A NOTE ON SPHERICAL F–TILINGS BY RIGHT TRIANGLES

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Abstract. In this paper we present some spherical f-tilings by two (distinct) right triangles. We classify the group of symmetries of the presented tilings and the transitivity classes of isohedrality are also determined. The combinatorial structure is given in Table 1.

1. INTRODUCTION

Let $S^2$ be the Riemannian sphere of radius 1. By a dihedral folding tiling (f-tiling for short) of the sphere $S^2$ whose prototiles are spherical right triangles, $T_1$ and $T_2$, we mean a polygonal subdivision $\tau$ of $S^2$ such that each cell (tile) of $\tau$ is congruent to $T_1$ or $T_2$, and the vertices of $\tau$ satisfy the angle-folding relation, i.e., each vertex of $\tau$ is of even valency and the sums of alternating angles around each vertex are $\pi$. In this paper we shall discuss dihedral f-tilings by two spherical right triangles.

F-tilings are intrinsically related to the theory of isometric foldings of Riemannian manifolds introduced by S. A. Robertson [4] in 1977. In fact, the set of singularities of any spherical isometric folding corresponds to a folding tiling of the sphere.

We shall denote by $\Omega(T_1,T_2)$ the set, up to an isomorphism, of all dihedral f-tilings of $S^2$ whose prototiles are $T_1$ and $T_2$. From now $T_1$ is a spherical right triangle of internal angles $\pi/2$, $\alpha$ and $\beta$ with edge lengths $a$ (opposite to $\beta$), $b$ (opposite to $\alpha$) and $c$ (opposite to $\pi/2$), and $T_2$ is a spherical right triangle of internal angles $\pi/2$, $\gamma$ and $\delta$ with edge lengths $d$ (opposite to $\delta$), $e$ (opposite to $\gamma$) and $f$ (opposite to $\pi/2$) (see Figure 1). We will assume throughout the text that $T_1$ and $T_2$ are distinct triangles, i.e., $(\alpha, \beta) \neq (\gamma, \delta)$ and $(\alpha, \beta) \neq (\delta, \gamma)$. The case $\alpha = \beta$ or $\gamma = \delta$ was analyzed in [2], and so we will assume further that $\alpha \neq \beta$ and $\gamma \neq \delta$.

It follows straightway that

\begin{equation}
\alpha + \beta > \frac{\pi}{2} \text{ and } \gamma + \delta > \frac{\pi}{2}.
\end{equation}

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A spherical isometry σ is a symmetry of a spherical tiling τ if σ maps every tile of τ into a tile of τ. The set of all symmetries of τ is a group under composition of maps denoted by G(τ). In this paper the group of symmetries of each f-tiling τ ∈ Ω(T₁, T₂) will also be presented.

We say that the tiles T and T' of τ are in the same transitivity class, if the symmetry group G(τ) contains a transformation that maps T into T'. If there are k transitivity classes of tiles, then τ is k-isohedral or k-tile-transitive.

If there are k transitivity classes of tiles, then τ is k-transitive or k-tile-transitive.

Dihedral f-tilings are k-isohedral for k ≥ 2. In this paper we also determine the transitivity classes of isohedrality of each presented tiling.

A fundamental region of τ is a part of S² as small or irredundant as possible which determines τ based on its symmetries. More precisely, the image of a point in S² under the symmetry group of τ forms an orbit of the action. A fundamental region of τ is a subset of S² which contains exactly one point from each orbit, therefore if |G(τ)| = n, then the area of a fundamental region of τ is \(4\pi n\).

It is well known that any spherical isometry is either a reflection, a rotation or a glide-reflection which consists of reflecting through some spherical great circle, and then rotating around the line containing the great circle by ±v'.

In what follows, \(R^\theta \), \(R^\phi \) and \(R^\psi \) denote the rotations through an angle \(\theta \) around the xx axis, yy axis and zz axis, respectively. The reflections on the coordinate planes xy, xz and yz are denoted, respectively, by \(\rho^x\), \(\rho^z\) and \(\rho^y\) with the notation used in [1]. For instance, it follows that \(R^\theta \rho^y R^\phi = R^\phi \rho^y R^\theta \), \(R^\phi R^\theta = R^\theta R^\phi\) and \(\rho^y \rho^z = \rho^z \rho^y = R^y \). Besides, \(2k\) is the smallest positive integer such that \((\rho^y R^\theta)^{2k} = id\).

The nth dihedral group \(D_n\) (group of symmetries of the planar regular n-gon) consists of n rotations and n symmetries (reflections). If a is a rotation of order n and b is a symmetry, then \((a, b : a^n = 1, b^2 = 1, ba = a^{n-1}b)\) is a group.

Figure 1. Prototiles: spherical right triangles T₁ and T₂.
A NOTE ON SPHERICAL F–TILINGS BY RIGHT TRIANGLES

presentation for $D_n$. Moreover, the elements $1, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b$ are pairwise disjoints.

In the next section we present some examples of f-tilings by right-triangles on a case of adjacency. The complete classification of all f-tilings by the considered prototiles is far from being achieved. We believe that this very hard study leads to infinite families of f-tilings (with discrete or continuous parameters) without no patterns, precisely due to the fact that both prototiles are right triangles. In contrast with other prototiles (consider, for instance, the rectangle illustrated in Figure 2(a) with $\alpha > \frac{\pi}{2}$ that cannot be subdivided in two tiles of the same family), the right triangles are special prototiles since they can be subdivided into two new right triangles (therefore within the same family of prototiles), and so on (see Figure 2(b)).

Figure 2. Prototiles subdivisions.

2. SOME EXAMPLES OF F–TILINGS BY RIGHT–TRIANGLES
ON A CASE OF ADJACENCY

We will suppose that any element of $\Omega (T_1, T_2)$ has at least two cells congruent, to $T_1$ and $T_2$, respectively, such that they are in adjacent positions as illustrated in Figure 3.

Figure 3. Case of adjacency.

As $a = e$, using spherical trigonometric formulas, we obtain

$$\frac{\cos \beta}{\sin \alpha} = \frac{\cos \gamma}{\sin \delta}$$

(2)
We will assume that all the edges of $T_1$ and $T_2$ are pairwise distinct (except $a$ and $e$). The study of right triangular spherical dihedral f–tilings on this case of adjacency, where the prototiles have two pairs of congruent sides, was already presented in [3].

In order to pursue any dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we start by considering one of its local configurations, beginning with a common vertex to two tiles of $\tau$ in adjacent positions, and then enumerating the following tiles according to the angles and edges relations until a complete f-tiling or an impossibility is achieved.

**Lemma 2.1.** With the previous assumptions vertex $v$ (Figure 3) has valency four.

**Proof.** Suppose that vertex $v$ has valency greater than four (Figure 4(a)). Then, if

- $\beta > \alpha$, we must have $\frac{\pi}{2} + \beta + k\gamma = \pi$ or $\frac{\pi}{2} + \beta + k\delta = \pi$ for some $k \geq 1$. In the first case an incompatibility between sides cannot be avoided around vertex $v$. In the last case we obtain $\frac{\pi}{2} + \beta + k\delta = \pi = \frac{\pi}{2} + \alpha + \gamma + (k-1)\delta$ which is not possible (observe that $\frac{\pi}{2} + \frac{\pi}{2} + \alpha + \beta + \gamma + \delta > 2\pi$).
- $\beta < \alpha$, one gets $\frac{\pi}{2} + k_1\beta + k_2\gamma = \pi$ or $\frac{\pi}{2} + k_1\beta + k_2\delta = \pi$ for some $k_1, k_2 \geq 1$. Analogously, in the first case an incompatibility between sides cannot be avoided around vertex $v$. In the last case we obtain $\frac{\pi}{2} + k_1\beta + k_2\delta = \pi = \frac{\pi}{2} + \alpha + \gamma + (k_1 - 1)\beta + (k_2 - 1)\delta$ which is not possible.

Therefore, vertex $v$ has valency four as illustrated in Figure 4(b). □

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Local configurations.

Now, we analyze the cases when the valency of the vertices $v_1, v_2, v_3$ and $v_4$ is four. As there was not imposed any strict order relation between the angles, it is enough to consider vertices $v_1$ and $v_2$, for instance. As previously mentioned, when the valency of these vertices is greater than four, the study is not complete and we only present some examples of f–tilings.
Proposition 2.2. Let $T_1$ and $T_2$ be spherical right triangles such that they are in adjacent positions as illustrated in Figure 4(b). If at least one of the vertices $v_1$ and $v_2$ has valency four, then $\Omega(Q,T) \neq \emptyset$ iff $\alpha + \delta = \pi$ and $\beta = \gamma = \frac{\pi}{2}$ for some $k \geq 3$. In this case, for each $k \geq 3$, there is a family of f-tilings denoted by $\mathcal{R}_k^\alpha$ with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$. Planar and 3D representations are given in Figure 6.

Proof. Suppose that any element of $\Omega(T_1,T_2)$, has at least two cells congruent, respectively, to $T_1$ and $T_2$, such that they are in adjacent positions as illustrated in Figure 4(b).

1. If $v_1$ has valency four, taking into account the edge lengths, we must have $\alpha + \delta = \pi$ and the last configuration is extended to the one illustrated in Figure 5. Repeating the same argument, we get $\beta = \gamma = \frac{\pi}{k}$ with $k \geq 3$. The extension of the last configuration gives rise to the “closed” planar representation, see Figure 6(a). We denote this family of f-tilings by $\mathcal{R}_k^\alpha$ with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$ and $k \geq 3$. A 3D representation of $\mathcal{R}_k^\alpha$ for $k = 3$, is given in Figure 6(b).

In the planar representation, the dark region corresponds to a fundamental region of $\mathcal{R}_3^\alpha$. In fact, any symmetry of this f-tiling fixes the north vertex $N$; the symmetries that fix $N$ are the rotations $\text{id} = R_0^\alpha$, $R_{2\pi}^\alpha$, $R_{4\pi}^\alpha$, and the reflections $\rho^{\pi\gamma}$, $\rho_1 = \rho^{\pi\gamma} \circ R_{2\pi}^\alpha$, $\rho_2 = \rho^{\pi\gamma} \circ R_{4\pi}^\alpha$, and so the symmetry group of $\mathcal{R}_3^\alpha$ is isomorphic to $D_3$ generated by $R_{2\pi}^\alpha$ and $\rho^{\pi\gamma}$. Similarly, we prove that $G(\mathcal{R}_k^\alpha)$ is isomorphic.
It follows immediately that $R^k_\alpha$ is 2-isohedral with respect to the symmetry group.

2. Suppose now that $v_2$ has valency four. As $\gamma \neq \frac{\pi}{2}$ (otherwise $a = f$), it follows that $\delta + \gamma = \pi$ (see Figure 7). At vertex $v_1$ we must have $\gamma + k\alpha = \pi$, $k \geq 1$. However, an incompatibility between sides cannot be avoided around this vertex for all $k$. $\square$

Figure 6. f-tilings $R^k_\alpha$, with $\alpha \in \left(\frac{(k-2)\pi}{2k}, \frac{(k+2)\pi}{2k}\right)$ and $k \geq 3$. 

Figure 7. Local configuration.
Figure 8. f-tiling $\mathcal{D}_a$. (a) Planar representation of $\mathcal{D}_a$. (b) 3D representation of $\mathcal{D}_a$. 
Given the reasons referred earlier, when the valency of all the vertices \( v_i \), \( i = 1, 2, 3, 4 \), is greater than four, we present examples of \( f \)-tilings satisfying:

(i) \( \alpha + \delta = \frac{\pi}{2}, \beta = \frac{\pi}{4}, \gamma = \frac{\pi}{4} \) and \( \delta = \arctan \sqrt{2}, \)

(ii) \( \alpha + 2\delta = \pi, \gamma = \frac{\pi}{2k} \) and \( \beta = \frac{\pi}{2}, k = 2, 3, \)

(iii) \( \delta + \alpha + \gamma = \pi, \beta = \frac{\pi}{2} \) and \( \alpha \in \left( \frac{\pi}{6}, \frac{\pi}{2} \right), \)

(iv) \( \alpha + 3\delta = \pi, \gamma = \frac{\pi}{4} \) and \( \beta = \frac{\pi}{2}. \)

In the first case we consider three \( f \)-tilings, say \( D_a, D_b \) and \( D_c. \)

In Figure 8 a planar and 3D representations of \( D_a \) are given. The dark region in the planar representation corresponds to a fundamental region of \( D_a \). In fact, the dark line corresponding to a great circle composed by 12 segments of length \( \frac{\pi}{6} \) is invariant under any symmetry. Thus, any symmetry of \( D_a \) fixes \( N \) or maps \( N \) into \( S \). The symmetries that fix \( N \) are generated by the rotation \( R^{\bar{c}}_a \) and the symmetries that send \( N \) into \( S \) are \( R^z_a, R^{\bar{c}z}_a = R^z_a \circ R^{\bar{c}}_a \) and \( R^{\bar{c}2z}_a = R^z_a \circ R^{\bar{c}2}_a. \)

Note that \( (R^i_n)^2 = \text{id}, i = 1, 2, \) and the symmetry group of \( D_a \) is \( G(D_a) = \langle R^z_a, R^c_a \rangle \cong D_3; D_b \) is 16-isohedral (8 transitivity classes of triangles \( T_1 \) and 8 transitivity classes of triangles \( T_2 \)).

The \( f \)-tilings \( D_b \) and \( D_c \) (Figure 9 and Figure 10) are obtained from \( D_a \) rotating the southern hemisphere \( \frac{\pi}{2} \) for the left and the right, respectively.

A planar and 3D representations of \( D_b \) are illustrated in Figure 9. The three great circles \( x = 0, y = 0 \) and \( z = 0 \) depicted in the planar representation have 8 segments of length \( \frac{\pi}{4} \) and there exist exactly six vertices surrounded by 8 angles \( \gamma \) (say \( N, S, W, E, C, L \)). The symmetries that fix vertex \( N \) are generated by \( R^z_a \) and \( \rho^{yz} \) with 8 symmetries (for the other vertices the analysis is similar). And so the symmetry group contains 48 symmetries. It follows immediately that \( G(D_b) \) is isomorphic to \( C_2 \times S_4 \), the octahedral group, and \( D_b \) is 2-isohedral.

Concerning to the tiling \( D_c \), the dark line (corresponding to the equator) is invariant under any symmetry. Thus, any symmetry of \( D_c \) fixes \( N \) (and \( S \)) or maps \( N \) into \( S \) (and vice-versa). The symmetries that fix this points are generated by the rotation \( R^{\bar{c}}_a \) and the reflection \( \rho^{yz} \). On the other hand, the reflection \( \rho^{yz} \) sends \( N \) into \( S \) and commutes with the previous symmetries. It follows that \( G(D_c) \) is isomorphic to \( C_2 \times D_3 \) and \( D_c \) is 8-isohedral. Finally, the dark region in the planar representation of \( D_c \) corresponds to a fundamental region of this tiling.

In the case (ii) we consider four \( f \)-tilings denoted by \( L, M, Q^2 \) and \( Q^3 \).

A planar representation of \( L \) is illustrated in Figure 11(a). One has \( \beta = \frac{\pi}{4}, \gamma = \frac{\pi}{4}, \delta = \arccos \frac{\sqrt{2}}{4} \approx 69.3^\circ \) and \( \alpha = \pi - 2\delta \approx 41.4^\circ. \) Its 3D representation is given in Figure 11(b).

Similarly to the previous case, any symmetry of \( L \) fixes \( N \) or maps \( N \) into \( S \). The symmetries that fix \( N \) (and \( S \)) are generated, for instance, by the rotation \( R^{\bar{c}2z}_a \) and the reflection \( \rho^{yz} \) giving rise to a subgroup \( G \) of \( G(L) \) isomorphic to \( D_3 \). To obtain the symmetries that send \( N \) into \( S \), it is enough to compose each
element of $G$ with $a = R_{x}^{2}\rho^{y}$. Now, one has

$$a^{5}\rho^{yz} = R_{x}^{z\rho^{y}}\rho^{y}\rho^{yz} = R_{x}^{z\rho^{y}}R_{z}^{y} = R_{z}^{y} = \rho^{y}\rho^{y}\rho^{y}\rho^{y}\rho^{y} = \rho^{y}a.$$
Moreover, \(|a| = 6\) and \(\rho^{\pi z} \notin <a>\). Therefore, \(<a, \rho^{\pi z}> = G(L) \cong D_6\) and \(L\) is 2-isohedral. Observe that \(R_{\pi z} \in G(L)\); in fact, \(R_{\pi z} = \rho^{\pi z} R_{2\pi} \circ R_{\pi z} \rho^{\pi y}\).
A planar representation of $\mathcal{M}$ is illustrated in Figure 12(a). One has $\beta = \frac{\pi}{5}$, $\gamma = \frac{\pi}{5}$, $\delta = \arccos \left( 4 \cos \frac{\pi}{5} \right) \approx 74.3^\circ$ and $\alpha = \pi - 2\delta \approx 31.4^\circ$. Its 3D representation is given in Figure 12(b).

$G(\mathcal{M})$ contains a subgroup $S$ isomorphic to $D_4$ generated by $R_2^\pi$ and $\rho^{\nu_2}$. On the other hand, $a = \rho^{2\nu_2}R_2^\pi$ is also a symmetry of $\mathcal{M}$ that maps $N$ into $S$. Since $a$ has order 4, then $G(\mathcal{M})$ is isomorphic to $D_8$ generated by $a$ and $\rho^{\nu_2}$. Finally, $\mathcal{M}$ is 9-isohedral.

We illustrate planar and 3D representations of $Q^2$ and $Q^3$ in Figure 13 and Figure 14, respectively. One has $\beta = \frac{\pi}{5}$, $\gamma = \frac{\pi}{5}$, $\delta = \arccos \left( 4 \cos \frac{\pi}{5} \right)$ and $\alpha = \pi - 2\delta$, $k = 2, 3$.

The great circle $x = 0$ depicted in the planar representation of $Q^2$ has 6 segments of length $\frac{\pi}{3}$. Any symmetry of $Q^2$ fixes $C$ or maps $C$ into $L$. As before, the symmetries that fix $C$ are generated, for instance, by the rotation $R_2^{2\nu_2}$ and the reflection $\rho^{2\nu_2}$, and so $G(Q^2)$ contains a subgroup $G$ isomorphic to $D_3$. In order to obtain all the symmetries that send $C$ into $L$, it is enough to compose each element of $G$ with $\rho^{2\nu_2}$ which commutes with all elements of $G$. And so $G(Q^2) \cong C_2 \times D_3$. It follows immediately that $Q^2$ is 3-isohedral.

The tiling $Q^3$ has exactly four vertices surrounded by 12 angles $\gamma$ and denoted by $v_i$, $i = 1, 2, 3, 4$. Any symmetry of $Q^3$ that sends $v_i$ into $v_j$, $i \neq j$, consists of a reflection on the great circle containing the remaining vertices. On the other hand, the symmetries of $Q^3$ fixing one of these four vertices form a subgroup $G$
isomorphic to $D_3$. Thus, $G(Q^3)$ contains exactly 24 symmetries. Now, we easily conclude that $G(Q^3)$ is the group of all symmetries of the regular tetrahedron or the group of all permutations of four objects, $S_4$. Finally, $Q^3$ is 3-tile-transitive with respect to this group.
In the case (iii) we consider a family of $f$-tilings denoted by $G_\alpha$, $\alpha \in (\pi/6, \pi/3)$. The corresponding planar and 3D representations are illustrated in Figure 15. Due to the condition (2) we have $\alpha = \gamma$, and so $\delta = \pi - 2\alpha$.

$G(G_\alpha)$ contains a subgroup $G$ isomorphic to $D_3$ generated by $R_{2\pi/3}$ and $\rho^{\pi/2}$. On the other hand, $a = \rho^{\pi/2}R_{2\pi/3}$ is a symmetry of $G_\alpha$ that maps $N$ into $S$. Similarly to some previous cases, we conclude that $G(G_\alpha)$ is isomorphic to $D_6$ generated by $a$ and $\rho^{\pi/2}$. Finally, $G_\alpha$ is 2-isohedral.

In the last case we consider an $f$-tiling denoted by $C$ whose planar and 3D representations are presented in Figure 16. One has $\beta = \pi/3$, $\gamma = \pi/4$, $\delta = \frac{1}{2} \arccos \sqrt{\frac{3}{4}} \approx 49.2^\circ$ and $\alpha = \pi - 3\delta \approx 32.4^\circ$.

The dark line in the planar representation corresponds to the equator that is invariant under any symmetry of $C$. With the labeling used in this figure, the symmetries that fix $N$ are generated, for instance, by the rotation $R_{\frac{2\pi}{3}}$ and the reflection $\rho^{\pi/2}$ giving rise to a subgroup $G$ of $G(C)$ isomorphic to $D_4$. On the other hand, $a = \rho^{\pi/2}R_{2\pi/3}$ is also a symmetry of $C$ that maps $N$ into $S$. The symmetry group of $C$ is then $G(C) = <a, \rho^{\pi/2}> \simeq D_8$, $C$ is 12-isohedral.
Figure 14. f-tiling $Q^3$.

(a) Planar representation of $Q^3$
(b) 3D representation of $Q^3$

Figure 15. f-tiling $G_a$, $\alpha \in (\frac{\pi}{6}, \frac{\pi}{3})$. 

(a) Planar representation of $G_a$, $\alpha \in (\frac{\pi}{6}, \frac{\pi}{3})$
(b) 3D representation of $G_a$, $\alpha \in (\frac{\pi}{6}, \frac{\pi}{3})$
Figure 16. f-tiling $C$. 

(a) Planar representation of $C$

(b) 3D representation of $C$
3. Summary

In Table 1 is shown a list of the presented spherical dihedral f-tilings whose prototiles are spherical right triangles, $T_1$ and $T_2$, of internal angles $\frac{\pi}{2}$, $\alpha$, $\beta$, and $\frac{\pi}{2}$, $\gamma$, $\delta$, respectively, in the case of adjacency illustrated in Figure 3. Our notation is as follows:

- $|V|$ is the number of distinct classes of congruent vertices,
- $N_1$ and $N_2$, respectively, are the number of triangles congruent to $T_1$ and $T_2$, respectively, used in each tiling.
- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega(T_1,T_2)$; by $C_n$ we mean the cyclic group of order $n$; $D_n$ is the dihedral group of order $2n$; $S_4$ is the group of all permutations of four distinct objects; the octahedral group is $O_h \cong C_2 \times S_4$ (the symmetry group of the cube).
- $I(\tau)$ corresponds to the number of isohedrality classes of each tiling $\tau \in \Omega(T_1,T_2)$.

The distinct classes of congruent vertices of each f-tiling are illustrated in Figure 17.

The distinct classes of congruent vertices of each f-tiling are illustrated in Figure 17.
A NOTE ON SPHERICAL F–TILINGS BY RIGHT TRIANGLES

Table 1. Combinatorial structure of some dihedral f–tilings of $S^2$ by right triangles on the case of adjacency of Figure 3.

| f-tiling | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $|V|$ | $N_1$ | $N_2$ | $G(\tau)$ | $I(\tau)$ |
|----------|----------|----------|----------|----------|-------|-------|-------|-----------|-----------|
| $R^k_n$  | $d(k - 2)\pi / (k + 2)\pi$ | $\pi / k$ | $\pi / k$ | $\pi - \alpha$ | 4     | 2k    | 2k    | $D_k$     | 2         |
| $D_a$    | $\pi / 2 - \delta$ | $\pi / 3$ | $\pi / 4$ | $\arctan \sqrt{2}$ | 5     | 48    | 48    | $D_3$    | 16        |
| $D_h$    | $\pi / 2 - \delta$ | $\pi / 3$ | $\pi / 4$ | $\arctan \sqrt{2}$ | 5     | 48    | 48    | $C_2 \times S_4$ | 2         |
| $D_c$    | $\pi / 2 - \delta$ | $\pi / 3$ | $\pi / 4$ | $\arctan \sqrt{2}$ | 6     | 48    | 48    | $C_2 \times D_3$ | 8         |
| $L$      | $\pi - 2\delta$ | $\pi / 3$ | $\pi / 4$ | $\arccos \sqrt{2} / 4$ | 4     | 12    | 24    | $D_6$    | 2         |
| $M$      | $\pi - 2\delta$ | $\pi / 3$ | $\pi / 4$ | $\arccos \left(4 \cos \frac{\pi}{8}\right)$ | 5     | 48    | 96    | $D_8$    | 9         |
| $Q^2$    | $\pi - 2\delta$ | $\pi / 3$ | $\pi / 4$ | $\arccos \left(4 \cos \frac{\pi}{4}\right)$ | 4     | 12    | 24    | $C_2 \times D_3$ | 3         |
| $Q^3$    | $\pi - 2\delta$ | $\pi / 3$ | $\pi / 4$ | $\arccos \left(4 \cos \frac{\pi}{6}\right)$ | 4     | 24    | 48    | $S_4$    | 3         |
| $C_\alpha$ | $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ | $\pi / 3$ | $\pi / 4$ | $\pi - 2\alpha$ | 3     | 12    | 12    | $D_6$    | 2         |
| $C$      | $\pi - 3\delta$ | $\pi / 3$ | $\pi / 4$ | $\frac{1}{2} \arccos \frac{\sqrt{2} - 2}{4}$ | 5     | 48    | 144   | $D_8$    | 12        |

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