ASSOCIATED PRIMES OF TOP LOCAL HOMOLOGY MODULES WITH RESPECT TO AN IDEAL

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Abstract. Let \((R, \mathfrak{m})\) be a local ring, \(a\) be an ideal of \(R\) and \(M\) be a non-zero Artinian \(R\)-module with \(\text{Ndim}_R M = n\). In this paper we determine the associated primes of the top local homology module \(H^a_n(M)\).

1. Introduction

Throughout this paper assume that \((R, \mathfrak{m})\) is a commutative Noetherian local ring, \(a\) is an ideal of \(R\) and \(M\) is an \(R\)-module. In [2] Cuong and Nam defined the local homology modules \(H^a_i(M)\) with respect to \(a\) by

\[ H^a_i(M) = \lim_{\to} \text{Tor}_R^i(R/\mathfrak{a}^n, M). \]

This definition is dual to Grothendieck’s definition of local cohomology modules and coincides with the definition of Greenless and May in [6] for an Artinian \(R\)-module \(M\). For basic results about local homology we refer the reader to [2, 3] and [13]; for local cohomology see [1].

In [8] Macdonald and Sharp studied the top local cohomology module with respect to the maximal ideal and showed that \(\text{Att}(H^\mathfrak{m}_n(N)) = \{ p \in \text{Ass} N : \dim R/p = n \}\), where \(N\) is a finitely generated \(R\)-module of dimension \(n\). Cuong and Nam proved in [2] a dual result stating that

\[ \text{Ass}_R(H^\mathfrak{m}_n(M)) = \{ p \in \text{Att}_R(M) : \dim R/p = d \} \]

for a non-zero Artinian \(R\)-module \(M\) of Noetherian dimension \(d\). In this paper we study the top local homology module \(H^a_n(M)\), where \(M\) is a non-zero Artinian \(R\)-module of Noetherian dimension \(n\) and \(a\) is an arbitrary ideal of \(R\). The module \(H^a_n(M)\) is called a top local homology module because \(\max\{ i : H^a_i(M) \neq 0 \} \leq n \) by [2, Proposition 4.8].

A non-zero \(R\)-module \(M\) is called secondary if the multiplication map by any element \(a\) of \(R\) is either surjective or nilpotent. A secondary representation of the \(R\)-module \(M\) is an expression for \(M\) as a finite sum of secondary modules. If such a representation exists, we will say that \(M\) is representable. A prime ideal \(p\) of \(R\)
is said to be an attached prime of $M$ if $p = (N :_R M)$ for some submodule $N$ of $M$. If $M$ admits a reduced secondary representation $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\text{Att}_R(M)$ of $M$ is equal to $\{\sqrt{0:_R S_i} : i = 1, \ldots, n\}$. Note that every Artinian $R$-module $M$ is representable and minimal elements of the set $V(\text{Ann}(M))$, the set of prime ideals of $R$ containing ideal $\text{Ann}(M)$, belong to $\text{Att}(M)$. It is well known that if $N$ is a submodule of Artinian $R$-module $M$, then $\text{Att}(M/N) \subseteq \text{Att}(M) \subseteq \text{Att}(N) \cup \text{Att}(M/N)$ (See [9, Section 6]).

We now recall the concept of Noetherian dimension $\text{Ndim}_R(M)$ of an $R$-module $M$. For $M = 0$ we define $\text{Ndim}_R(M) = -1$. Then by induction, for any integer $t \geq 0$, we define $\text{Ndim}_R(M) = t$ when

i) $\text{Ndim}_R(M) < t$ is false, and

ii) for every ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of $M$ there exists an integer $m_0$ such that $\text{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geq m_0$.

Thus $M$ is non-zero and finitely generated if and only if $\text{Ndim}_R(M) = 0$. If $M$ is Artinian module, then $\text{Ndim}_R(M) < \infty$. (For more details see [7] and [11]).

Following [5], for any $R$-module $M$, we define the cohomological dimension of $M$ with respect to $a$ as

$$\text{cd}(a, M) = \max\{i : H^i_a(M) \neq 0\}.$$ 

By [1, Theorem 6.1.2 and Theorem 6.1.4], we have $\text{cd}(a, M) \leq \dim M$ and $\text{cd}(m, M) = \dim M$. We will call

$$\text{hd}(a, M) := \max\{i : H^i_a(M) \neq 0\}$$

the homological dimension of $M$ with respect to $a$. It follows from [2, Propositions 4.8 and 4.10] that if $M$ is an Artinian $R$-module, then $\text{hd}(a, M) \leq \text{Ndim}_R(M)$ and $\text{hd}(m, M) = \text{Ndim}_R(M)$.

Throughout the paper, for an $R$-module $M$, $E(R/m)$ denotes the injective envelope of $R/m$ and $D(.)$ denotes the Matlis duality functor $\text{Hom}_R(., E(R/m))$. It is well known that $\dim D(M) = \dim M$. Also, if $M$ is an Artinian $R$-module, then $M \simeq D D(M)$ and $D(M)$ is a Noetherian $\hat{R}$-module. (See [1, Theorem 10.2.19] and [10, Theorem 1.6(5)]).

Note that if $M$ is an Artinian $R$-module, then $H^i_a(M) \simeq D(H^i_D(M))$ for all $i$ (See [2, Proposition 3.3(ii)]), and therefore $\text{hd}(a, M) = \text{cd}(a, D(M))$. Thus $\text{hd}(a, M) \leq \dim D(M) = \dim M$.

The main result of this paper shows that if $M$ is a non-zero Artinian $R$-module such that $\text{Ndim}_R(M) = n$, then

$$\text{Ass}_R(H^n_a(M)) = \{\mathfrak{p} \cap R : \mathfrak{p} \in \text{Att}_R M \text{ and } \text{cd}(a \hat{R}, \hat{R}/\mathfrak{p}) = n\}.$$ 

2. THE RESULTS

To prove our main result, we need the following lemmas.

**Lemma 2.1.** Let $(R, m)$ be a local ring, $a$ be an ideal of $R$ and $0 \to L \to M \to N \to 0$ be an exact sequence of Artinian $R$-modules. Then $\text{hd}(a, M) = \text{Max}\{\text{hd}(a, L), \text{hd}(a, N)\}$.
Proof. Since $D(M)$ is Noetherian $\hat{R}$-module, by \cite[Corollary 2.3(i)]{5}, $\cd(aR, D(M)) \leq \cd(a, D(M))$. Hence by the Independence Theorem (\cite[Theorem 4.2.1]{1}), $\cd(a, D(N)) \leq \cd(a, D(M))$. Therefore $\hd(a, N) \leq \hd(a, M)$. From the long exact sequence

$$
\mathcal{H}^{i+1}_a(L) \rightarrow \mathcal{H}^i_a(M) \rightarrow \mathcal{H}^i_a(N) \rightarrow \mathcal{H}^i_a(L) \rightarrow \mathcal{H}^i_a(M) \rightarrow \ldots
$$

we deduce that $\hd(a, L) \leq \hd(a, M)$. Hence $\Max\{\hd(a, L), \hd(a, N)\} \leq \hd(a, M)$. From the above long exact sequence we also infer that $\hd(a, M) \leq \Max\{\hd(a, L), \hd(a, N)\}$ and the proof is complete.

Lemma 2.2. Let $(R, m)$ be a complete local ring, $a$ be an ideal of $R$ and $M$ be a non-zero Artinian module. Then $\cd(a, R/p) \leq \hd(a, M)$ for all $p \in \Att(M)$.

Proof. Since $D(M)$ is a Noetherian $R$-module and $\Supp(R/p) \subseteq \Supp(D(M))$ for all $p \in \Ass D(M)$, by \cite[Theorem 2.2]{5} we infer that $\cd(a, R/p) \leq \cd(a, D(M))$ for all $p \in \Ass D(M)$. Since $\Att(M) = \Ass D(M)$ and $\cd(a, D(M)) = \hd(a, M)$, we obtain $\cd(a, R/p) \leq \hd(a, M)$ for all $p \in \Att(M)$.

Lemma 2.3. Let $(R, m)$ be a local ring, $a$ be an ideal of $R$ and $M$ be an Artinian $R$-module. Then $\hd(a, M) \leq \cd(a, R/\Ann M)$.

Proof. Let $\hat{R} := R/\Ann M$. By \cite[Theorem 3.3]{12}, $\mathcal{H}^i_a(M) \simeq \mathcal{H}^{\hat{a}R_i}_a(M)$ for all $i$. Thus $\hd(a, M) = \hd(aR, M)$. Since $\hd(aR, M) \leq \cd(aR, \hat{R})$ (see \cite[Corollary 3.2]{6}) and $\cd(aR, \hat{R}) = \cd(a, \hat{R})$ (see \cite[Lemma 2.1]{5}), we conclude that $\hd(a, M) \leq \cd(a, \hat{R})$.

Lemma 2.4. Let $(R, m)$ be a complete local ring, $a$ be an ideal of $R$ and $M$ be a non-zero Artinian module of dimension $n$ with $\hd(a, M) = n$. Then the set

$$
\Sigma := \{N' : N' \text{ is a submodule of } M \text{ and } \hd(a, M/N') < n\}
$$

has a smallest element $N$. The module $N$ has the following properties:

i) $\hd(a, N) = \dim N = n$.

ii) $N$ has no proper submodule $L$ such that $\hd(a, N/L) < n$.

iii) $\Att(N) = \{p \in \Att(M) : \cd(a, R/p) = n\}$.

iv) $\mathcal{H}^i_a(N) \simeq \mathcal{H}^i_a(M)$.

Proof. It is clear that $M \in \Sigma$ and thus $\Sigma$ is not empty. Since $M$ is an Artinian $R$-module, the set $\Sigma$ has a minimal member $N$. By Lemma 2.1, if $N_1, N_2 \in \Sigma$, then $\hd(a, M/N_1 \cap N_2) < n$. Since the intersection of any two members of $\Sigma$ is again in $\Sigma$, it follows that $N$ is contained in every member of $\Sigma$ implying that $N$ is the smallest element of $\Sigma$.

i) Since $\hd(a, M/N) < n$, from the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ and Lemma 2.1 we obtain $\hd(a, N) = n$. From $n = \hd(a, N) \leq \dim N \leq \dim M = n$ we derive $\dim N = n$.

ii) Suppose that $L$ is a submodule of $N$ such that $\hd(a, N/L) < n$. From the exact sequence

$$
0 \rightarrow N/L \rightarrow M/L \rightarrow M/N \rightarrow 0
$$

we deduce that $\hd(a, L) \leq \hd(a, M)$. Hence $\Max\{\hd(a, L), \hd(a, N)\} \leq \hd(a, M)$. From the above long exact sequence we also infer that $\hd(a, M) \leq \Max\{\hd(a, L), \hd(a, N)\}$ and the proof is complete.
and Lemma 2.1 we infer \( \text{hd}(a, M/L) < n \). Hence \( L \in \Sigma \) and \( L = N \).

iii) If \( p \in \text{Att}(N) \), then \( p = \text{Ann}(N/L) \), where \( L \) is a submodule of \( N \). By (ii), \( \text{hd}(a, N/L) = n \). Hence \( n = \text{hd}(a, N/L) \leq \dim R/p \leq \dim(M) = n \). Thus \( \dim(R/p) = \dim(M) \). Since \( \dim(M) = \dim(R/\text{Ann}(M)) \), we conclude that \( p \) is a minimal element of the set \( \mathcal{V}(\text{Ann}(M)) \). Thus \( p \in \text{Att}(M) \).

On the other hand, using Lemma 2.3, we derive \( n = \text{hd}(a, N/L) \leq \text{cd}(a, R/p) \leq \dim(R/p) \leq \dim(M) = n \). Therefore \( \text{cd}(a, R/p) = n \).

Now suppose that \( p \in \text{Att}(M) \) and \( \text{cd}(a, R/p) = n \). Since \( \text{hd}(a, M/N) < n \) and \( \text{cd}(a, R/p) = n \), Lemma 2.2 implies that \( p \notin \text{Att}(M/N) \). Therefore \( p \in \text{Att}(N) \).

iv) The exact sequence \( 0 \to N \to M \to M/N \to 0 \) induces the exact sequence

\[
H^{n+1}_{a}(M/N) \to H_{a}^{n}(N) \to H_{a}^{n}(M) \to H_{a}^{n}(M/N) \to .
\]

Since \( \text{hd}(a, M/N) < n \), \( H^{n+1}_{a}(M/N) = H_{a}^{n}(M/N) = 0 \). Therefore \( H_{a}^{n}(N) \cong H_{a}^{n}(M) \).

**Theorem 2.5.** Let \( (R, m) \) be a complete local ring, \( a \) be an ideal of \( R \) and \( M \) be a non-zero Artinian module of dimension \( n \). Then

\[
\text{Ass}(H_{a}^{n}(M)) = \{ p \in \text{Att}(M) : \text{cd}(a, R/p) = n \}. 
\]

**Proof.** If \( n = 0 \), then \( M \) has a finite length and therefore \( a^{k}M = 0 \) for some \( k \in \mathbb{N} \). Hence

\[
\text{Ass}(H_{a}^{n}(M)) = \{ m \} = \text{Att}(M) = \{ p \in \text{Att}(M) : \text{cd}(a, R/p) = 0 \}. 
\]

Thus we can assume that \( n > 0 \). If \( H_{a}^{n}(M) = 0 \), then \( \text{hd}(a, M) < n \). Hence by Lemma 2.2 \( \text{cd}(a, R/p) < n \) for all \( p \in \text{Att}(M) \). This implies \( \{ p \in \text{Att}(M) : \text{cd}(a, R/p) = n \} = \emptyset = \text{Ass}(H_{a}^{n}(M)) \) and the result has been proved in this case.

Now assume that \( n > 0 \) and \( H_{a}^{n}(M) \neq 0 \). Then \( \text{hd}(a, M) = \dim(M) = n \). By Lemma 2.4, we can assume that \( M \) has no proper submodule \( L \) with \( \text{hd}(a, M/L) < n \) and we must show that \( \text{Ass}(H_{a}^{n}(M)) = \text{Att}(M) \).

If \( r \notin \cup_{p \in \text{Att}(M)} p \), then the exact sequence \( 0 \to (0 : M) r \to M \to M/r \to 0 \) induces the exact sequence \( H^{n}_{a}(0 : M) r \to H^{n}_{a}(M) \to H^{n}_{a}(M/r) \). Using [3, Lemma 4.7], we obtain \( \text{Ndim}_{R}(0 : M/r) \leq n - 1 \), and therefore \( H^{n}_{a}(0 : M/r) = 0 \). Since \( 0 \to H^{n}_{a}(M) \to H^{n}_{a}(M/r) \) is exact, we infer \( r \notin \cup_{p \in \text{Ass}(H^{n}_{a}(M))} p \). Since \( \text{Att}(M) \) is a finite set, every \( p \in \text{Ass}(H^{n}_{a}(M)) \) is included in some \( q \in \text{Att}(M) \). For such \( q \) there exists a submodule \( L \) of \( M \) satisfying \( q = \text{Ann}(M/L) \). Hence \( n = \text{hd}(a, M/L) \leq \dim(M/L) \leq \dim R/q \leq \dim R/p \leq n \). This shows \( q = p \) and \( \text{Ass}(H^{n}_{a}(M)) \subseteq \text{Att}(M) \).

To prove the reverse inclusion, assume \( p \in \text{Att}(M) \). There exists a submodule \( L \) of \( M \) such that \( \text{Att}(L) = \{ p \} \). Since we have assumed that \( M \) has no proper submodule \( U \) with \( \text{hd}(a, M/U) < n \), Lemma 2.4 implies that \( \text{cd}(a, R/p) = n \). Hence by Lemma 2.2, we have \( \text{hd}(a, L) = n \) and \( H^{n}_{a}(L) \neq 0 \). Since \( \text{cd}(a, R/p) = n \) and \( \text{Att}(L/U) \subseteq \text{Att}(L) = \{ p \} \) for all submodules \( U \), Lemma 2.2 shows that \( L \) cannot have any proper submodule \( U \) such that \( \text{hd}(a, L/U) < n \). Analogously as above, we obtain \( \text{Ass}(H^{n}_{a}(L)) \subseteq \text{Att}(L) = \{ p \} \). Since \( H^{n}_{a}(L) \neq 0 \), we establish that \( \text{Ass}(H^{n}_{a}(L)) = \{ p \} \). However, from the exact sequence \( 0 \to H^{n}_{a}(L) \to H^{n}_{a}(M) \to . \)
that completes the proof. □

**Corollary 2.6.** Let $(R, m)$ be a complete local ring, $a$ be an ideal of $R$ and $M$ be a non-zero Artinian module of dimension $n$. Then

\[ \text{Ass}(H^n(M)) = \{ p \in \text{Att}(M) : \dim(R/p) = n \} \]

**Proof.** Since $\text{cd}(m, R/p) = \dim R/p$, it follows from Theorem 2.5. □

The following Theorem is the main result of this paper.

**Theorem 2.7.** Let $(R, m)$ be a local ring, $a$ be an ideal of $R$ and $M$ be a non-zero Artinian $R$-module with $\text{Ndim}_RM = n$. Then

\[ \text{Ass}_R(H^n(M)) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Att}_R M \text{ and } \text{cd}(aR, \tilde{R}/\mathfrak{P}) = n \} \]

**Proof.** Since $\dim_{\tilde{R}}D(M) = \dim_{\tilde{R}}M = \text{Ndim}_RM = n$ (for details consult [4]), by [1, Theorem 7.1.6], $H^n_{aR}(D(M))$ is an Artinian local cohomology module and $D(H^n_{aR}(D(M))) \simeq H^n_{aR}(M)$ is a Noetherian $\tilde{R}$-module. It is well known that $\text{Ass}_R(L) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Ass}_{\tilde{R}} L \}$ for each finitely generated $\tilde{R}$-module $L$ (See [9, Exercise 6.7]). Thus $\text{Ass}_R(H^n_{aR}(M)) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Ass}_{\tilde{R}}(H^n_{aR}(M)) \}$. Since by [13, Proposition 4.3], $H^n_{aR}(M) \simeq H^n_{aR}(M)$ as $R$-modules, we conclude that $\text{Ass}_R(H^n_{aR}(M)) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Ass}_{\tilde{R}}(H^n_{aR}(M)) \}$. According to Theorem 2.5, $\text{Ass}_R(H^n_{aR}(M)) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Att}_R M \text{ and } \text{cd}(aR, \tilde{R}/\mathfrak{P}) = n \}$. Therefore $\text{Ass}_R(H^n_{aR}(M)) = \{ \mathfrak{P} \cap R : \mathfrak{P} \in \text{Att}_R M \text{ and } \text{cd}(aR, \tilde{R}/\mathfrak{P}) = n \}$. □

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**References**


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