

Payoff space in C^1 -games

David Carfi

Abstract. In this paper we give a general method to determine the payoff space, and consequently, in some particular cases, the Pareto boundaries, of certain type of normal form game with n -persons having payoff functions of class C^1 . Specifically, we consider n -person games in which the strategy set of any player is a compact interval of the real line, and in which the payoff functions are C^1 , in the sense that they are restrictions of C^1 functions defined in open neighborhoods of the strategy profile space of the game. We face the problem of determining the payoff space and its Pareto optimal boundaries and, finally, of finding some classical compromise solutions.

M.S.C. 2000: 91A05, 91A06, 91A10, 91A12, 91A44.

Key words: Differentiable game, payoff space, cooperative solutions, Pareto boundaries.

1 Introduction

In the current literature the study of a game in normal form mainly consists of finding the Nash equilibria in mixed strategies and in the analysis of their stability properties (see [7], [8] and [9]). This does not give a complete and global view of the game, since, for instance, it should be interesting to know the positions of the payoff profiles corresponding to the Nash equilibria in the payoff space of the game, since the knowledge of these positions requires the knowledge of the entire payoff space. This need becomes inevitable when the problem to be solved in the game is a bargaining one: in fact, the determination of a bargaining solution (or of compromise solutions) needs the analytical determination of the Pareto boundaries. In our paper we shall present a general method to find an explicit expression of the topological boundary of the payoff space of the game. Resuming, the motivation of the paper resides upon the fact that a complete and deep study of a game in normal form requires the knowledge of the payoff space, or at least of its topological boundary, especially when one passes to the cooperative phase of the game, since to find bargaining solutions or other compromise solutions, the knowledge of the Pareto boundaries is necessary.

Remark. In this paper we follow the way shown in [3], [4], [5], [6] to construct theoretical bases for Decisions in Economics and Finance by means of algebraic, topological and differentiable structures.

2 Preliminaries and notations

We shall consider n -person games in normal form. We give the definition used in this work for ease of the reader. The form of definition we give is particularly useful for our purpose.

Definition 1. (definition of a game in normal form). Let $E = (E_i)_{i=1}^n$ be an ordered family of non-empty sets. We call n -person game in normal form upon the support E each function $f : \times E \rightarrow \mathbb{R}^n$, where $\times E$ denotes the cartesian product $\times_{i=1}^n E_i$ of the family E . The set E_i is called the *strategy set of player i* , for every index i of the family E , and the product $\times E$ is called the *strategy profile space*, or the n -strategy space, of the game.

Terminology. With this choice of definition for games in normal form, we have to introduce several notions:

- the set $\{i\}_{i=1}^n$ of the first n positive integers is called *the set of the players* of the game;
- each element of the cartesian product $\times E$ is called a *strategy profile* of the game;
- the image of the function f , i.e., the set of all real n -vectors of type $f(x)$, with x in the strategy profile space $\times E$, is called the *n -payoff space*, or simply the *payoff space*, of the game f .

We recall, further, for completeness (and ease of the reading), the definition of Pareto boundary we shall use in the paper.

Definition 2 (Pareto boundary). The *Pareto maximal boundary* of a game f is the subset of the n -strategy space of those n -strategies x such that the corresponding payoff $f(x)$ is maximal in the n -payoff space, with respect to the usual order of the Euclidean n -space \mathbb{R}^n . We shall denote the maximal boundary of the n -payoff space by $\bar{\partial}f(S)$ and the maximal boundary of the game by $\bar{\partial}_f(S)$ or by $\bar{\partial}(f)$. In other terms, the maximal boundary $\bar{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the *minimal Pareto boundary*.

3 The method

The context. We deal with a type of normal form game f defined on the product of n compact non-degenerate intervals of the real line, and such that f is the restriction to the n -strategy space of a C^1 -function defined on an open set of \mathbb{R}^n containing the n -strategy space S (which, in this case, is a compact non-degenerate n -interval of the n -space \mathbb{R}^n).

Before stating the main result of the method, we recall some basic notions.

3.1 Topological boundary

We recall that the topological boundary of a subset S of a topological space (X, \mathcal{T}) is the set defined by the following three equivalent properties:

- it is the closure of S without the interior of S : $\partial S = \text{cl}(S) \setminus \text{int}(S)$;
- it is the intersection of the closure of S with the closure of its complement

$$\partial S = \text{cl}(S) \cap \text{cl}(X \setminus S);$$

- it is the set of those points x of X such that every neighborhood of x contains at least one point of S and at least one point in the complement of S .

The key theorem of our method is the following one.

Theorem 1. Let f be a C^1 function defined upon an open set O of the euclidean space \mathbb{R}^n and with values in \mathbb{R}^n . Then, for every part S of the open O , the topological boundary of the image of S by the function f is contained in the union $f(\partial S) \cup f(C)$, where C is the critical set of f in S , that is the set of the points x of S such that the Jacobian matrix $J_f(x)$ is not invertible.

The full comprehension of the proof requires some important preliminary notions.

3.2 Local diffeomorphism and the local inversion theorem

Let X and Y be two open subsets of the euclidean space \mathbb{R}^n , let $f : X \rightarrow Y$ be a function and let x_0 be a point of X . The function f is said a *local homeomorphism* (respectively, a *local diffeomorphism*) *at the point* x_0 if there is an open neighborhood U of x_0 such that the restriction of f to the pair of subsets $(U, f(U))$ is a homeomorphism (respectively diffeomorphism).

The following fundamental theorem is a consequence of the Dini's theorem.

Theorem (local inversion theorem for C^1 -functions). *Let X and Y be two open subsets of the euclidean space \mathbb{R}^n , and let $f : X \rightarrow Y$ be a C^1 -function. Then, for every point $x_0 \in X$ such that the derivative $f'(x_0)$ is a bijective linear application, f is a local diffeomorphism at x_0 .*

In the conditions of the above theorem, we shall say that a point x_0 of X is a regular point of the function f if the derivative $f'(x_0)$ is a bijective linear application.

Proof of Theorem 1. The theorem derives from the local inversion theorem for C^1 -functions, and it is based on the fact that a C^1 function f is a local diffeomorphism at every point (of its domain) in which f has invertible Jacobian matrix. More precisely, since f is a local diffeomorphism at the points not belonging to the critical part of f , it is also a local homeomorphism at those points, and then it sends the neighborhoods of a regular point x_0 onto neighborhoods of the image $f(x_0)$ and, consequently, regular interior points to interior points of Y . So, let X be a payoff in the topological boundary of the image $f(S)$. The payoff X cannot be the transformation of a regular interior point of S , hence it must belong to the part $f(\partial S)$ or to the part $f(C)$. ■

To determine the payoff space of the game we have to do some further topological remarks.

Remark 1. Taking into account that f is a continuous function (since it is C^1), the set $f(S)$ is compact since S is compact, and it is connected since S is connected.

Remark 2. The critical part C is a closed set, since it is the level of a continuous functional (the Jacobian functional $\det J_f$), so the complement of C in S is relatively open in S .

Remark 3 (the openness of local homeomorphisms). Let X and Y be two open subsets of the euclidean n -space, let $f : X \rightarrow Y$ be a local homeomorphism and let O be an open subset of X . Clearly, $f(O)$ is open; indeed, let y_0 be a point of $f(O)$, we must prove that y_0 is an interior point. Let x_0 be a reciprocal image of y_0 , by definition of local homeomorphism, there is a neighborhood U of x_0 , such that $f(U)$ is an open neighborhood of y_0 .

Remark 4 (transformation of simply connected parts). Let A be a relatively open and simply connected set in one of the connected components of $S \setminus C$, suppose that f is injective on A . Since the restriction to the pair $(S \setminus C, f(S \setminus C))$ is an open mapping, the restriction to the pair $(A, f(A))$ is a homeomorphism. Consequently, the image $f(A)$ is simply connected.

Conclusions. So, it is enough to determine the critical part of the game and transform it together with the sides of the n -strategy space, but it is necessary to taking into account the above topological considerations.

4 Example in dimension 2

Description of the game. We consider a loss-game $G = (f, \leq)$, with strategy sets $E = F = [0, 1]$ and biloss (disutility) function defined by

$$f(x, y) = (-4xy, x + y),$$

for every bistrategy (x, y) of the game.

Remark 5. This game can be viewed as the mixed extension of the finite bimatrix game

$$M = \begin{pmatrix} (-4, 2) & (0, 1) \\ (0, 1) & (0, 0) \end{pmatrix}.$$

Classification. The game is not linear, it is, utterly, bilinear. It is not symmetric (with respect to the players), since $f_1(x, y) \neq f_2(y, x)$, but it is symmetric with respect to the bistrategies, since $f_i(x, y) = f_i(y, x)$, for every player i . It is not invertible, since there are two different equivalent bistrategies: $f(1, 0) = f(0, 1) = (0, 1)$.

4.1 The critical space of the game

In the following we shall denote by A , B , C and D the vertices of the square $E \times F$, starting from the origin and going anticlockwise.

The Jacobian matrix. The Jacobian matrix is

$$J_f(x, y) = \begin{pmatrix} -4y & -4x \\ 1 & 1 \end{pmatrix},$$

for every bistrategy (x, y) . The Jacobian determinant is

$$\det J_f(x, y) = -4y + 4x,$$

for every pair (x, y) .

The critical space. The critical zone is the subset of the bistrategy space of those bistrategies (x, y) verifying the equality $-y + x = 0$. In symbols, the critical zone is the segment

$$\mathcal{C}(f) = \{(x, y) \in [0, 1]^2 : x = y\} = [A, C].$$

The transformation of critical space. Let us determine the image $f([A, C])$. The segment $[A, C]$ is defined by the relations

$$\begin{cases} x = y \\ y \in [0, 1]. \end{cases}$$

The value of the biloss function upon the generic point (y, y) is $f(y, y) = (-4y^2, 2y)$. Setting

$$\begin{cases} X = -4y^2 \\ Y = 2y, \end{cases}$$

we have

$$\begin{cases} X = -Y^2 \\ Y \in [0, 2]. \end{cases}$$

Thus, the image of the critical zone is the parabolic segment of equation $X = -Y^2$ with end points $A' = (0, 0)$ and $C' = (-4, 2)$.

4.2 The biloss (disutility) space

Transformation of the topological boundary of the bistrategy space. We start from the image $f([A, B])$. The segment $[A, B]$ is defined by the relations

$$\begin{cases} y = 0 \\ x \in [0, 1]. \end{cases}$$

The value of the biloss function upon the generic point of this segment is the biloss $f(x, 0) = (0, x)$. Setting $X = 0$ and $Y = x$, we have

$$\begin{cases} X = 0 \\ Y \in [0, 1]. \end{cases}$$

Thus the image of the segment $[A, B]$ is the segment of end points $A' = (0, 0)$ and $B' = (0, 1)$. *Image of $f([D, C])$.* The segment $[D, C]$ is defined by the relations

$$\begin{cases} y = 1 \\ x \in [0, 1]. \end{cases}$$

The image of the generic point is $f(x, 1) = (-4x, x + 1)$. Setting

$$\begin{cases} X = -4x \\ Y = x + 1, \end{cases}$$

we have

$$\begin{cases} X = 4 - 4Y \\ Y \in [1, 2]. \end{cases}$$

Thus the image is the segment of end points $D' = (0, 1)$ and $C' = (-4, 2)$. *Transformation* $f([C, B])$. The segment $[C, B]$ is defined by

$$\begin{cases} x = 1 \\ y \in [0, 1]. \end{cases}$$

The image of the generic point is $f(1, y) = (-4y, 1 + y)$. Setting

$$\begin{cases} X = -4y \\ Y = 1 + y, \end{cases}$$

we obtain

$$\begin{cases} X = 4 - 4Y \\ X \in [-4, 0]. \end{cases}$$

So the image is the segment of end points $C' = (-4, 2)$ and $B' = (0, 1)$. Finally, let's determine the image $f([A, D])$. The segment $[A, D]$ is defined by

$$\begin{cases} x = 0 \\ y \in [0, 1]. \end{cases}$$

The image of the generic point is $f(0, y) = (0, y)$. Setting

$$\begin{cases} X = 0 \\ Y = y, \end{cases}$$

we obtain

$$\begin{cases} X = 0 \\ Y \in [0, 1]. \end{cases}$$

So the image is the segment of end points $A' = (0, 0)$ and $D' = (0, 1)$.

Extrema of the game. The extrema of the game are

$$\alpha := \inf G = (-4, 2) \notin G,$$

and

$$\beta := \sup G = (0, 2) \notin G.$$

They are both shadow extremes.

Pareto boundaries. The Pareto boundaries of the biloss space are

$$\underline{\partial}f(E \times F) = f([A, C])$$

(the image of the critical zone of the game, that is a parabolic arc) and

$$\bar{\partial}f(E \times F) = [B', C'];$$

consequently the Pareto boundaries of the bistrategy space are

$$\underline{\partial}_f(E \times F) = [A, C]$$

and

$$\bar{\partial}_f(E \times F) = [B, C] \cup [D, C].$$

4.3 Cooperative phase: selection of Pareto bistrategies

We shall examine the most common cooperative solutions. *The Kalai Smorodinsky solution (the elementary best compromise)*. The elementary best compromise biloss (X, Y) is the intersection of the segment joining the threat biloss $v^\#$ (see [1] and [2]) with the infimum of the game, thus it satisfies the system

$$\begin{cases} Y = (1/4)X + 1 \\ X = -Y^2 \\ X \in [-4, 0] \\ Y \in [0, 2], \end{cases}$$

leading to the resolvent equation $X^2 + 24X + 16 = 0$, its acceptable solution is $a = \sqrt{128} - 12$, so the biloss $K' = (a, a/4 + 1)$ is the best compromise biloss. The Kalai Smorodinsky solution is the unique corresponding bistrategy solving of the system

$$\begin{cases} -4xy = a \\ x + y = a + 1, \end{cases}$$

i.e., the strategy profile

$$K = \left(\frac{a+1}{2}, \frac{a+1}{2} \right).$$

Core best compromise. The core best compromise biloss is the intersection of the segment joining the threat biloss $v^\#$ with the infimum of the core, thus it satisfies the system

$$\begin{cases} Y = X + 1 \\ X = -Y^2 \\ X, Y \in [0, 1], \end{cases}$$

putting $\gamma = (-1 + \sqrt{5})/2$ the solution is the biloss $P' = (-\gamma^2, \gamma)$, it is the unique core best compromise biloss. The core best compromise solution solves the system

$$\begin{cases} -4xy = -\gamma^2 \\ x + y = \gamma, \end{cases}$$

taking into account that this solution must belong to the core, we known also that $x = y$, and then $x = y = \gamma/2$.

Nash bargaining solution with $v^\#$ as disagreement point. The possible Nash bargaining bilosses, with disagreement point represented by the conservative bivalue $v^\#$, are the possible solutions of the following optimization problem:

$$\begin{cases} \max (X - v_1^\#)(Y - v_2^\#) = \max X(Y - 1) \\ \text{sub } X = -Y^2. \end{cases}$$

The section of the objective Nash bargaining function upon the constraint is defined by

$$g(Y) = -Y^2(Y - 1) = -Y^3 + Y^2,$$

for every Frances' loss Y . The derivative

$$g'(Y) = -3Y^2 + 2Y,$$

is non-negative when $Y(3Y - 2) \leq 0$, that is on the interval $[0, 2/3]$, consequently the maximum point of g is the loss $Y = 2/3$, with corresponding Emil's loss $X = -4/9$ by the constraint. Concluding the point $F' = (-4/9, 2/3)$ is the unique Nash bargaining biloss. The set of Nash bargaining solutions is the reciprocal image of this biloss by the biloss function f .

Minimum aggregate loss (maximum collective utility). The possible bilosses with maximum collective utility are the possible solutions of the following optimization problem:

$$\begin{cases} \min (X + Y) \\ \text{sub } X = -Y^2. \end{cases}$$

We immediately see that the unique biloss with these two properties is $C' = (-4, 2)$, with collective utility 2. The unique maximum utility solution of the game is then the corresponding bistrategy C .

5 Another example in dimension 2

We shall study the mixed extension of the finite game with payoff bimatrix

$$M = \begin{pmatrix} (0, 0) & (0, 1) \\ (1, 0) & (a, b) \end{pmatrix},$$

where $a, b \in [0, 1]$ and $a + b < 1$.

Payoff functions. The payoff functions of the mixed extension are defined on the biprobabilistic space $[0, 1]^2$, by

$$\begin{aligned} f_1(p, q) &= p(1 - q) + apq = p - (1 - a)pq, \\ f_2(p, q) &= q(1 - p) + bpq = q - (1 - b)pq, \end{aligned}$$

for every probabilistic profile (p, q) . The payoff function of the game is defined by

$$f(p, q) = (p - a'pq, q - b'pq) = (p(1 - a'q), q(1 - b'p)),$$

where $a' = 1 - a$ and $b' = 1 - b$ are the complements with respect to 1 of a and b , respectively. The complements with respect to 1 cannot be zero since $a + b < 1$.

Critical zone of the game. The Jacobian determinant of the function f at the bistrategy (p, q) is

$$J_f(p, q) = \begin{pmatrix} 1 - q + aq & -p + ap \\ -q + bq & 1 - p + bp \end{pmatrix},$$

i.e.,

$$J_f(p, q) = \begin{pmatrix} 1 - (1 - a)q & -(1 - a)p \\ -(1 - b)q & 1 - (1 - b)p \end{pmatrix} = \begin{pmatrix} 1 - a'q & -a'p \\ -b'q & 1 - b'p \end{pmatrix}.$$

The Jacobian determinant at (p, q) is

$$\begin{aligned} \det J_f(p, q) &= (1 - a'q)(1 - b'p) - a'b'pq = \\ &= 1 - a'q - b'p. \end{aligned}$$

It vanishes upon the line r of equation

$$a'q + b'p = 1.$$

This line r intersects the bistrategic space $[0, 1]^2$ iff $a/b' \leq 1$, that is, iff $a + b \leq 1$; since we assumed $a + b < 1$ this intersection must be non-empty, it shall be a segment. The end points of this segment are the points $H = (a/b', 1)$ and $K = (1, b/a')$ (note that the relation $a/b' \leq 1$ is equivalent to the relation $b/a' \leq 1$). Consequently, the critical zone of the game is the segment $[H, K]$, and its first and second projections are, respectively, the interval $[a/b', 1]$ and the interval $[b/a', 1]$.

Remark 6. If $a/b' > 1$, that is $a + b > 1$, the critical zone of the game is void.

Transformation of the critical zone. Let (p, q) be a bistrategy of the critical zone $[H, K]$, we have

$$\begin{aligned} f(p, q) &= (p - a'pq, q - b'pq) = \\ &= (p(1 - a'q), q(1 - b'p)) = (b'p^2, a'q^2). \end{aligned}$$

Hence the first projection of the image of the critical zone is the interval

$$b' [(a/b')^2, 1] = [a^2/b', b'].$$

The second projection is analogously $[b^2/a', a']$. The image of the critical zone is the set

$$f([H, K]) = \{(X, Y) \in [a^2/b', b'] \times [b^2/a', a'] : X = b'p^2, Y = a'q^2\}.$$

Note that the images of the two points H and K are, respectively,

$$\begin{aligned} f(H) &= f(a/b', 1) = ((a/b')(1 - a'), (1 - b'(a/b'))) = \\ &= ((a/b')a, (1 - a)) = (a^2/b', a'), \end{aligned}$$

and $f(K) = f(1, b/a') = ((1 - b), (b/a')(1 - b')) = (b', b^2/a')$.

Explicit expression of the image of the critical zone. For a point $(b'p^2, a'q^2)$ of the image of the critical zone, we have

$$\begin{aligned} (b'p^2, a'q^2) &= (b'p^2, q(1 - b'p)) = \left(b'p^2, \frac{1}{a'}(1 - b'p)^2 \right) = \\ &= \left(X, (1/a') \left(1 - \sqrt{b'X} \right)^2 \right), \end{aligned}$$

where we put $X = b'p^2$. Hence the explicit equation of the image of the critical zone is

$$Y = \frac{1}{a'} \left(1 - 2\sqrt{b'X} + b'X \right),$$

where X is in the interval $[a^2/b', b']$.

Transformation of the sides of the bistrategic square. Let A , B , C and D be the four vertices of the bistrategic square $[0, 1]^2$ starting from the origin and going anticlockwise. Let us transform the side $[B, C]$. The image of the generic point $(1, q)$ of the side $[B, C]$ is

$$f(1, q) = (1 - a'q, q - b'q) = (1 - a'q, bq) = \left(1 - \frac{a'}{b}Y, Y \right),$$

where we put $Y = bq$. One has $Y \in [0, b]$ and $X = 1 - (a'/b)Y$ that is

$$Y = b/a' - (b/a')X.$$

Explicit equation of the Pareto boundary. The Pareto boundary, in the case $a/b' \leq 1$ is the set of payments pairs (X, Y) such that

$$Y = \begin{cases} 1 - (b'/a)X & \text{iff } 0 \leq X \leq a^2/b' \\ (1/a') \left(1 - 2\sqrt{b'X} + b'X \right) & \text{iff } a^2/b' < X < b' \\ (b/a') - (b/a')X & \text{iff } b' \leq X \leq 1 \end{cases}.$$

If, on the contrary, $a > b'$, it is the union of the two segments $[(0, 1), (a, b)]$ and $[(a, b), (1, 0)]$.

Remark 7. Resuming, the interesting cases are those for which $a \leq b'$, when the boundary is the union of the two segments $[(0, 1), f(H)]$, $[f(K), (1, 0)]$ and of the arc Γ of equation

$$Y = (1/a') \left(1 - \sqrt{b'X} \right)^2,$$

with end points $f(H) = (a^2/b', a')$ and $f(K) = (b', b^2/a')$. *Nash bargaining solutions.* We have to maximize the function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$G(X, Y) = (X - a)(Y - b),$$

for every pair (X, Y) of the plane, constrained to payments space $f([0, 1]^2)$. There are two cases. If $a/b' \geq 1$ the game-payment corresponding to the Nash bargaining solution is $C' = (a, b)$, since the cone of the upper bounds of (a, b) intersects the payments space only in the point (a, b) itself. If $a/b' < 1$ the maximum is attained

on the arc Γ - since, say A' and B' the payments $(1, 0)$ and $(0, 1)$, the segments $[A', C']$ and $[C', B']$ are not contained in the cone of the upper bounds of C' - and consequently the zone in which lies the maximum of the function G is contained in the remaining part of Pareto boundary, that is the arc Γ . Moreover, the function G vanishes on the frontier of the cone of upper bounds of the point C' , and so the maximum shall be in the interior of the curve Γ (not at its end points) thus we can apply the Lagrange theorem.

Determination of the payments associated with Nash bargaining solution in the case $a/b' < 1$. The function G has the same values of the function g defined by

$$g(p, q) = G(b'p^2, a'q^2),$$

for every pair (p, q) of the segment $[H, K]$. The Lagrange function is defined by

$$\begin{aligned} L(p, q, \lambda) &= g(p, q) + \lambda(b'p + a'q - 1) = \\ &= (b'p^2 - a)(a'q^2 - b) - \lambda(b'p + a'q - 1). \end{aligned}$$

Applying the Lagrange theorem, we can conclude that the maximum point (p, q) of the g on the constraint must verify the system

$$\begin{cases} 2p(a'q^2 - b) - \lambda = 0 \\ 2(b'p^2 - a)q - \lambda = 0 \\ b'p + a'q - 1 = 0, \end{cases}$$

it conducts, at once, to a third degree equation in p or in q . Precisely, the equation

$$2t^3 - 3t^2 + (1 - b - a + 2ab)t + ab' = 0,$$

with $t = b'p$.

Solution in the particular case $a/b' < 1$ and $a = b$. If $a = b$, the Nash bargaining solution (p, q) must be a stationary point of the Lagrange function

$$L(p, q, \lambda) = a'p^2a'q^2 - \lambda(a'p + a'q - 1),$$

thus it solves the system

$$\begin{cases} 2pa'q^2 - \lambda = 0 \\ 2a'p^2q - \lambda = 0 \\ a'p + a'q - 1 = 0, \end{cases}$$

that is, the system

$$\begin{cases} q = p \\ a'p + a'q - 1 = 0, \end{cases}$$

so we immediately deduce the unique Nash bargaining solution

$$(p, q) = \left(\frac{1}{2a'}, \frac{1}{2a'} \right),$$

with payment $f(p, q) = \left(\frac{1}{4a'}, \frac{1}{4a'} \right)$.

Note that the payment $1/(4a')$ is greater or equal to a , for each $a \in [0, 1]$, and it is strictly greater than a , if $a \neq 1/2$, in fact the relation $1/(4a') > a$ is equivalent to the inequality $(2a - 1)^2 > 0$, and if $a/a' < 1$, we have just $a < 1/2$.

6 An example in dimension 3

The game. Let consider the three person game $f : [-1, 1]^3 \rightarrow \mathbb{R}^3$ defined by

$$f(x) = (x_1x_2, x_2x_3, -x_3x_1),$$

for every strategy triple x .

Critical part of the game. The Jacobian of the game at a strategy triple x is

$$J_f(x) = \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ -x_3 & 0 & -x_1 \end{pmatrix}.$$

The Jacobian determinant is $D(x) = -2x_1x_2x_3$, for each strategy triple x . The functional D vanishes only upon the three coordinate planes.

Image of the critical part. The image of the critical part of the game is “the star” union of the three segments $[-e_i, e_i]$, with $i = 1, 2, 3$, where e_i is the i -th vector of the canonical basis of the 3-space \mathbb{R}^3 .

Image of the topological boundary of the strategy space. The images of the six sides of the strategy cube are pairwise coincident. The boundary of the n -payoff space is the union of the three supports of the parametric surfaces

$$s_i : [-1, 1]^2 \rightarrow \mathbb{R}^3,$$

defined by

$$s_3(y) = (y_1y_2, y_2, -y_1), s_2(y) = (y_1, y_2, -y_1y_2),$$

and $s_1(y) = (y_1, y_1y_2, -y_1)$, for each y in the square $[-1, 1]^2$.

We consider the game as a gain-game - rigorously we consider the function f endowed with the usual majoring order \geq of the euclidean space - so we are interested in the part of the payoff in the first orthant.

Remark 8. It is evident that the payoff space is concave: for example, the point $(1/3)(1, 1, 1)$ does not belong to the payoff space, but it is convex combination of the canonical basis e , whose elements are in the payoff space.

Remark 9. If the game is with transferable utility, being the maximum cumulative utility of the game 1, the players can agree on the payoff $(1/3)(1, 1, 1)$, that is the barycentric payoff on the maximum utility triangle $\text{conv}(e)$ (convex envelope of the canonical basis) of equation $u_1 + u_2 + u_3 = 1$.

Remark 10. If the game is without transferable utility, the players can agree to use mixed correlated strategy profiles. The convex hull of the 3-payoff space (which is the payoff space of the correlated mixed extension of the game) has the same plane, of equation $\sum u = 1$, as Pareto boundary, now the Kalai-Smorodinsky payoff in this situation is evidently $(1/3)(1, 1, 1)$.

References

- [1] J. Aubin, *Mathematical Methods of Game and Economic Theory*, Dover Publications; Revised edition, 2007.
- [2] J. Aubin, *Optima and Equilibria*, Springer-Verlag Berlin and Heidelberg, 1993.
- [3] D. Carfi, Giovanni Caristi, *Financial dynamical systems*, Differential Geometry - Dynamical Systems (DGDS), 10 (2008), 71-85; <http://vectron.mathem.pub.ro/dgds/v10/D10-CA.pdf>
- [4] D. Carfi, *S-Linear Algebra in Economics and Physics*, Applied Sciences (APPS) 9 (2007), 44 - 66; <http://www.mathem.pub.ro/apps/v9/a9.html>.
- [5] D. Carfi, *Structures on the space of financial events*, Atti della Accademia Peloritana dei Pericolanti, classe di Scienze Fisiche Matematiche e Naturali, Vol. LXXXVI issue 2 (2008), 1-12; <http://antonello.unime.it/atti/>
- [6] D. Carfi, *Optimal boundaries for decisions*, Atti della Accademia Peloritana dei Pericolanti, classe di Scienze Fisiche Matematiche e Naturali, Vol. LXXXVI issue 1 (2008), 1-12; <http://antonello.unime.it/atti/>
- [7] M. J. Osborne, A. Rubinstein, *A Course in Game Theory*, Academic Press, 2001.
- [8] G. Owen, *Game Theory*, Academic Press, 2001.
- [9] R. B. Myerson, *Game Theory*, Harvard University Press, 1991.

Author's address:

David Carfi
Faculty Economics - University of Messina,
Via dei Verdi, 75, Messina, Italy.
E-mail: davidcarfi71@yahoo.it