RESULTS ON DIMENSION THEORY AND SOME GENERALIZATIONS OF COMPACT SPACES

H. Z. HDEIB and K. Y. AL-ZOUBI

Abstract. In this paper we introduce $G_\delta$-sequential spaces as a generalization of sequential spaces, and obtain some product theorems for $[n,m]$-compact spaces and for spaces with large inductive dimension $\leq n$.

1. Introduction

Dimension theory dates back at least to the work of P. Urysohn [11] and K. Menger [8]. Since then many mathematicians have contributed to the development of this theory. There are three notions of dimension of a topological space $X$, small inductive dimension (denoted by $\text{ind}(X)$), large inductive dimension (denoted by $\text{Ind}(X)$) and covering dimension (denoted by $\text{dim}(X)$). If $\text{ind}(X) = 0$, then $X$ is called a zero-dimensional space. If $\text{dim}(X) = 0$, then $X$ is called a strongly zero-dimensional space.

In Section 2, we introduce $G_\delta$-sequential spaces as a generalization of sequential spaces, and obtain some product theorems for $[n,m]$-compact spaces and for spaces with large inductive dimension $\leq n$. Theorems 2.9, 2.10, 2.11, 2.13 and 2.17 formulate the main results of this paper. In this paper, all spaces are assumed to be $T_1$ topological spaces. For terminology not defined here, see Engelking [3] and Willard [12].

2. Product theorems

Franklin [4] introduced sequential spaces as generalization of first countable spaces. In this section, we define $G_\delta$-sequential spaces as a generalization of sequential spaces. We also obtain some product theorems for $[n,m]$-compact spaces and spaces with large inductive dimension $\leq n$.

Definition 2.1 ([4]). A subset $A$ of a space $X$ is called sequentially open if each sequence in $X$ converging to a point in $A$ is eventually in $A$. A space $X$ is called a sequential space if every sequentially open subset of $X$ is open.

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Definition 2.2. A space $X$ is called $G_δ$-sequential if every sequentially open subset is a $G_δ$-set.

Definition 2.3. Let $X$ be an arbitrary space. The $G_δ$-topology of $X$ is the topology generated by the $G_δ$-sets of $X$.

Definition 2.4 ([7]). A space $X$ is called scattered if every non-empty closed subset $A$ of $X$ has an isolated point.

Definition 2.5 ([1]). A space $X$ is called $[n,m]$-compact if every open cover $U$ of $X$ with $|U| \leq m$ has a subcover of cardinality $< n$. If $X$ is $[n,m]$-compact for all $m > n$, then it is called $[n,\infty]$-compact. $[\aleph_0, m]$-compact spaces will be called simply $m$-compact.

Definition 2.6 ([2]). A space $X$ is called paracompact if every open cover $U$ of $X$ has a locally finite open refinement.

Definition 2.7. A mapping $f$ from a space $X$ onto a space $Y$ is called $\sigma$-closed if $f$ maps closed sets onto $F_σ$-sets.

It is clear that every sequential space is $G_δ$-sequential. However a $G_δ$-sequential space may fail to be sequential (see Arens-Fort example [10, page 54]).

Kramer [6] showed that if $X$ is a sequential space and $Y$ is a countably compact space, then the projection mapping $P: X \times Y \rightarrow X$ is closed. A similar theorem concerning $\sigma$-closed mappings can be obtained using $G_δ$-sequential spaces. For this purpose we need the following lemma which can be obtained by modifying the proof of Kramer [6, Lemma 5.3].

Lemma 2.8. Let $X$ be a $G_δ$-sequential space and $Y$ be a countably compact space. Let $F$ be a closed subset of $X \times Y$ and $V$ be an open subset of $Y$. Let $x$ be a point of $X$ such that $F(x) = \{y \in Y \mid (x, y) \in F\} \subset V$. Then there is a $G_δ$-set $U$ containing $x$ such that $z \in U$ implies $F(z) \subset V$.

Theorem 2.9. Let $X$ be a $G_δ$-sequential space and $Y$ be a countably compact space. Then the projection mapping $P: X \times Y \rightarrow X$ is $\sigma$-closed.

The proof follows from Lemma 2.8 by taking $x \in X - P(F)$ and $V = \phi$.

Theorem 2.10. Let $f$ be a continuous $\sigma$-closed mapping from a space $X$ onto a space $Y$ such that $f^{-1}(y)$ is $m$-compact for each $y \in Y$. Then $X$ is $[n,m]$-compact if the $G_δ$-topology of $Y$ is so.

Proof. Let $U = \{U_\alpha \mid \alpha \in \Lambda\}$, $|\Lambda| \leq m$ be an open cover of $X$. Let $Γ$ denote the family of all finite subsets of $\Lambda$. Then $|Γ| \leq m$. Since $f^{-1}(y)$ is $m$-compact, we have that for each $y \in Y$, there exists a finite subset $\gamma$ of $\Lambda$ such that $f^{-1}(y) \subset \bigcup \{U_\alpha \mid \alpha \in \gamma\}$. Let $V_\gamma = Y - f(X - \bigcup_{\alpha \in \gamma} U_\alpha)$. Then $y \in V_\gamma$, $V_\gamma$ is a $G_δ$-set and $f^{-1}(V_\gamma) \subset \bigcup \{U_\alpha \mid \alpha \in \gamma\}$. Thus $\{V_\gamma \mid \gamma \in Γ\}$ cover of $Y$, of which each element is a $G_δ$-set, and $|Γ| \leq m$. Since the $G_δ$-topology of $Y$ is $[n,m]$-compact, $\{V_\gamma \mid \gamma \in Γ\}$ has a subcover of cardinality $< n$. Therefore $X$ is the union of less than $n$ members of $\{f^{-1}(V_\gamma) \mid \gamma \in Γ\}$. But for each $\gamma \in Γ$, the set $f^{-1}(V_\gamma)$ is contained in the union of finitely many members of $U$. Hence $X$ is $[n,m]$-compact. □
Theorem 2.11. Let $X$ be a scattered, paracompact Hausdorff space. Then the $G_δ$-topology of $X$ is paracompact.

Proof. Let $U$ be a cover of $X$ by $G_δ$-sets. Let

$$F = \{x \in X \mid x \in U \text{ and } U \text{ is open implies } U \text{ cannot be covered by a } \sigma\text{-locally finite open refinement of } U\}. $$

Obviously $F$ is closed. Suppose $F \neq \emptyset$. Since $X$ is scattered, $F$ has an isolated point $x$. Thus there exists an open set $V \subseteq X$ such that $V \cap F = \{x\}$. Choose $U^* \subseteq U$ such that $x \in U^*$. Without loss of generality we can assume that $U^* = \bigcap \{V_n \mid n = 1, 2, \ldots\}$ where $V_n$ is open for each $n = 1, 2, \ldots$, and $V_{n+1} \subseteq V_n \subseteq V$. For each $n = 1, 2, \ldots$, $(V_n - V_{n+1}) \subseteq X - F$. Therefore each $y \in (V_n - V_{n+1})$ has a neighborhood $M_y$ which can be covered by a $\sigma$-locally finite open refinement of $U$.

Now $\mathcal{M} = \{M_y \mid y \in (V_n - V_{n+1})\}$ is an open cover of $V_n - V_{n+1}$. Since $V_n - V_{n+1}$ is closed and $X$ is paracompact, $\mathcal{M}$ has a locally finite (in $X$) open (in $X$) refinement, say $\mathcal{H}_n = \{H_α \mid α ∈ \Lambda_n\}$. For each $α ∈ \Lambda_n$, $H_α$ is covered by a $\sigma$-locally finite open refinement of $U$, say $\bigcup_{i=1}^{\infty} A_α^i$. Let $B^*_i = \{H_α ∩ A \mid A \in A_α^i\}$ and $K^*_n = \{B \mid B ∈ B^*_i, α ∈ \Lambda_n\}$. Then $K^*_n$ is a locally finite open refinement of $U$, because if $x ∈ X$, there exists an open set $N_x$ such that $N_x ∩ H_α = \emptyset$ for all except finitely many indices, say $α_1, α_2, \ldots, α_m$. Each one of the collections $B^*_1, B^*_2, \ldots, B^*_n$ is locally finite. Hence for each $j = 1, 2, \ldots, n$, there exists an open set $W^*_j$ and each $W^*_j$ intersects at most finitely many members of $B^*_1$.

Hence $W^*_1 ∩ \ldots ∩ W^*_n$ is a locally finite open neighborhood of $x$ which intersects finitely many members of $K^*_n$.

Now $\bigcup_{i=1}^{\infty} K^*_i$ is an open $\sigma$-locally finite open refinement of $U$ which covers $V_n - V_{n+1}$. Consequently, $(\bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} K^*_n) \cup \{U^*\}$ is an open $\sigma$-locally finite open refinement of $U$ which covers $V$. This contradicts the fact that $x \in V$. Thus $F = \emptyset$. Therefore, for each $x ∈ V$, there is an open neighborhood $G_x$ of $x$ such that $G_x$ can be covered by a $\sigma$-locally finite open refinement of $U$. Since $X$ is paracompact, $\{G_x \mid x ∈ X\}$ has a locally finite open refinement $\{D_β \mid β ∈ Γ\}$ where for each $β ∈ Γ$, $D_β$ is covered by a $\sigma$-locally finite open refinement of $U$, say $\bigcup_{i=1}^{\infty} C^*_β$.

Let $G_β = \{C \mid C ∈ C^*_β, β ∈ Γ\}$. Then it is easy to see that $G_β$ is locally finite. Therefore $\bigcup_{i=1}^{\infty} G_β$ is a $\sigma$-locally finite open refinement of $U$ which covers $X$. Hence the $G_δ$-topology of $X$ is paracompact. \hfill $\square$

Theorem 2.12 ([5]). Let $X$ be an $[n,∞]$-compact scattered space. Then the $G_δ$-topology of $X$ is $[n,∞]$-compact.

The proof follows by a similar method used in Theorem 2.11.

Theorem 2.13. Let $Y$ be an $m$-compact space and $X$ be a $G_δ$-sequential scattered space. Then $X × Y$ is $[n, m]$-compact if $X$ is $[n,∞]$-compact.

Proof. By Theorem 2.9, the projection mapping $P : X × Y → X$ is closed. By Theorem 2.10, $X × Y$ is $[n, m]$-compact. \hfill $\square$
Definition 2.14. An open (closed) rectangle in \( X \times Y \) is a set of the form \( U \times V \) where \( U \) is an open (closed) subset of \( X \) and \( V \) is an open (closed) subset of \( Y \).

The following definition was introduced by Nagata [9] to study the dimension of the products.

Definition 2.15. Let \( X \) and \( Y \) be two spaces. Then the product space \( X \times Y \) is called an \( F \)-product if whenever \( H \) and \( K \) are disjoint closed sets in \( X \times Y \), then there is an open cover \( U = \{U_\alpha | \alpha \in \Lambda\} \) of \( X \times Y \) and a closed cover \( F = \{F_\alpha | \alpha \in \Lambda\} \) of \( X \times Y \) such that:

(i) \( F \) consists of closed rectangles and \( U \) consists of open rectangles.
(ii) \( U \) is \( \sigma \)-locally finite.
(iii) \( F_\alpha \subset U_\alpha \) for all \( \alpha \in \Lambda \).
(iv) \( U \) refines \( \{(X \times Y) - H, (X \times Y) - K\} \).

Kramer [6] proved that if \( X \) is sequential, paracompact and Hausdorff while \( Y \) is countably compact and normal, then \( X \times Y \) is an \( F \)-product.

In case \( X \) is a \( G_\delta \)-sequential space, we have the following theorems

Theorem 2.16. Let \( X \) be a \( G_\delta \)-sequential, paracompact, scattered and Hausdorff space. Let \( Y \) be a countably compact normal space. Then \( X \times Y \) is an \( F \)-product.

The proof follows from Theorem 2.11 and a similar technique used in the proof of the above Theorem of Kramer.

Nagata [9] showed that if \( X \) and \( Y \) are non-empty with \( \text{Ind}(X) \leq n \) while \( \text{Ind}(Y) \leq m \) and \( X \times Y \) is a totally normal \( F \)-product, then \( \text{Ind}(X \times Y) \leq n + m \).

Using this result together with Theorem 2.16, we get the following theorem.

Theorem 2.17. Suppose \( X \) and \( Y \) are given as in Theorem 2.16. If \( \text{Ind}(X) \leq n \), \( \text{Ind}(Y) \leq m \) and \( X \times Y \) is a totally normal, then \( \text{Ind}(X \times Y) \leq n + m \).

References


H. Z. Hdeib, Department of Mathematics, Faculty of science, University of Jordan, Amman-Jordan, e-mail: Hdeibza@ju.edu.sci.jo

K. Y. Al-Zoubi, Department of Mathematics, Faculty of science, Yarmouk University, Irbid-Jordan, e-mail: Khalidz@yu.edu.jo