LYAPUNOV OPERATOR INEQUALITIES FOR EXPONENTIAL STABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS IN BANACH SPACES

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Abstract. In the present paper we prove a sufficient condition and a characterization for the stability of linear skew-product semiflows by using Lyapunov function in Banach spaces. These are generalizations of the results obtained in [1] and [12] for the case of \( C_0 \)-semigroups. Moreover, there are presented the discrete variants of the results mentioned above.

1. Introduction

The theorem of A. M. Lyapunov establishes that if \( A \) is a \( n \times n \) complex matrix then \( A \) has all its characteristics roots with real parts negative if and only if for any positive definite Hermitian matrix \( H \), there exists a positive definite Hermitian matrix \( W \) satisfying the equation

\[
A^*W + WA = -H
\]

(\( L_H \))

(\( * \) denotes the conjugate transpose of a matrix) (see [2]).

The use of the above Lyapunov operator equation is extended on the infinite-dimensional framework by Daleckij and Krein [4] for the case of semigroups \( T(t) = e^{tA} \), where \( A \) is a bounded linear operator. The authors prove in [4] that \( \{e^{tA}\}_{t \geq 0} \), with \( A \in \mathcal{B}(X) \) is exponentially stable if and only if there exists \( W \in \mathcal{B}(X) \), \( W >> 0 \) (i.e., there exists \( m > 0 \) such that \( \langle Wx, x \rangle \geq m\|x\|^2 \) for any \( x \in X \), solution of the Lyapunov equation \( A^*W + WA = -I \).

This result is extended by R. Datko [5], for the general case of \( C_0 \)-semigroups as it follows.

**Theorem 1.1** ([5]). A \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) is exponentially stable if and only if there exists \( W \in \mathcal{B}(X) \), \( W = W^* \), \( W \geq 0 \) such that

\[
(Ax, Wx) + (Wx, Ax) = -\|x\|^2
\]

(\( L \))

for all \( x \in D(A) \), where \( A \) denotes the infinitesimal generator of \( \{T(t)\}_{t \geq 0} \).
C. Chicone [3], Y. Latushkin [3], A. Pazy [9], J. Goldstein [6] and L. Pandolfi [8] studied the Lyapunov operator equations with unbounded $A$. All the above results are given in the setting of one-parameter semigroups acting on Hilbert spaces.

Moreover, in [10], an attempt to establish an equivalence between the solvability of the Lyapunov operator equation and the exponential stability of a $C_0$-semigroup in the general context of Banach spaces is presented.

Also in [12], C. Preda and P. Preda studied the case of the Lyapunov operator equation for the exponential stability of one-parameter semigroups acting on Banach spaces by using the idea of N.U. Ahmed (see [1]).

For the case of linear skew-product semiflows on real Hilbert spaces, a result which presents an equality of Lyapunov type can be found in [15]. In that paper, Pham Viet Hai and Le Ngoc Thanh present some characterizations for the uniform exponential stability of linear skew-product semiflows using a variant of Lyapunov equality.

Some necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows in Banach spaces are given in the paper [7]. The authors use Banach function spaces to obtain generalizations of some well-known results of Datko, Neerven, Rolewicz and Zabcyk.


In the present paper, we try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows acting on Banach spaces.

This paper extends for the case of linear skew-product semiflows the results obtained in [12] for the case of strongly continuous, one-parameter semigroups acting on Banach spaces by using analogous techniques.

In order to do that, we need to recall some notions about the adjoint of a linear operator on a Banach space.

Let $X$ be a real or complex Banach space and $X'$ its (dual) conjugate space consisting of all bounded and antilinear functionals on $X$. Also $X^*$ will denote the classic dual space of all bounded and linear functionals on $X$.

If $Y$ is also a Banach space, we will denote by $\mathcal{B}(X,Y)$ the Banach space of all linear and bounded operators from $X$ to $Y$. If $X = Y$, we will write $\mathcal{B}(X)$.

The norms on $X$, $X'$, $Y$ and $\mathcal{B}(X,Y)$ will be denoted by the symbol $\| \cdot \|$.

We will use the symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ to denote the set of real, nonnegative real and natural numbers respectively and $\mathbb{N}^* = \mathbb{N} - \{0\}$.

We will present some definitions in what follows.

Let $\Theta$ be a metric space.

**Definition 1.1.** A map $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ is said to be a continuous semiflow on $\Theta$ if the following conditions hold

i) $\sigma(\theta,0) = \theta$ for all $\theta \in \Theta$;
ii) $\sigma(\theta, t + s) = \sigma(\sigma(\theta, s), t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$;

iii) $(\theta, t) \mapsto \sigma(\theta, t)$ is continuous on $\Theta \times \mathbb{R}_+$.

If iii) holds for any $t, s \in \mathbb{R}$ then $\sigma$ is said to be a flow on $\Theta$.

**Definition 1.2.** Let $\sigma$ be a continuous semiflow on $\Theta$. A strongly continuous cocycle over the continuous semiflow $\sigma$ is an operator-valued function $\Phi: \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$, $(\theta, t) \mapsto \Phi(\theta, t)$ that satisfies the following properties

i) $\Phi(\theta, 0) = I$ (the identity operator on $X$) for all $\theta \in \Theta$;

ii) $(\theta, t) \mapsto \Phi(\theta, t)x$ is continuous for each $\theta \in \Theta$ and $x \in X$;

iii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$ (the cocycle identity);

If, in addition,

iv) there exist constants $M, \omega > 0$ such that $\|\Phi(\theta, t)\| \leq M e^{\omega t}$ for $t \geq 0$ and $\theta \in \Theta$,

then the strongly continuous cocycle is exponentially bounded.

**Definition 1.3.** The linear skew-product semiflow (LSPS) associated with the above cocycle is the dynamical system $\pi = (\Phi, \sigma)$ on $\varepsilon = X \times \Theta$ defined by

$$\pi: X \times \Theta \times \mathbb{R}_+ \to X \times \Theta, \quad (x, \theta, t) \mapsto \pi(x, \theta, t) = (\Phi(\theta, t)x, \sigma(\theta, t)).$$

We will give some examples of LSPS. First of all, we will define some notions used in the following examples.

**Definition 1.4.** A family $\{T(t)\}_{t \geq 0}$ of linear and bounded operators acting on $X$ is said to be a $C_0$-semigroup or a strongly continuous semigroup on $X$ if the following conditions hold:

i) $T(0) = I$;

ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$;

iii) there exists $\lim_{t \to t_+} T(t)x = x$ for all $x \in X$.

If the second property holds for any $t, s \in \mathbb{R}$, then $\{T(t)\}_{t \in \mathbb{R}}$ is called a $C_0$-group.

For a general presentation of the theory of $C_0$-semigroups, we refer the reader to [9].

**Definition 1.5.** A family of linear and bounded operators $\{U(t, s)\}_{t \geq s \geq 0}$ is said to be a two-parameter evolution family if the following conditions hold:

i) $U(t, t) = I$ for all $t \geq 0$;

ii) $U(t, t_0)U(t_0, s) = U(t, s)$ for all $t \geq t_0 \geq s \geq 0$;

iii) $U(\cdot, s)x$ is continuous on $[s, \infty)$ for all $s \geq 0$, $x \in X$;

iv) there exist $M, \omega > 0$ such that $\|U(t, s)\| \leq M e^{\omega(t-s)}$ for all $t \geq s \geq 0$.
For a general presentation of the theory of two-parameter evolution families, we refer the reader to [3] or [4].

**Example 1.1.** Let $Θ$ be a metric space, $σ$ a semiflow on $Θ$ and $\{T(t)\}_{t≥0}$ a $C_0$-semigroup on $X$. The pair $π_T = (Φ_T, σ)$ where $Φ_T(θ, t) = T(t)$, for all $(θ, t) ∈ Θ × R_+$ is a linear skew-product semiflow over $σ$ on $Θ × X$.

**Example 1.2.** Let $Θ = R_+, σ(θ, t) = θ + t$ and let $\{U(t, s)\}_{t≥s}$ be an evolution family on the Banach space $X$. We define $Φ_U(θ, t) = U(t + θ, θ)$ for all $(θ, t) ∈ Θ × R_+$. Then $\{Φ_U(θ, t)\}_{θ ∈ Θ, t≥0}$ is an exponentially bounded, strongly continuous cocycle (over the above semiflow $σ$) and the linear skew-product semiflow associated with it is the pair $π = (Φ_U, σ)$.

Therefore, we can say that the notion of a cocycle generalizes the classic notion of a two-parameter evolution family.

**Example 1.3.** Let $Θ$ be a metric space, $σ$ a semiflow on $Θ$, $X$ a Banach space and $A : Θ → B(X)$ a continuous mapping. The problem

\[
\begin{align*}
\dot{x}(t) &= A(σ(θ, t))x(t) \\
x(t_0) &= x_0
\end{align*}
\]

has an unique solution for all $t_0 ∈ R_+$ and $x_0 ∈ X$. For details we refer the reader to [13].

**Definition 1.6.** A linear skew-product semiflow (LSPS) $π = (Φ, σ)$ on a Banach bundle $ε = X × Θ$ is said to be exponentially stable if there exist constants $N, ν > 0$ such that

\[∥Φ(θ, t)x∥ ≤ Ne^{νt}∥x∥\] for all $t ≥ 0, θ ∈ Θ, x ∈ X$.

All the results concerning the Lyapunov inequality for the exponential stability of linear skew-product semiflows (LSPS), were acting on Hilbert spaces. We will try to go more general and find variants of Lyapunov operator equation for the exponential stability of linear skew-product semiflows (LSPS) acting on Banach spaces. This requires to recall some facts about the adjoint of a linear operator on a Banach space (see [12]).

**Definition 1.7.** Let $X, Y$ be two Banach spaces and $A ∈ B(X, Y)$. Then there exists an unique operator $A^* ∈ B(Y', X')$ that satisfies $y(Ax) = A^*y(x)$ for all $x ∈ X$ and $y ∈ Y'$. $A^*$ will be called the adjoint of $A$.

It can be easily checked that

- $∥A∥ = ∥A^*∥$;
- $(A + B)^* = A^* + B^*$;
- $(λA)^* = \overline{λ}A^*$;
- If $X, Y$ are reflexive, then $A^{**} = A$. 
It is worth to note that the above notion of the adjoint of a linear and bounded operator between two Banach spaces allows us to create a definition of the adjoint that directly generalizes the definition of the adjoint of an operator on Hilbert spaces. In other words, if $X$ and $Y$ are Hilbert spaces and $A \in \mathcal{B}(X,Y)$, then there is no difference of the adjoint between the adjoint $A^*$ defined by considering $X,Y$ to be Hilbert spaces, and the adjoint $A^*$ defined by considering $X,Y$ to be Banach spaces. If we chose that $A^*: Y^* \to X^*$, then we would obtain a different definition compared to the Hilbert space definition.

For defining the concept of a self-adjoint operator on a Banach space, we recall that $X$ is isomorphic and isometric with a subspace of $X''$.

**Definition 1.8.**

(i) An operator $A \in \mathcal{B}(X,X')$ is said to be self-adjoint if the restriction of $A^*$ to $X$ is $A$, and therefore,

$$Ay(x) = Ax(y) \quad \text{for all } x,y \in X.$$  

(ii) $A \in \mathcal{B}(X,X')$ is said to be positive if $A$ is self-adjoint and $Ax(x) \geq 0$ for all $x \in X$.

**Remark 1.1.** It is easy to see that $A \in \mathcal{B}(X,X')$ is positive if and only if $Ax(x)$ is a positive real number for all $x \in X$.

In the following we will denote by $\mathcal{B}^+(X,X') = \{ A \in \mathcal{B}(X,X') : A \text{ is positive} \}$.

Following Lyapunov’s idea, we obtain a Lyapunov-type operatorial equation for the case of linear skew-product semiflows acting on Banach spaces. Indeed, from the equation $(L^H)$ and $(L)$, taking into account the fact that any $C_0$-semigroup is a particular case of linear skew-product semiflows, we obtain for the case of Hilbert spaces that (see [15])

$$\langle A(\sigma(\theta,t))x, W(\sigma(\theta,t))x \rangle + \langle W(\sigma(\theta,t))x, A(\sigma(\theta,t))x \rangle = -\|x\|^2.$$  

If we assume that $(L^*)$ holds for some conditions, let $f$ be the function defined by

$$f(t) = \langle W(\sigma(\theta,t))\Phi(\theta,t)x, \Phi(\theta,t)x \rangle.$$  

It can be easily seen that $f'(t) = -\|\Phi(\theta,t)x\|^2$. Integrating with respect to $\tau$ on the interval $[0,t]$, we have

$$\langle W(\sigma(\theta,t))\Phi(\theta,t)x, \Phi(\theta,t)x \rangle - \langle W(\theta)x, x \rangle = -\int_0^t \|\Phi(\theta,\tau)x\|^2d\tau,$$

which implies

$$\Phi^*(\theta,t)W(\sigma(\theta,t))\Phi(\theta,t)x + \int_0^t \Phi^*(\theta,\tau)\Phi(\theta,\tau)x d\tau = W(\theta)x.$$
If we rewrite the equation above to the case of Banach spaces, using the considerations about the adjoint of an operator in Banach spaces, we have

\[(L')\quad W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) + \int_{0}^{t} \|\Phi(\theta,\tau)x\|^2 d\tau = W(\theta)x(x).\]

**Remark 1.2.** The bounded function \(W: \Theta \to B^+(X,X')\) from the equation \((L')\) is said Lyapunov function corresponding to linear skew-product semiflow \(\pi = (\Phi,\sigma)\).

2. Results

In what follows it will be presented a sufficient condition for the exponential stability of linear skew-product semiflows acting on Banach spaces in terms of Lyapunov inequality.

**Theorem 2.1.** Let \(\pi = (\Phi,\sigma)\) be a linear skew-product semiflow (LSPS). If there exists \(W: \Theta \to B^+(X,X')\) bounded such that

\[(1)\quad W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) + \int_{0}^{t} \|\Phi(\theta,\tau)x\|^2 d\tau \leq W(\theta)x(x)\]

for all \(t \geq 0, \theta \in \Theta\) and \(x \in X\), then \(\pi = (\Phi,\sigma)\) is exponentially stable.

**Proof.** Let \(x \in X, \theta \in \Theta\) and \(t \geq 0\). From (1) we have that

\[
\int_{0}^{t} \|\Phi(\theta,\tau)x\|^2 d\tau \leq W(\theta)x(x) - W(\sigma(\theta,t))\Phi(\theta,t)x(\Phi(\theta,t)x) \\
\leq W(\theta)x(x) = |W(\theta)x(x)| \leq K\|x\|^2
\]

for all \(\theta \in \Theta, x \in X\) and \(t \geq 0\), where \(K = \sup_{\theta \in \Theta} \|W(\theta)\| > 0\).

Thus we get that

\[
\int_{0}^{t} \|\Phi(\theta,\tau)x\|^2 d\tau \leq K\|x\|^2
\]

for all \(\theta \in \Theta, x \in X\) and \(t \geq 0\), which implies the following relation for \(t \to \infty\)

\[
\int_{0}^{\infty} \|\Phi(\theta,\tau)x\|^2 d\tau \leq K\|x\|^2 \quad \text{for all} \ \theta \in \Theta \ \text{and} \ x \in X.
\]

From [15, Lemma 2.4], it results that the linear skew-product semiflow \(\pi = (\Phi,\sigma)\) is exponentially stable. 

In what follows, it will be presented the necessary condition which needs a stronger hypothesis.
Theorem 2.2. Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$, for all $x \in X$, there exists $W : \Theta \rightarrow \mathcal{B}^+(X, X')$ bounded such that

$$W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau = W(\theta)x(x)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

Proof. The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable and therefore we have from Definition 1.6 that there exist the constants $N, \nu > 0$ such that

$$\|\Phi(\theta, t)x\| \leq Ne^{-\nu t}\|x\|$$

for all $t \geq 0$, $\theta \in \Theta$, $x \in X$.

Now we consider $x, y \in X$, $\theta \in \Theta$ and

$$W(\theta)x(y) = \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)d\tau.$$

Next we will show that $W \in \mathcal{B}^+(X, X')$.

Thus we have that

$$|W(\theta)x(y)| = \left| \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)d\tau \right| \leq \int_0^\infty |\Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)|d\tau$$

$$\leq \|\Gamma\| \int_0^\infty \|\Phi(\theta, \tau)x\|\|\Phi(\theta, \tau)y\|d\tau \leq \|\Gamma\| N^2 \int_0^\infty e^{-2\nu \tau} \|x\|\|y\|d\tau$$

$$= \frac{N^2}{2\nu} \|\Gamma\|\|x\|\|y\|,$$

which shows that $W$ is linear and bounded.

On the other hand,

$$W(\theta)y(x) = \int_0^\infty \Gamma(\Phi(\theta, \tau)y)(\Phi(\theta, \tau)x)d\tau$$

$$= \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)y)d\tau = W(\theta)x(y)$$

for all $x, y \in X$ and $\theta \in \Theta$. Thus, $W$ is self-adjoint.

Moreover,

$$W(\theta)x(x) = \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau \geq \gamma \int_0^\infty \|\Phi(\theta, \tau)x\|^2d\tau \geq 0,$$

which implies the fact that $W$ is positive.
It results that $W \in B^+(X, X')$. Now we have that

$$W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x)$$

$$= \int_0^\infty \Gamma(\Phi(\sigma(\theta, t), \tau)x)(\Phi(\sigma(\theta, t), \tau)\Phi(\theta, t)x)d\tau$$

$$= \int_0^\infty \Gamma(\Phi(\theta, t + \tau)x)(\Phi(\theta, t + \tau)x)d\tau$$

$$= \int_0^\infty \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau - \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau$$

$$= W(\theta)x(x) - \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau$$

and therefore, we get the relation (2) and the proof is complete. \[\square\]

As a result of the last two theorems, we now obtain the necessary and sufficient conditions for the exponential stability of a linear skew-product semiflow (LSPS) as follows.

**Corollary 2.1.** The linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable if and only if for all $\Gamma \in B^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma \|x\|^2$ for all $x \in X$, there exists $W : \mathbb{R}_+ \to B^+(X, X')$ bounded such that

$$W(\sigma(\theta, t))\Phi(\theta, t)x(\Phi(\theta, t)x) + t \int_0^t \Gamma(\Phi(\theta, \tau)x)(\Phi(\theta, \tau)x)d\tau = W(\theta)x(x)$$

for all $t \geq 0$, $\theta \in \Theta$ and $x \in X$.

**Proof.** Necessity results from Theorem 2.2.

Sufficiency results analogously with Theorem 2.1, by considering in addition $\Gamma \in B^+(X, X')$ with the same property as in Theorem 2.2. \[\square\]

In what follows we will also present the discrete versions of the above results.

**Theorem 2.3.** Let $\pi = (\Phi, \sigma)$ be linear skew-product semiflow. If there exists $W : \mathbb{N} \to B^+(X, X')$ bounded such that

$$W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \|\Phi(\theta, k)x\|^2 \leq W(\theta)x(x)$$

for all $\theta \in \Theta$, $n \in \mathbb{N}^*$ and $x \in X$, then the linear skew-product semiflow is exponentially stable.
Proof. We take $n \in \mathbb{N}^*$ and $x \in X$. From relation (4), we have that
\[
\sum_{k=0}^{n-1} \|\Phi(\theta,k)x\|^2 \leq W(\theta)x(x) - W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x) \\
\leq W(\theta)x(x) = |W(\theta)x(x)| \leq L\|x\|^2
\]
for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$, where $L = \sup_{\theta \in \Theta} \|W(\theta)\| > 0$.

For $n \to \infty$ in the previous relation we obtain that
\[
\sum_{k=0}^{\infty} \|\Phi(\theta,k)x\|^2 \leq L\|x\|^2 < \infty
\]
for all $\theta \in \Theta$ and $x \in X$.

Applying [15, Lemma 2.1 and Lemma 2.2], we get that the linear skew product semiflow $\pi = \Phi, \sigma$ is exponentially stable. \qed

The sufficient condition is given in the following theorem

**Theorem 2.4.** Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow (LSPS) exponentially stable. Then for all $\Gamma \in \mathcal{B}^+(X, X')$ with the property that there exists $\gamma > 0$ such that $\Gamma x(x) \geq \gamma\|x\|^2$ for all $x \in X$, there exists $W: \Theta \to \mathcal{B}^+(X, X')$ bounded such that
\[
W(\sigma(\theta,n))\Phi(\theta,n)x(\Phi(\theta,n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)x) = W(\theta)x(x)
\]
for all $n \in \mathbb{N}^*$, $\theta \in \Theta$ and $x \in X$.

Proof. As the linear skew-product semiflow $\pi = (\Phi, \sigma)$ is exponentially stable, we have from Definition 1.6 that there exist the constants $N$, $\nu > 0$ such that $\|\Phi(\theta,n)x\| \leq Ne^{-\nu n}\|x\|$, for all $n \in \mathbb{N} \theta \in \Theta$ and $x \in X$.

We take now $x, y \in X$, $n \in \mathbb{N}^*$ and
\[
W(\theta)x(y) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y).
\]
Next it will be shown that $W \in \mathcal{B}^+(X, X')$.

Therefore, we have that
\[
|W(\theta)x(y)| = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta,k)x)(\Phi(\theta,k)y) \leq \sum_{k=0}^{\infty} \|\Gamma\|\|\Phi(\theta,k)x\||\Phi(\theta,k)y|| \\
\leq \|\Gamma\|\sum_{k=0}^{\infty} \|\Phi(\theta,k)x\||\Phi(\theta,k)y|| \leq \|\Gamma\|N^2 \sum_{k=0}^{\infty} e^{-2\nu k}\|x\||y|| \\
\leq \frac{N^2}{1 - e^{-2\nu}}\|\Gamma\||x||y||,
\]
which shows that $W$ is linear and bounded.
Moreover,
\[
W(\theta)y(x) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)y)(\Phi(\theta, k)x)
\]
\[
= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)y) = W(\theta)x(y)
\]
for all \(x, y \in X\) and \(\theta \in \Theta\). Thus, \(W\) is self-adjoint.

On the other hand,
\[
W(\theta)x(x) = \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) \geq \gamma \sum_{k=0}^{\infty} \|\Phi(\theta, k)x\|^2 \geq 0,
\]
which implies the fact that \(W\) is positive.

It results that \(W \in B^+(X, X')\). Thus we have that
\[
W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x)
\]
\[
= \sum_{k=0}^{\infty} \Gamma(\Phi(\sigma(\theta, n), k)\Phi(\theta, n)x)(\Phi(\sigma(\theta, n), k)\Phi(\theta, n)x)
\]
\[
= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, n+k)x)(\Phi(\theta, n+k)x)
\]
\[
= \sum_{k=0}^{\infty} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x)
\]
\[
= W(\theta)x(x) - \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x)
\]
and therefore, we get the relation (5). \(\square\)

As a result of Theorems 2.3 and 2.4, it can be obtained the following corollary

**Corollary 2.2.** The linear skew-product semiflow \(\pi = (\Phi, \sigma)\) is exponentially stable if and only if for all \(\Gamma \in B^+(X, X')\) with the property that there exists \(\gamma > 0\) such that \(\Gamma x(x) \geq \gamma \|x\|^2\) for all \(x \in X\), there exists \(W : \mathbb{R}_+ \rightarrow B^+(X, X')\) bounded such that

\[
W(\sigma(\theta, n))\Phi(\theta, n)x(\Phi(\theta, n)x) + \sum_{k=0}^{n-1} \Gamma(\Phi(\theta, k)x)(\Phi(\theta, k)x) = W(\theta)x(x)
\]

for all \(n \in \mathbb{N}^*, \theta \in \Theta\) and \(x \in X\).

**Proof.** Necessity results from Theorem 2.4.
Sufficiency results analogously with Theorem 2.3. \(\square\)

**Remark 2.1.** As a conclusion, we can mention here that it is interesting to note that the sufficient condition can be easily obtained, but for the necessary condition, we need a stronger hypothesis. Thus, in terms of the the existence of \(\Gamma \in B^+(X, X')\) with the properties presented above, the exponential stability of
a linear skew-product semiflow implies the existence of a Lyapunov function that verifies the Lyapunov-type equation. Also, the sufficient condition holds in terms of the existence of $\Gamma \in B^+(X, X')$.

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