Properties of a new integral operator

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Abstract
In this paper, we derive sufficient conditions for the univalence, starlikeliness, convexity and some other properties in the class \( \mathcal{N}(\rho) \), for a new integral operator defined on the space of normalized analytic functions in the open unit disk.

1 Introduction
Let \( \mathcal{A} \) be the class of functions which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \) given by
\[
    f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U.
\] (1.1)
Consider \( S \) the subclass of \( \mathcal{A} \) consisting of univalent functions. We denote by \( S^*(\alpha) \) the class of starlike univalent functions of order \( \alpha \) \((0 \leq \alpha < 1)\),
\[
    S^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left[ \frac{zf'(z)}{f(z)} \right] > \alpha, \ z \in U \right\}.
\]
By \( K(\alpha) \) we denote a subclass of \( \mathcal{A} \) consisting of convex univalent functions of order \( \alpha \) \((0 \leq \alpha < 1)\) defined as
\[
    K(\alpha) = \left\{ f \in \mathcal{A} : \Re \left[ \frac{zf''(z)}{f'(z)} + 1 \right] > \alpha, \ z \in U \right\}.
\]
Clearly, we have
(i) $S^*(0) = S^*$ the class of all starlike functions with respect to the origin;
(ii) $K(0) = K$ the class of all convex functions;
(iii) $K \subset S^* \subset S$, $K(\alpha) \subset S^*(\alpha)$, $K(\alpha) \subset K$ and $S^*(\alpha) \subset S^*$.

A function $f \in A$ is said to be in the class $R_\lambda$ if and only if
$$\text{Re} \left[ f'(z) \right] > \lambda,$$
for some $\lambda$, $0 \leq \lambda < 1$.

Recently, Frasin and Jahangiri [4] define the family $B(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$ consisting of functions $f \in A$ satisfying the condition
$$\left| f'(z) \left[ \frac{z}{f(z)} \right]^{\mu} - 1 \right| < 1 - \lambda,$$  \hspace{1cm} (1.2)
for all $z \in U$.

It is obvious that: (i) $B(0, \lambda) = R_\lambda$; (ii) $B(1, \lambda) = S^*(\lambda)$;
(iii) $B(2, \lambda) = B(\lambda)$ (see Frasin and Darus [5]);
(iv) $B(2, 0) = S$ (see Ozaki and Nunokawa [3]).

Let $N(\rho)$ be the subclass of $A$ that contains all the functions $f$ which satisfy the inequality
$$\text{Re} \left[ \frac{zf''(z)}{f'(z)} + 1 \right] < \rho, \hspace{0.5cm} \rho > 1, z \in U.$$

Uralegaddi, Ganigi and Sarangi in [11] and Owa and Srivastava in [7] introduced and studied the class $N(\rho)$.

In the present paper, we introduce a new integral operator
$$J_\alpha : A \times A \rightarrow A$$
defined by:
$$J_\alpha(f, g)(z) = \int_0^z \left[ \frac{e^{f(t)}}{g'(t)} \right]^\alpha dt,$$  \hspace{1cm} (1.3)
where parameter $\alpha$ is a complex number, with $\text{Re} \alpha \geq 1$.

In this paper our purpose is to obtain univalence conditions, starlikeness properties, the order of convexity for the integral operator abovementioned and to show that the operator $J_\alpha(f, g)(z)$ is in the class $N(\rho)$, by using functions from the class $B(\mu, \lambda)$. Recently, various types of integral operators were studied by different authors (see [10, 2]), and some of them motivated us to come up with the integral operator defined in (1.3).

In the proof of our main results, we need to recall here the following:
Theorem 1.1. **(Becker [1])** If the function \( f \) is regular in the unit disk \( U \), \( f(z) = z + a_2 z^2 + \cdots \) and
\[
(1 - |z|^2) \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]
for all \( z \in U \), then the function \( f \) is univalent in \( U \).

Lemma 1.1. **(General Schwarz Lemma [6])** Let \( f \) be regular function in the disk \( U_R = \{z \in \mathbb{C} : |z| < R\} \) with \( |f(z)| < M \), \( M \) fixed. If \( f \) has in \( z=0 \) one zero with multiply bigger than \( m \), then
\[
|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R.
\]
The equality case hold only if \( f(z) = e^{i\theta} \frac{M}{R^m} z^m \), where \( \theta \) is constant.

Lemma 1.2. **[9]** Let the functions \( p \) and \( q \) be analytic in \( U \) with \( p(0) = q(0) = 0 \), and let \( \delta \) be a real number. If the function \( q \) maps the unit disk \( U \) onto a region which is starlike with respect to the origin, the inequality
\[
\text{Re} \left[ \frac{p'(z)}{q'(z)} \right] > \delta, \quad \text{for all } z \in U
\]
implies that
\[
\text{Re} \left[ \frac{p(z)}{q(z)} \right] > \delta, \quad \text{for all } z \in U.
\]

2 **Main results**

The univalence condition for the operator \( I_\alpha(f,g) \) defined in (1.3) is proved in the next theorem, by using the Becker univalence criterion.

**Theorem 2.1.** Let \( \alpha \) be a complex number, with \( \Re \alpha \geq 1 \), \( f \in B(\mu, \lambda) \) and \( g \in A \). Suppose also that positive real numbers \( M \) (\( M \geq 1 \)) and \( N \) (\( N \geq 1 \)) are so constrained that
\[
|f(z)| < M \quad \text{and} \quad \left| \frac{g''(z)}{g'(z)} \right| \leq N, \quad z \in U.
\]
If
\[
|\alpha| \leq \frac{3\sqrt{3}}{2[(2 - \lambda)M^\mu + N]},
\]
then the function \( J_\alpha(f,g) \) is in the class \( S \).
Proof. Let the function \( h \) be defined by
\[
h(z) := J_\alpha(f, g)(z), \quad z \in U.
\] (2.3)

Obviously \( h \) is regular in \( U \) and \( h(0) = h'(0) - 1 = 0 \). From (2.3) we obtain
\[
\frac{zh''(z)}{h'(z)} = \alpha \left[ zf'(z) - \frac{zg''(z)}{g'(z)} \right].
\] (2.4)

Hence, we get
\[
(1 - |z|^2) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq (1 - |z|^2) \cdot |z| \cdot |\alpha| \left[ |f'(z)| + \left| \frac{g''(z)}{g'(z)} \right| \right],
\] (2.5)

\[
\leq (1 - |z|^2) \cdot |z| \cdot |\alpha| \left[ \left( f'(z) \left( \frac{z}{f(z)} \right)^\mu - 1 \right) \left| \frac{f(z)}{z} \right|^\mu + \left| \frac{g''(z)}{g'(z)} \right| \right].
\] (2.6)

By using the hypothesis of the theorem and applying the General Schwarz Lemma, we have
\[
(1 - |z|^2) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq (1 - |z|^2) \cdot |z| \cdot |\alpha| \left[ (2 - \lambda)M^\mu + N \right].
\] (2.7)

Considering the function
\[
t : [0, 1) \to \mathbb{R},
\]
\[
t(x) = x(1 - x^2), \quad x = |z|,
\]
we find that
\[
t(x) \leq \frac{2}{3\sqrt{3}}, \quad \text{for all } x \in [0, 1).
\] (2.8)

From (2.7), (2.8) and (2.6) we obtain
\[
(1 - |z|^2) \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{2|\alpha|}{3\sqrt{3}} \left[ (2 - \lambda)M^\mu + N \right] \leq 1.
\] (2.9)

Finally, by applying Theorem 1.1 in (2.9) we yield that the function \( J_\alpha(f, g) \) is in the class \( S \).

In the following theorem we give sufficient conditions such that the integral operator \( J_\alpha(f, g) \in S^* \).
Theorem 2.2. Let \( \alpha \) be a complex number, with \( \Re \alpha \geq 1 \), \( f \in B(\mu, \lambda) \) and \( g \in A \). Suppose also that positive real number \( M \) \( (M \geq 1) \) is so constrained that
\[
|f(z)| < M \quad \text{and} \quad \left| \frac{zg''(z)}{g'(z)} \right| < 1, \quad z \in U. \tag{2.10}
\]
If
\[
|\alpha| \leq \frac{1}{(2 - \lambda)M^\mu + 1}, \tag{2.11}
\]
then the function \( J_\alpha (f, g) \) is in the class \( S^* \).

Proof. For the function \( h \) be given by (2.3) we obtain
\[
\frac{zh'(z)}{h(z)} = z e^{\alpha f(z)} \int_0^z \left[ e^{\alpha f(t)} \right]^\mu dt. \tag{2.12}
\]
Setting
\[
p(z) = zh'(z) \quad \text{and} \quad q(z) = h(z),
\]
we find that \( p(0) = q(0) = 0 \), and \( q \) satisfies the starlikeness condition of Lemma 1.2. Since,
\[
\frac{p'(z)}{q'(z)} = 1 + \alpha \left[ z f'(z) - \frac{zg''(z)}{g'(z)} \right]
\]
we obtain
\[
\left| \frac{p'(z)}{q'(z)} - 1 \right| \leq |\alpha| \left[ \left( \left| f'(z) \left( \frac{z}{f(z)} \right)^\mu \right| - 1 \right) + 1 \right] \left| \frac{f(z)}{z} \right|^\mu + \left| \frac{zg''(z)}{g'(z)} \right|. \tag{2.13}
\]
Also, since \( |f(z)| < M \), \( z \in U \), applying the Schwarz Lemma, we have
\[
\left| \frac{f(z)}{z} \right| \leq M, \quad \text{for all} \quad z \in U. \tag{2.14}
\]
By using the hypothesis of the Theorem and replacing (2.14) in inequation (2.13), we obtain
\[
\left| \frac{p'(z)}{q'(z)} - 1 \right| \leq |\alpha| \left[ \left( \left| f'(z) \left( \frac{z}{f(z)} \right)^\mu \right| - 1 \right) + 1 \right] M^\mu |z| + 1 \leq |\alpha|[1 + (2 - \lambda) \cdot M^\mu] \leq 1.
\]
Thus, we have
\[
\Re \left[ \frac{p'(z)}{q'(z)} \right] > 0, \quad z \in U \tag{2.15}
\]
and, applying Lemma 1.2, we find that
\[
\text{Re} \left[ \frac{p(z)}{q(z)} \right] > 0, \quad z \in U.
\] (2.16)

This completes the proof of the theorem. \qed

Letting \( \mu = 1 \) in Theorem 2.2, we have

**Corollary 2.1.** Let \( \alpha \) be a complex number, with \( \text{Re} \alpha \geq 1 \), \( f \in S^*(\lambda) \) and \( g \in A \). Suppose also that positive real number \( M, M \geq 1 \) is so constrained that
\[
|f(z)| < M \quad \text{and} \quad \left| \frac{z g''(z)}{g'(z)} \right| < 1, \quad z \in U.
\]

If
\[
|\alpha| \leq \frac{1}{1 + (2 - \lambda)M},
\]
then the function \( J_\alpha(f, g) \) is in the class \( S^* \).

Letting \( \lambda = 0 \) in Corollary 2.1, we obtain

**Corollary 2.2.** Let \( \alpha \) be a complex number, with \( \text{Re} \alpha \geq 1 \), \( f \in S^* \) and \( g \in A \). Suppose also that positive real number \( M, M \geq 1 \) is so constrained that
\[
|f(z)| < M \quad \text{and} \quad \left| \frac{z g''(z)}{g'(z)} \right| < 1, \quad z \in U.
\]

If
\[
|\alpha| \leq \frac{1}{1 + 2M},
\]
then the function \( J_\alpha(f, g) \) is in the class \( S^* \).

**Theorem 2.3.** Let \( \alpha \) be a complex number, with \( \text{Re} \alpha \geq 1 \), \( f \in B(\mu, \lambda) \) and \( g \in A \). Suppose also that positive real numbers \( M \) \( (M \geq 1) \) and \( N \) \( (N \geq 1) \) are so constrained that
\[
|f(z)| < M \quad \text{and} \quad \left| \frac{g''(z)}{g'(z)} \right| < N, \quad z \in U.
\]

Then the function \( J_\alpha(f, g) \) is in the class \( K(\delta) \), where
\[
\delta = 1 - |\alpha|\left[ N + (2 - \lambda) \cdot M^\mu \right] \quad \text{and} \quad 0 < |\alpha|\left[ N + (2 - \lambda) \cdot M^\mu \right] \leq 1.
\]
Proof. By letting the function $h$ defined in (2.3), from equation (2.18) we find that

$$
\left| \frac{zh''(z)}{h'(z)} \right| \leq |z| \cdot |\alpha| \left[ \left| f'(z) \right| + \left| \frac{g''(z)}{g'(z)} \right| \right]
\leq |z| \cdot |\alpha| \left[ \left| f'(z) \right| \left( \left| \frac{z}{f(z)} \right|^\mu - 1 \right) + \left| f(z) \right|^\mu + \left| \frac{g''(z)}{g'(z)} \right| \right].
$$

(2.17)

From the hypothesis and applying the Schwarz Lemma in inequation (2.17), we obtain

$$
\left| \frac{zh''(z)}{h'(z)} \right| \leq |\alpha|[N + (2 - \lambda) \cdot M^\mu] = 1 - \delta.
$$

This evidently completes the proof.

Letting $\mu = 1$ in Theorem 2.3, we have

**Corollary 2.3.** Let $\alpha$ be a complex number, with $\text{Re}\alpha \geq 1$, $f \in S^\ast(\lambda)$ and $g \in A$. Suppose also that positive real numbers $M$ ($M \geq 1$) and $N$ ($N \geq 1$) are so constrained that

$$
|f(z)| < M \quad \text{and} \quad \left| \frac{g''(z)}{g'(z)} \right| < N, \quad z \in U.
$$

Then the function $J_\alpha(f, g)$ is in the class $K(\delta)$, where

$$
\delta = 1 - |\alpha|[N + (2 - \lambda) \cdot M^\mu] \quad \text{and} \quad 0 < |\alpha|[N + (2 - \lambda)M] \leq 1.
$$

Letting $\delta = \lambda = 0$ in Corollary 2.3, we obtain

**Corollary 2.4.** Let $\alpha$ be a complex number, with $\text{Re}\alpha \geq 1$, $f \in S^\ast$ and $g \in A$. Suppose also that positive real numbers $M$ ($M \geq 1$) and $N$ ($N \geq 1$) are so constrained that

$$
|f(z)| < M \quad \text{and} \quad \left| \frac{g''(z)}{g'(z)} \right| < N, \quad z \in U.
$$

Then the function $J_\alpha(f, g)$ is in the class $K$, where

$$
|\alpha| = \frac{1}{2M + N}.
$$
Theorem 2.4. Let the functions $f, g \in A$, with $f$ in the class $B(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$, and $\alpha$ a complex number, with $\text{Re} \alpha \geq 1$. If $|f(z)| < M$, for $M$ a positive real number, $M \geq 1$, $z \in U$ and $\left| \frac{g''(z)}{g'(z)} \right| < 1$, then the integral operator $J_\alpha(f, g)$ defined by (1.3) is in the class $N(\rho)$, where

$$
\rho = |\alpha| [1 + (2 - \lambda) M^\mu] + 1.
$$

Proof. From (2.4) we obtain that

$$
\frac{z J''_\alpha(f, g)(z)}{J'_\alpha(f, g)(z)} = \alpha z \left[ f'(z) - \frac{g''(z)}{g'(z)} \right]
$$

So,

$$
\text{Re} \left[ \frac{z J''_\alpha(f, g)(z)}{J'_\alpha(f, g)(z)} + 1 \right] = \text{Re} \left[ \alpha z \left( f'(z) - \frac{g''(z)}{g'(z)} \right) + 1 \right]
$$

$$
\leq |z| \cdot |\alpha| \left[ |f'(z)| + \left| \frac{g''(z)}{g'(z)} \right| + 1 \right]
$$

$$
\leq |z| \cdot |\alpha| \left[ |f'(z)| \left( \frac{z}{f(z)} \right)^\mu \left| \frac{f(z)}{z} \right|^\mu + 1 \right] + 1. \quad (2.18)
$$

Since $f$ is in the class $B(\mu, \lambda)$, $|f(z)| < M$, from General Schwarz Lemma and from (2.18), we find that

$$
\text{Re} \left[ \frac{z J''_\alpha(f, g)(z)}{J'_\alpha(f, g)(z)} + 1 \right] < |\alpha| \left[ 1 + \left( \left| f'(z) \left( \frac{z}{f(z)} \right)^\mu \right| - 1 \right) M^\mu \right] + 1
$$

$$
< |\alpha| [1 + (2 - \lambda) M^\mu] + 1 = \rho. \quad (2.19)
$$

We yield that the function $J_\alpha(f, g)$ is in the class $N(\rho)$.

For $\mu = 0$ in Theorem 2.4 we obtain:

Corollary 2.5. Let the functions $f, g \in A$, with $f$ in the class $R_\lambda$, $0 \leq \lambda < 1$, and $\alpha$ a complex number, with $\text{Re} \alpha \geq 1$. If $|f(z)| < M$, for $M$ a positive real number, $M \geq 1$, $z \in U$ and $\left| \frac{g''(z)}{g'(z)} \right| < 1$, then the integral operator $J_\alpha(f, g)$ defined by (1.3) is in the class $N(\rho)$, where $\rho = |\alpha| (3 - \lambda) + 1$. 

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