees.

A caterpillar tree is a tree $T$ with the property that it contains a path $P$ such that any vertex of $T$ is either a vertex of $P$ or it is adjacent to a vertex of $P$. Clearly, any caterpillar tree is a $C_{\ell}$-graph for some positive integer $\ell$.

Caterpillar trees were first studied by Harary and Schwenk [9]. These graphs have applications in chemistry and physics [4]. In Figure 3, an example of caterpillar tree is displayed. Note that any caterpillar tree is a narrow graph in the sense of Cox and Erskine [2]. Conversely, one may easily see that any narrow graph is a caterpillar tree. Moreover, as it was observed in [11], a tree is
a caterpillar graph if and only if it is weakly closed in the sense of definition given in [11].

In the next theorem we characterize the trees $T$ with $\text{reg}(S/J_T) = \ell$ where $\ell$ is the length of the longest induced path of $T$.

**Theorem 4.1.** Let $T$ be a tree on the vertex set $[n]$ whose longest induced path $P$ has length $\ell$. Then $\text{reg}(S/J_T) = \ell$ if and only if $T$ is caterpillar.

**Proof.** Let $T$ be a caterpillar tree whose longest induced path has length $\ell$. Then, by the definition of a caterpillar tree, it follows that $T$ is a $C_\ell$-graph. Hence, $\text{reg}(S/J_T) = \ell$ by Theorem 3.4. Conversely, let $\text{reg}(S/J_T) = \ell$ and assume that $T$ is not caterpillar. Then $T$ contains an induced subgraph $H$ with $\ell + 3$ vertices as in Figure 4.

Then, by [15, Theorem 27], it follows that $\text{reg}(S/J_H) = \ell + 1$. Thus, since $\text{reg}(S/J_H) \leq \text{reg}(S/J_G)$ (see [10, Corollary 2.2]), it follows that $\text{reg}(S/J_G) \geq \ell + 1$, contradiction to our hypothesis. □

**References**


Faryal CHAUDHRY,
Abdus Salam School of Mathematical Sciences,
GC University, Lahore
68-B, New Muslim Town, Lahore 54600, Pakistan.
Email: chaudhryfaryal@gmail.com
Ahmet DOKUYUCU,
Faculty of Mathematics and Computer Science,
Ovidius University of Constanta,
Bd. Mamaia 124, 900527 Constanta, Romania,
and Department of Information Technology,
Lumina-The University of South-East Europe,
Sos. Colentina nr. 64b, Bucharest, Romania.
Email: ahmet.dokuyucu@lumina.org

Rida IRFAN,
Abdus Salam School of Mathematical Sciences,
GC University, Lahore
68-B, New Muslim Town, Lahore 54600, Pakistan.
Email: ridairfan_88@yahoo.com