ON M-PROJECTIVE CURVATURE TENSOR
OF A GENERALIZED SASAKIAN SPACE FORM

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Abstract. In the present paper, we have studied M-projectively flat generalized Sasakian space form, \( \eta \)-Einstein generalized Sasakian space form and irrotational M-projective curvature tensor on a Sasakian space form.

1. Introduction

A Riemannian manifold with constant sectional curvature \( C \) is known as real-space-form and its curvature tensor is given by

\[
R(X,Y)Z = C\{g(Y,Z)X - g(X,Z)Y\}.
\]

A Sasakian manifold \((M,\phi,\xi,\eta,g)\) is said to be a Sasakian space form [3], if all the \( \phi \)-sectional curvatures \( K(X \wedge \phi X) \) are equal to a constant \( C \), where \( K(X \wedge \phi X) \) denotes the sectional curvature of the section spanned by the unit vector field \( X \), orthogonal to \( \xi \) and \( \phi X \). In such a case, the Riemannian curvature tensor of \( M \) is given by

\[
R(X,Y)Z = \frac{C+3}{4}\{g(Y,Z)X - g(X,Z)Y\}
\]

\[
+ \frac{C-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}
\]

\[
+ \frac{C-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X
\]

\[
+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\.
\]

As a natural generalization of these manifolds, P. Alegre, D. E. Blair and A. Carriazo [3], [1] introduced the notion of generalized Sasakian space form.

Sasakian space form and Generalized Sasakian space form have been studied by several authors, viz., [3], [2], [6], [14], [10].

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In 1971, G. P. Pokhariyal and R. S. Mishra [13] defined a tensor field $W^*$ on a Riemannian manifold as

\begin{equation}
W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}(S(Y,Z)X - S(X,Z)Y) + g(Y,Z)QX - g(X,Z)QY
\end{equation}

Such a tensor field $W^*$ is known as $M$-projective curvature tensor.

The properties of the $M$-projective curvature tensor in Sasakian and Kaehler manifold were studied by R. H. Ojha [11] [12]. He showed that it bridges the gap between the conformal curvature tensor, coharmonic curvature tensor and concircular curvature tensor. S. K. Chaubey and R. H. Ojha [8] studied the properties of the $M$-projective curvature tensor in Riemannian and Kenmotsu manifold. S. K. Chaubey [9] also studied the properties of $M$-projective curvature tensor in LP-Sasakian manifold. C. S. Bagewadi, E. Girish Kumar and Venkatesha [4] studied irrotational $D$-conformal curvature tensor in Kenmotsu and trans-Sasakian manifolds. C. S. Bagewadi, Venkatesha and N. S. Basavarajappa [5] proved that if pseudo projective curvature tensor in a LP-Sasakian manifold is irrotational, then the manifold is Einstein. Motivated by these ideas, in the present paper, we made an attempt to study the properties of $M$-projective curvature tensor in generalized Sasakian space form. The present paper is organized as follows.

In Section 2, we review some preliminary results. In Section 3, we study $M$-projectively flat generalized Sasakian space form and obtain necessary and sufficient conditions for a generalized Sasakian space form to be $M$-projectively flat. And in Section 4, we study $\eta$-Einstein generalized Sasakian space form satisfying $W^*(\xi,X) \cdot R = 0$. Finally in Section 5, we prove that $M$-projective curvature tensor in an $\eta$-Einstein generalized Sasakian space form is irrotational if and only if $f_3 = \frac{3f_2}{(1-2n)}$.

2. Preliminaries

An odd-dimensional Riemannian manifold $(M,g)$ is called an almost contact manifold if there exists a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ on $M$, such that

\begin{align*}
\phi^2(X) &= -X + \eta(X)\xi, \\
\eta(\phi X) &= 0, \\
g(\phi X, \phi Y) &= g(X,Y) - \eta(X)\eta(Y), \\
\phi \xi &= 0, \quad \eta(\xi) = 0, \quad g(X,\xi) = \eta(X),
\end{align*}

for any vector fields $X, Y$ on $M$.

If in addition, $\xi$ is a Killing vector field, then $M$ is said to be a $K$-contact manifold. It is well known that a contact metric manifold is a $K$-contact manifold if and only if

\begin{equation}
(\nabla_X \xi) = -\phi(X)
\end{equation}

for any vector field $X$ on $M$. 
On the other hand, the almost contact metric structure on $M$ is said to be normal if $[\phi, \phi](X,Y) = -2d\eta(X,Y)\xi$ for any $X,Y$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of $\phi$ given by

$$[\phi, \phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

A normal contact metric manifold is called a Sasakian manifold. It can be proved that Sasakian manifold is $K$-contact, and that an almost contact metric manifold is Sasakian if and only if

$$\nabla_X(\phi(\eta(Y))X) = g(X,Y)\xi - \eta(Y)X.$$  \hfill (2.6)

Given an almost contact metric manifold $(M, \phi, \xi, \eta, g)$, we say that $M$ is an generalized Sasakian space form if there exists three functions $f_1, f_2$ and $f_3$ on $M$ such that

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}$$  \hfill (2.7)

for any vector fields $X, Y, Z$ on $M$, where $R$ denotes the curvature tensor of $M$. This kind of manifold appears as a natural generalization of the well-known Sasakian space form $M(C)$, which can be obtained as particular cases of generalized Sasakian space form by taking $f_1 = \frac{C+3}{4}$ and $f_2 = f_3 = \frac{C-1}{4}$.

Further in a $(2n + 1)$-dimensional generalized Sasakian space form, we have [1]

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi,$$  \hfill (2.8)

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y),$$  \hfill (2.9)

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3,$$  \hfill (2.10)

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y],$$  \hfill (2.11)

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X],$$  \hfill (2.12)

$$\eta(R(X,Y)Z) = (f_1 - f_3)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$$  \hfill (2.13)

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X).$$  \hfill (2.14)

3. $M$-projectively flat generalized Sasakian space form

For a $(2n + 1)$-dimensional $(n > 1)$ $M$-projectively flat generalized Sasakian space form, from (1.2), we have

$$R(X,Y)Z = \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$  \hfill (3.1)
In view of (2.8) and (2.9), the equation (3.1) takes the form

\[
R(X, Y)Z = \frac{1}{4n} \left[ 2(2nf_1 + 3f_2 - f_3)\{g(Y, Z)X - g(X, Z)Y\} \\
- (3f_2 + (2n - 1)f_3)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\
+ g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \right].
\]

(3.2)

Using (2.7), the equation (3.2) reduces to

\[
f_1\{g(Y, Z)X - g(X, Z)Y\} \\
+ f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi
\]

(3.3) = \frac{1}{4n} \left[ 2(2nf_1 + 3f_2 - f_3)\{g(Y, Z)X - g(X, Z)Y\} \\
- (3f_2 + (2n - 1)f_3)\{\eta(Y)\eta(Z)X - \eta(Y)\eta(Z)Y\} + g(Y, Z)\eta(Y)\xi - g(X, Z)\eta(Y)\xi \right].

Replacing Z by \(\phi Z\) in (3.3), we obtain

\[
f_1\{g(Y, \phi Z)X - g(X, \phi Z)Y\} \\
+ f_2\{g(X, \phi^2 Z)\phi Y - g(Y, \phi^2 Z)\phi X + 2g(X, \phi Y)\phi Z\} \\
+ f_3\{g(X, \phi Z)\eta(Y)\xi - g(Y, \phi Z)\eta(X)\xi
\]

(3.4) = \frac{1}{4n} \left[ 2(2nf_1 + 3f_2 - f_3)\{g(Y, \phi Z)X - g(X, \phi Z)Y\} \\
- (3f_2 + (2n - 1)f_3)\{g(Y, \phi Z)\eta(X)\xi - g(X, \phi Z)\eta(Y)\xi\} \right].

Putting X = \xi in (3.4), we get

\[
4nf_1g(Y, \phi Z)\xi - 4nf_3g(Y, \phi Z)\xi
\]

(3.5) = [4nf_1 + 3f_2 - (1 + 2n)f_3]g(Y, \phi Z)\xi.

Simplifying (3.5), we get

\[
[(1 - 2n)f_3 - 3f_2]g(Y, \phi Z)\xi = 0.
\]

(3.6) Since \(g(Y, \phi Z) \neq 0\), it follows from (3.6) that

\[
f_3 = \frac{3f_2}{(1 - 2n)}.
\]

(3.7) Conversely, suppose that

\[
f_3 = \frac{3f_2}{(1 - 2n)}
\]

holds. Then in view of (2.7) and (2.9), we can write the equation (1.2) as

\[
\hat{W}^* (X, Y, Z, W) = f_2\{g(X, \phi Z)g(\phi Y, W) - g(Y, \phi Z)g(\phi X, W) \\
+ 2g(X, \phi Y)g(\phi Z, W)\} + f_3\{\eta(X)\eta(Z)g(Y, W) \\
- \eta(Y)\eta(Z)g(X, W) + g(X, Z)\eta(Y)\eta(W) \\
- g(Y, Z)\eta(X)\eta(W) + g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}.
\]

(3.8)
where \( \hat{W}^*(X, Y, Z, W) = g(W^*(X, Y)Z, W) \).

Replacing \( X \) by \( \phi X \) and \( Y \) by \( \phi Y \) in (3.8), we get

\[
\hat{W}^*(\phi X, \phi Y, Z, W) = f_2\{g(\phi X, \phi Z)g(\phi^2 Y, W) - g(\phi Y, \phi Z)g(\phi^2 X, W) \\
+ 2g(\phi X, \phi^2 Y)g(\phi Z, W)\} + f_3\{g(\phi Y, Z)g(\phi X, W) \\
- g(\phi X, Z)g(\phi Y, W)\}.
\]

(3.9)

Putting \( Y = W = e_i \) where \( \{e_i\} \), is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over \( i \) (1 \( \leq i \leq 2n+1 \)), we get

\[
\sum_{i=1}^{2n+1} \hat{W}^*(\phi X, \phi e_i, Z, e_i) = f_2\{-g(\phi X, \phi Z)g(\phi e_i, \phi e_i) \\
+ g(\phi^2 Z, \phi^2 X) + 2g(\phi^2 X, \phi^2 Z)\} \\
- f_3g(\phi Z, \phi X).
\]

(3.10)

Putting \( X = Z = e_i \), where \( e_i \), is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over \( i \) (1 \( \leq i \leq 2n+1 \)), we get after simplification that \( f_2 = 0 \). But then \( f_3 = 0 \) by (3.7).

Therefore,

\[
R(X, Y)Z = f_1[g(Y, Z)X - g(X, Z)Y].
\]

(3.11)

The above equation gives

\[
S(X, Y) = 2nf_1g(X, Y).
\]

(3.12)

Hence in view of (1.2), we have \( W^*(X, Y)Z = 0 \). This leads us to state the following.

**Theorem 3.1.** A \((2n+1)\)-dimensional \((n > 1)\) generalized Sasakian space form is M-projectively flat if and only if \( f_3 = \frac{3f_2}{1-2n} \).

But in [14], the author proved that if a \((2n+1)\)-dimensional \((n > 1)\) generalized Sasakian space form is Ricci semisymmetric, then \( f_3 = \frac{3f_2}{1-2n} \). Hence we conclude the following.

**Corollary 3.1.** If a \((2n+1)\)-dimensional \((n > 1)\) generalized Sasakian space form is Ricci semisymmetric, then it is M-projectively flat.

4. An \( \eta \)-Einstein generalized Sasakian space form satisfying

\[
W^*(\xi, X)R = 0
\]

In view of (2.4), (2.8), (2.9) and (2.12), (1.2) becomes

\[
W^*(\xi, X)Y = \frac{1}{4n}\{[1 - 2n)f_3 - 3f_2](g(X, Y)\xi - \eta(Y)X)\}.
\]

(4.1)

Now we have

\[
(W^*(\xi, X)R)(Y, Z)U = W^*(\xi, X)R(Y, Z)U - R(W^*(\xi, X)Y, Z)U \\
- R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U.
\]

(4.2)
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But as we assume $W^*(\xi, X)R = 0$, (4.2) takes the form

\begin{equation}
\end{equation}

Using (2.4), (2.11), (2.12), (2.13) and (4.1) in (4.3), we get

\begin{equation}
\frac{1}{4n}[(1 - 2n)f_3 - 3f_2][\hat{R}(X, Y, Z, U)\xi + \eta(Y)R(X, Z)U + \eta(Z)R(Y, U)X + \eta(U)g(Y, Z)\eta(Y)X
- g(Y, U)\eta(Z)X + g(X, Y)g(Z, U)\xi - g(X, Y)\eta(U)Z
- g(X, Z)g(Y, U)\xi + g(X, Z)\eta(Y)Y + g(X, U)\eta(Z)Y
- g(X, U)\eta(Y)Z] = 0,
\end{equation}

where

\begin{equation}
\hat{R}(X, Y, Z, U) = g(X, R(Y, Z)U).
\end{equation}

Taking inner product of (4.4) with respect to the Riemannian metric $g$ and then using (2.4) and (2.13), we have

\begin{equation}
\frac{1}{4n}[(1 - 2n)f_3 - 3f_2][\hat{R}(X, Y, Z, U) - (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}] = 0.
\end{equation}

Then

\begin{equation}
f_3 = \frac{3f_2}{(1 - 2n)}
\end{equation}
or

\begin{equation}
\hat{R}(X, Y, Z, U) = (f_1 - f_3)\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U)\}.
\end{equation}

Using (2.4) and (4.5) in (4.7), we get

\begin{equation}
R(Y, Z)U = (f_1 - f_3)\{g(Z, U)Y - g(Y, U)Z\}.
\end{equation}

Contracting (4.8) with respect to the vector field $Y$, we find

\begin{equation}
S(Z, U) = 2n(f_1 - f_3)g(Z, U).
\end{equation}

Therefore,

\begin{equation}
QZ = 2n(2n + 1)(f_1 - f_3)Z.
\end{equation}

Hence,

\begin{equation}
r = 2n(2n + 1)(f_1 - f_3) \quad \text{and so} \quad f_3 = \frac{3f_2}{(1 - 2n)}.
\end{equation}

Thus, we state following theorem.

**Theorem 4.1.** A $(2n + 1)$-dimensional $(n > 1)$ $\eta$-Einstein generalized Sasakian space form satisfies the condition $W^*(\xi, X)R = 0$ if and only if $f_3 = \frac{3f_2}{(1 - 2n)}$.

In the light of Theorems 3.1 and 4.1, we state next corollary.
**Corollary 4.1.** A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi, X)R = 0$ if and only if it is $M$-projectively flat.

5. The irrotational $M$-projective curvature tensor

**Definition 5.1.** The rotation (curl) of $M$-projective curvature tensor $W^*$ on a Riemannian manifold is given by

\[
\text{Rot} W^* = (\nabla_U W^*)(X, Y)Z + (\nabla_X W^*)(U, Y)Z \\
+ (\nabla_Y W^*)(X, U)Z - (\nabla_Z W^*)(X, Y)U.
\]  

(5.1)

By virtue of second Bianchi identity, we have

\[
(\nabla_U W^*)(X, Y)Z + (\nabla_X W^*)(U, Y)Z + (\nabla_Y W^*)(X, U)Z = 0.
\]

Therefore, (5.1) becomes

\[
\text{Rot} W^* = - (\nabla_Z W^*)(X, Y)U.
\]  

(5.2)

If the $M$-projective curvature tensor is irrotational, then curl $W^* = 0$, and so by (5.2) we get

\[
(\nabla_Z W^*)(X, Y)U = 0.
\]  

(5.3)

Replacing $U = \xi$ in (5.3), we have

\[
(\nabla_Z W^*)(X, Y)\xi = W^*(\nabla_Z X, Y)\xi + W^*(X, \nabla_Z Y)\xi \\
+ W^*(X, Y)\nabla_Z \xi.
\]  

(5.4)

Now, substituting $Z = \xi$ in (1.2) and then using (2.4), (2.8), (2.11) and (2.14), we obtain

\[
(\nabla_Z W^*)(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],
\]

where

\[
k = \frac{1}{4n}[1 - 2n]f_3 - 3f_2.
\]  

(5.5)

Using (5.5) in (5.4), we obtain

\[
W^*(X, Y)\phi Z = k[g(Z, \phi X)Y - g(Z, \phi Y)X].
\]  

(5.7)

Replacing $Z$ by $\phi Z$ in (5.7) and simplifying by using (2.1) and (2.3), we get

\[
W^*(X, Y)Z = k[g(Z, Y)X - g(Z, X)Y].
\]  

(5.8)

Also equations (1.2) and (5.8) give

\[
k[g(Z, Y)X - g(Z, X)Y] = R(X, Y)Z - \frac{1}{4n}[S(Y, Z)X - S(X, Z)Y \\
+ g(Y, Z)QX - g(X, Z)QY].
\]  

(5.9)
Contracting the above equation with respect to the vector $X$ and then using (5.6), we find

\begin{equation}
S(Y, Z) = 2n(f_1 - f_3)g(Y, Z),
\end{equation}

which gives

\begin{equation}
r = 2n(2n + 1)(f_1 - f_3).
\end{equation}

In consequence of (1.2), (5.6), (5.8), (5.10) and (5.11) we can find

\begin{equation}
R(X, Y)Z = -(f_1 - f_3)[g(Y, Z)X - g(X, Z)Y].
\end{equation}

Therefore, we can state the following theorem.

**Theorem 5.1.** The $M$-projective curvature tensor in an $\eta$-Einstein generalized Sasakian space form is irrotational if and only if $f_3 = \frac{3}{1 - 2n}$.

Theorem 4.1 together with Theorem 5.1 lead to the following corollaries.

**Corollary 5.1.** A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space form satisfies the condition $W^*(\xi, X)R = 0$ if and only if the $M$-projective curvature tensor is irrotational.

**Corollary 5.2.** A $(2n + 1)$-dimensional $(n > 1)$ generalized Sasakian space form is irrotational if and only if it is $M$-projectively flat.

**References**


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