THE CESÁRO $\chi^2$ SEQUENCE SPACES DEFINED BY A MODULUS

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Abstract. In this paper we define the Cesáro $\chi^2$ sequence space $\text{Ces}_p^q \left( \chi^2_f \right)$ defined by a modulus and exhibit some general properties of the space.

1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

An initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19] and many others.

Let us define the following sets of double sequences:

\[ \mathcal{M}_u (t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \]

\[ \mathcal{C}_p (t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\}, \]

\[ \mathcal{C}_0p (t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\}, \]

\[ \mathcal{L}_u (t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}. \]

\[ \mathcal{C}_0p (t) := \mathcal{C}_p (t) \cap \mathcal{M}_u (t) \quad \text{and} \quad \mathcal{C}_0bp (t) := \mathcal{C}_0p (t) \cap \mathcal{M}_u (t) \]

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim’s sense. In case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, $\mathcal{M}_u (t)$, $\mathcal{C}_p (t)$, $\mathcal{C}_0p (t)$, $\mathcal{L}_u (t)$, $\mathcal{C}_0p (t)$ and $\mathcal{C}_0bp (t)$ are reduced to the sets

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Theorem and introduced the $M$-core for double sequences and determined those four dimensional matrices consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$, and $L_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha$-duals of the spaces $BS$, $BV$, $CS_{bp}$ and the $\beta$-duals of the spaces $CS_{bp}$ and $CS_r$ of double series. Further Basar and Sever [28] introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $\chi_M^2 (p, q, u)$ of double sequences and gave some inclusion relations.

Spaces that are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A$-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections among strong $A$-summability, strong $A$-summability with respect to a modulus, and $A$-statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]–[38] and [39] the four dimensional matrix transformation $(A x)_{k,l} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m+n} x_{mn}$ was studied extensively by Robison and Hamilton. This will be accomplished by presenting the following sequence spaces:

$$
\text{Ces}_p^q (\chi_M^2) = d (x, 0)
$$

$$
= \left\{ x \in \chi^2 := \lim_{m,n \to \infty} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \left| (m+n) x_{mn} \right| \right)^{\frac{1}{p_m n}} \right)^{\frac{1}{p_m n}} \right\} = 0
$$

and

$$
\text{Ces}_p^q (\Lambda_M^2) = d (x, 0)
$$

$$
= \left\{ x \in \chi^2 := \sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \left| x_{mn} \right| \right)^{\frac{1}{p_m n}} \right)^{\frac{1}{p_m n}} \right) < \infty \right\}
$$
where $f$ is a modulus function. Other implications, general properties and variations will also be presented.

In the sequel of the paper we need the following inequality

$$(a + b)^p \leq a^p + b^p$$

for $a, b, \geq 0$ and $0 < p < 1$. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$.

The vector space of all double analytic sequences will be denoted by $\Lambda$. The double sequence whose only non-zero term is a $\delta = 1$ is called double gai sequence if $(\delta_{m,n})$ is convergent, where $\delta_{m,n}$ denotes the double sequence whose only non-zero term is a $\frac{1}{m+n}$ in the $(i,j)$th place for each $i, j \in \mathbb{N}$.

An FDK-space (or a metric space) $X$ is said to have AK property if $(\mathbb{N})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \rightarrow x$.

An FK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \rightarrow (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail and proved that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_2(1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripa thy et al. [18], Rao and Subramanian [15] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$. If convexity of Orlicz function $M$ is replaced by subadditivity of $M$, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$ if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The $\Delta_2$-condition is equivalent to $M(\ell u) \leq K\ell M(u)$ for all values of $u$ and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$
The space $\ell_M$ with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$, $(1 \leq p < \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$.

If $X$ is a sequence space, we give the following definitions:

(i) $X'$ is the continuous dual of $X$;

(ii) $X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \text{ for each } x \in X \right\}$;

(iii) $X^\beta = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \text{ is convergent, for each } x \in X \right\}$;

(iv) $X^\gamma = \left\{ a = (a_{mn}) : \sup_{m,n} \left( \sum_{m,n=1}^{MN} a_{mn} x_{mn} \right) < \infty, \text{ for each } x \in X \right\}$;

(v) let $X$ be an FK-space $\supset \phi$; then $X^f = \left\{ f(3_{mn}) : f \in X' \right\}$;

(vi) $X^\delta = \left\{ a = (a_{mn}) : \sup_{m,n} |a_{mn} x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \right\}$.

$X^\alpha, X^\beta, X^\gamma$ are called $\alpha$- (or K"othe-Toeplitz)-dual of $X$, $\beta$- (or generalized- K"othe-Toeplitz)-dual of $X$, $\gamma$-dual of $X$, $\delta$-dual of $X$, respectively. $X^\alpha$ was defined by Gupta and Kamptan [20]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \left\{ x = (x_k) \in w : (\Delta x_k) \in Z \right\}$$

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here $c, c_0$ and $\ell_\infty$ denote the classes of convergent, null and bounded scalar valued single sequences, respectively. The difference space $\ell_{p\Delta}$ was introduced and studied in the case $1 \leq p \leq \infty$ by Basar and Altay in [42] and in the case $0 < p < 1$ by Altay and Basar in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and $\ell_{p\Delta}$ are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{L_{p\Delta}} = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \quad (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2$, $\chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. 
2. Definitions and Preliminaries

Ces$_p^q\left(\chi^2_f\right)$ and Ces$_p^q\left(\Lambda^2_f\right)$ denote the Pringsheim's sense of Cesàro double gai sequence space of modulus and Pringsheim's sense of Cesàro double analytic sequence space of modulus, respectively.

Definition 2.1. A modulus function was introduced by Nakano [12]. We recall that a modulus $f$ is a function from $[0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$,
3. $f$ is increasing,
4. $f$ is continuous from the right at $0$. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from here that $f$ is continuous on $[0, \infty)$.

Definition 2.2. Let $A = (a_{mn}^{kl})$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $k, \ell$-th term to $Ax$ is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{kl} x_{mn}.$$ Such transformation is said to be nonnegative if $a_{mn}^{kl}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is $P$-convergent is not necessarily bounded.

Definition 2.3. Let $p \in [1, \infty)$ and $q$ be a double gai sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i} \sum_{n=0}^{j} q_{mn}, \quad i, j \in \mathbb{N},$$

Ces$_p^q\left(\chi^2_f\right) = d(x, 0)$

$$\left\{ x \in \chi^2 : = \lim_{m,n \to \infty} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \left( (m+n) |x_{mn}| \right) \frac{1}{m+n} \right) \right)^{\frac{1}{m+n}} = 0 \right\}$$

If $q_{mn} = 1$ for all $m, n \in \mathbb{N}$, then Ces$_p^q\left(\chi^2_f\right)$ reduces to Ces$_p\left(\chi^2_f\right)$, and if $f(x) = x$, then Ces$_p^q\left(\chi^2_f\right)$ reduces to Ces$_p^q\left(\chi^2\right)$.

Definition 2.4. Let $p \in [1, \infty)$ and $q$ be a double analytic sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i} \sum_{n=0}^{j} q_{mn}, \quad i, j \in \mathbb{N},$$

Ces$_p^q\left(\Lambda^2_f\right)$ denotes the Pringsheim's sense of Cesàro double analytic sequence space of modulus.
\[ \text{Ces}_p^q \left( \Lambda^2 f \right) = d(x,0) \]
\[ = \left\{ x \in \chi^2 \colon \sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \left( |x_{mn}| \right)^{\frac{1}{p_m n}} \right) \right)^{p_m n} \right) \right\} < \infty \]
\[
\leq (1 - \beta) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \frac{(m + n)! |x_{mn}|}{\beta} \right) \right)^{p_{mn}} + \beta \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \frac{(m + n)! |y_{mn}|}{\beta} \right) \right)^{p_{mn}}
\]

\[
\leq \frac{1}{\beta} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \frac{(m + n)! |x_{mn}|}{\beta} \right) \right)^{p_{mn}} + \frac{\beta}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} 2q_{mn} f \left( \frac{(m + n)! |y_{mn}|}{\beta} \right) \right)^{p_{mn}}
\]

\[
\leq d(x, 0)^{p_{mn}} + \frac{\varepsilon}{2} + \frac{\beta}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \frac{(m + n)! |y_{mn}|}{\beta} \right) \right)^{p_{mn}}
\]

\[
\leq d(x, 0)^{p_{mn}} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\]

\[
\leq d(x, 0)^{p_{mn}} + \varepsilon.
\]

**Proposition 3.2.** For every \( p = (p_{mn}) \),

\[
\left[ \text{Ces}_{\beta}^{q} \left( \Lambda^{2}_{f} \right) \right]^{\beta} = \left[ \text{Ces}_{p}^{q} \left( \Lambda^{2}_{f} \right) \right]^{\alpha} = \left[ \text{Ces}_{p}^{q} \left( \Lambda^{2}_{f} \right) \right]^{\gamma} = \left[ \text{Ces}_{p}^{q} \left( \eta^{2}_{f} \right) \right]^{\beta},
\]

where

\[
\left[ \text{Ces}_{p}^{q} \left( \eta^{2}_{f} \right) \right] = \bigcap_{N \in \mathbb{N} - \{1\}} \left\{ x = x_{mn} : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( \frac{|x_{mn}| N^{\frac{m+n}{p_{mn}}}}{p_{mn}} \right) \right)^{1/p_{mn}} < \infty \right\}.
\]

**Proof.** First we show that \( \left[ \text{Ces}_{p}^{q} \left( \eta^{2}_{f} \right) \right] \subset \left[ \text{Ces}_{p}^{q} \left( \Lambda^{2}_{f} \right) \right]^{\beta} \).
Let \( x \in \left[ Ces_p^q (\eta_1^2) \right] \) and \( y \in \left[ Ces_p^q (\Lambda^2_j) \right]^\beta \). Then we can find a positive integer \( N \) such that
\[
\left( |y_{mn}|^{1/(m+n)} \right)
\]
\[
< \min \left( 1, \sup_{m, n \geq 1} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( |y_{mn}|^{\frac{1}{m+n}} \right) \right)^{\frac{\beta}{mn}} \right) < N
\]
for all \( m, n \).

Hence we may write
\[
\left| \sum_{m, n} x_{mn} y_{mn} \right| \leq \sum_{m, n} |x_{mn} y_{mn}| \leq \sum_{m, n} (f (|x_{mn}|)) \leq \sum_{m, n} \left( f (|x_{mn}| N^{m+n}) \right).
\]

Since \( x \in Ces_p^q (\eta_1^2) \). The series on the right side of the above inequality is convergent, whence \( x \in Ces_p^q (\Lambda^2_j) \). Hence \( [Ces_p^q (\eta_1^2)] \subset [Ces_p^q (\Lambda^2_j)]^\beta \).

Now we show that \( [Ces_p^q (\Lambda^2_j)]^\beta \subset [Ces_p^q (\eta_1^2)] \).

For this, let \( x \in Ces_p^q (\Lambda^2_j)^\beta \) and suppose that \( x \notin Ces_p^q (\Lambda^2_j) \). Then there exists a positive integer \( N > 1 \) such that \( \sum_{m, n} (f (|x_{mn}| N^{m+n})) = \infty \).

If we define \( y_{mn} = N^{m+n} \text{sgn} x_{mn} \), \( m, n = 1, 2, \cdots \), then \( y \in Ces_p^q (\Lambda^2_j) \).

But, since
\[
\left| \sum_{m, n} x_{mn} y_{mn} \right| = \sum_{m, n} (f (|x_{mn}|)) = \sum_{m, n} (f (|x_{mn}| N^{m+n})) = \infty,
\]
we get \( x \notin Ces_p^q (\Lambda^2_j)^\beta \), which contradicts the assumption \( x \in Ces_p^q (\Lambda^2_j)^\beta \).

Therefore \( x \in Ces_p^q (\eta_1^2) \) and \( [Ces_p^q (\Lambda^2_j)]^\beta = [Ces_p^q (\eta_1^2)] \).

(ii) and (iii) can be shown in a similar way of (i). Therefore, we omit it. \( \square \)

**Proposition 3.3.** Let \( p = (p_{mn}) \) be a Cesàro space of double analytic modulus sequence of strictly positive real numbers \( p_{mn} \). Then

(i) \( Ces_p^q (\Lambda^2_j) \) is a paranormed space with

\[
g(x) = \sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( |x_{mn}|^{\frac{1}{m+n}} \right) \right)^{\frac{p_{mn}}{\frac{1}{m+n}}} \right)
\]

if and only if \( h = \inf p_{mn} > 0 \), where \( M = \max (1, H) \) and \( H = \sup p_{mn} \).

(ii) \( Ces_p^q (\Lambda^2_j) \) is a complete paranormed linear metric space if the condition \( p \) in (3.1) is satisfied.
Proof. The proof of (i). Sufficiency. Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$.

The inequality $g(x + y) \leq g(x) + g(y)$ follows from the inequality (3.1), since $p_{mn}/M \leq 1$ for all positive integers $m, n$. We also may write $g(\lambda x) \leq max(\{\lambda, |\lambda|^{h/M}\}) g(x)$, since $p_{mn} \leq max(\{\lambda, |\lambda|^{h/M}\})$ for all positive integers $m, n$ and for any $\lambda \in C$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when $x$ is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or $\lambda$ is fixed and $x \to \theta$.

Necessity. Let $Ces_p^\theta \left( A^2_f \right)$ be a paranormed space with the paranorm

$$g(x) = \sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( |x_{mn}| \right) \right)^{\frac{p_{mn}}{p_{mm}}} \right)^{\frac{1}{p_{mm}}}$$

and suppose that $h = 0$. Since $|\lambda|^{p_{mn}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers $m, n$ and $\lambda \in C$ such that $0 < |\lambda| \leq 1$, we have

$$\sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( |\lambda| \right) \right)^{\frac{p_{mn}}{p_{mm}}} \right)^{\frac{1}{p_{mm}}} = 1.$$  

Hence it follows that

$$g(\lambda x) = \sup \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( |\lambda| \right) \right)^{\frac{p_{mn}}{p_{mm}}} \right)^{\frac{1}{p_{mm}}} = 1$$

for $x = (\alpha) \in Ces_p^\theta \left( A^2_f \right)$ as $\lambda \to 0$. But this contradicts the assumption $Ces_p^\theta \left( A^2_f \right)$ is a paranormed space with $g(x)$.

The proof of (ii) is clear. \qed

Corollary 3.4. $Ces_p^\theta \left( A^2_f \right)$ is a complete paranormed space with the natural paranorm if and only if $Ces_p^\theta \left( A^2_f \right) = Ces_p^\theta \left( A^2_f \right)$.

Proposition 3.5. For every $p = (p_{mn})$, $Ces_p^\theta \left( \eta_f^2 \right) \subset Ces_p^\theta \left( \chi_f^2 \right) \subset Ces_p^\theta \left( A^2_f \right)$.

Proof. The proof of (i). First, we show that $Ces_p^\theta \left( \eta_f^2 \right) \subset Ces_p^\theta \left( \chi_f^2 \right) \subset Ces_p^\theta \left( A^2_f \right)$.

We know that $\left[ Ces_p^\theta \left( \chi_f^2 \right) \right]^{\beta} \subset Ces_p^\theta \left( A^2_f \right)$.

Therefore, $Ces_p^\theta \left( \eta_f^2 \right) \subset Ces_p^\theta \left( \chi_f^2 \right) \subset Ces_p^\theta \left( A^2_f \right)$.
The proof of (ii). Now we show that \( \left[ \operatorname{Ces}_p^q \left( \chi_j^2 \right) \right]^{\beta} \subsetneq \operatorname{Ces}_p^q \left( \Lambda_j^2 \right) \).

Let \( y = (y_{mn}) \) be an arbitrary point \( \left[ \operatorname{Ces}_p^q \left( \chi_j^2 \right) \right]^{\beta} \). If \( y \) is not \( \operatorname{Ces}_p^q \left( \Lambda_j^2 \right) \), then for each natural number \( q \), we can find an index \( m_p n_q \) such that

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n q_i} f \left( \left( (m_q + n_q)! |y_{m_n q_i}| \frac{1}{\beta_{m_n q_i}} \right) \right) \right) \right) > q
\]

for \( (1, 2, 3, \cdots) \). Define \( x = \{x_{mn}\} \) by

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |x_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right) q^{\frac{1}{\beta_{m_n}}}
\]

for \( (mn) = (m_p n_q) \) and some \( q \in \mathbb{N} \); and

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |x_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right) = 0, \text{ otherwise.}
\]

Then \( x \) is \( \operatorname{Ces}_p^q \left( \chi_j^2 \right) \), but for infinitely \( mn \),

\[
(3.3) \quad \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |x_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right) > 1.
\]

Consider the sequence \( z = \{z_{mn}\} \), where

\[
Q_{11} (q_{11} f (2!z_{11})^{p_{mn}})^{p_{mn}} = Q_{11} (q_{11} f (2!x_{11})^{p_{mn}})^{p_{mn}} - s
\]

with

\[
s = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |x_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right) ;
\]

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |z_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right)
\]

\[
= \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |x_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right).
\]

The \( z \) is a point of \( \operatorname{Ces}_p^q \left( \chi_j^2 \right) \). Also

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{m_n f} \left( (m_n + n_m)! |z_{m_n}| \frac{1}{\beta_{m_n}} \right) \right) \right) = 0.
\]
Hence $z$ is in $\text{Ces}_q^p \left( \chi_2^j \right)$. But, by the equation (3.3),
\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f \left( ((m+n)! |z_{mn} y_{mn}|)^{p_{mn}} \right) \frac{1}{p_{mn}} \right) \right)
\]
does not converge and so $\sum \sum x_{mn} y_{mn}$ diverges. Thus, the sequence $y$ would not be $\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta$. This contradiction proves that

(3.4) $\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta \subset \text{Ces}_p^q \left( \Lambda_j^2 \right)$.

If we now choose $f = \text{id}$, where $\text{id}$ is the identity and
\[
\frac{1}{Q_{ij}} (q_{1n} ((m+n)! y_{1n})) = \frac{1}{Q_{ij}} (q_{1n} ((m+n)! x_{1n}))
\]
and
\[
\frac{1}{Q_{ij}} (q_{mn} ((m+n)! y_{mn})) = \frac{1}{Q_{ij}} (q_{mn} ((m+n)! x_{mn})) = 0, \quad (m,i > 1)
\]
for all $n,j$, then obviously $x \in \text{Ces}_p^q \left( \chi_j^2 \right)$ and $y \in \text{Ces}_p^q \left( \Lambda_j^2 \right)$, but

(3.5) $\sum \sum x_{mn} y_{mn} = \infty$.

Hence $y \notin \left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta$.

From (3.4) and (3.5), we are granted $\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta \subset \text{Ces}_p^q \left( \Lambda_j^2 \right)$. □

**Proposition 3.6.** In $\text{Ces}_p^q \left( \chi_j^2 \right)$ weak convergence does not imply strong convergence.

**Proof.** Assume that weak convergence implies strong convergence $\text{Ces}_p^q \left( \chi_j^2 \right)$. Then, we would have $\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta = \text{Ces}_p^q \left( \chi_j^2 \right)$ [see Wilansky]. But
\[
\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta \subset \text{Ces}_p^q \left( \Lambda_j^2 \right) \]
Thus $\left[ \text{Ces}_p^q \left( \chi_j^2 \right) \right]^\beta \not\subset \text{Ces}_p^q \left( \chi_j^2 \right)$. Hence weak convergence does not imply strong convergence in $\text{Ces}_p^q \left( \chi_j^2 \right)$. □

**Proposition 3.7.** Let $f$ be an modulus function which satisfies the $\Delta_2$-condition. Then $\text{Ces}_p^q \left( \chi_2^2 \right) \subset \text{Ces}_p^q \left( \chi_j^2 \right)$.

**Proof.** Let

(3.6) $x \in \text{Ces}_p^q \left( \chi_2^2 \right)$. 
Then
\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} ((m+n)! |x_{mn}|)^{p_{mn}} \right) \right) \leq \varepsilon
\]
for sufficiently large \(m, n\) and every \(\varepsilon > 0\).

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f ((m+n)! |x_{mn}|)^{p_{mn}} \right) \right) \leq f(\varepsilon)
\]
(because \(f\) is non-decreasing). This implies
\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f ((m+n)! |x_{mn}|)^{p_{mn}} \right) \right) \leq Kf(\varepsilon) < (\varepsilon)
\]
(by the \(\Delta_2\)-condition, for some \(K > 0\) and by defining \(f(\varepsilon) < \varepsilon K\)).

\[
\lim_{m,n \to \infty} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f ((m+n)! |x_{mn}|)^{p_{mn}} \right) \right)^{p_{mn}} = 0.
\]

Hence
\[
(3.7) \quad \lim_{m,n \to \infty} \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{1}{Q_{ij}} \sum_{m=1}^{i} \sum_{n=1}^{j} q_{mn} f ((m+n)! |x_{mn}|)^{p_{mn}} \right) \right)^{p_{mn}} = 0.
\]

From (3.6) and (3.8), we get \(\text{Ces}^q_p(\Lambda_f^2) \subseteq \text{Ces}^q_p(\chi_f^2)\).

Proposition 3.8. \(\text{Ces}^q_p(\Lambda_f^2) \subseteq \text{Ces}^q_p(\chi_f^2)\).

Proof. Let \((x_{mn}) \in \left[ \text{Ces}^q_p(\Lambda_f^2) \right]^\beta\). Assume that \((x_{mn}) \notin \text{Ces}^q_p(\chi_f^2)\). Then there exists a sequence of positive integers
\[
f((x_{m_r+n_r})) > \frac{1}{(m_r+n_r)!2^{(m_r+n_r)}}, \quad (r = 1, 2, 3, \ldots).
\]
Take
\[
y_{m_r+n_r} = \begin{cases} (2 (m_r+n_r)!)^{m_r+n_r} & \text{for } r = 1, 2, 3, \ldots, \\ y_{m_r+n_r} = 0 & \text{otherwise}. \end{cases}
\]
Then \((y_{mn}) \in \left[ \text{Ces}^q_p(\Lambda_f^2) \right]^\beta\). But
\[
\sum \sum |x_{mn} y_{mn}| = \sum_{r=1}^{\infty} |x_{m_r+n_r} y_{m_r+n_r}| = f \left( \sum_{r=1}^{\infty} |x_{m_r+n_r} y_{m_r+n_r}| \right) > 1 + 1 + 1 + \cdots.
\]
We know that the infinite series $1 + 1 + 1 + \ldots$ diverges. Hence $\sum \sum |x_{mn}y_{mn}|$ diverges. This contradicts (3.9). Hence $(x_{mn}) \in \text{Ces}_p^q \left( \chi_2^2 \right)$. Therefore,

(3.10) \[ \left[ \text{Ces}_p^q \left( A_2^2 \right) \right]^\beta \subset \text{Ces}_p^q \left( \chi_2^2 \right). \]

If we now choose $p = (p_{mn})$, it is a constant $f = \text{id}$, where id is the identity and and

\[
\frac{1}{Q_{ij}} (q_{1n} ((1 + n)! y_{1n})) = \frac{1}{Q_{ij}} (q_{1n} ((1 + n)! x_{1n})) \quad \text{and} \quad \frac{1}{Q_{ij}} (q_{mn} ((m + n)! y_{mn})) = \frac{1}{Q_{ij}} (q_{mn} ((m + n)! x_{mn})) = 0
\]

where $(m, i > 1)$ for all $n, j$, then obviously $x \in \text{Ces}_p^q \left( \chi_2^2 \right)$ and $y \in \text{Ces}_p^q \left( A_2^2 \right)$, but

(3.11) \[ \sum \sum x_{mn} y_{mn} = \infty. \]

Hence $y \notin \left[ \text{Ces}_p^q \left( \chi_2^2 \right) \right]^\beta$.

From (3.10) and (3.1), we are granted $\left[ \text{Ces}_p^q \left( A_2^2 \right) \right]^\beta \subset \text{Ces}_p^q \left( \chi_2^2 \right). \quad \square$

**Proposition 3.9.** Let $\left( \text{Ces}_p^q \left( \chi_2^2 \right) \right)^*$ denote the dual space of $\text{Ces}_p^q \left( \chi_2^2 \right)$. Then we have $\left( \text{Ces}_p^q \left( \chi_2^2 \right) \right)^* = \text{Ces}_p^q \left( A_2^2 \right)$.

**Proof.** We recall that

\[
x = \mathfrak{Z}_{mn} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{(m+n)!} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}
\]

with $\frac{1}{(m+n)!}$ in the $(m, n)^{th}$ position and zero otherwise, with

\[
x = \mathfrak{Z}_{mn} \left\{ \left( \sum_{n=1}^\infty \sum_{j=1}^\infty \left( \frac{1}{Q_{ij}} \sum_{m=1}^i \sum_{n=1}^j q_{mn} f \left( \frac{((m + n)! |x_{mn}|)^{m+n}}{m+n} \right) \right) \right) \right\}^{\frac{1}{p_{mn}}}
\]

\[
= \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \left(\frac{1}{(m+n)!}\right) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1/m+n & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}
\]

which is a $\text{Ces}_p^q \left( \chi_2^2 \right)$ sequence. Hence $\mathfrak{Z}_{mn} \in \text{Ces}_p^q \left( \chi_2^2 \right)$. Let us take $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn} y_{mn}$ with $x \in \text{Ces}_p^q \left( \chi_2^2 \right)$ and $f \in \left( \text{Ces}_p^q \left( \chi_2^2 \right) \right)^*$. Take $x = \ldots$
\((x_{mn}) = \mathcal{Z}_{mn} \in \text{Ces}_p^q \left(\chi^2_f\right)\). Then

\[ |y_{mn}| \leq \|f\|_d (\mathcal{Z}_{mn}, 0) < \infty \quad \text{for each} \ m, n. \]

Thus \((y_{mn})\) is a bounded sequence and hence an Cesàro double analytic sequence of modulus. In other words \(y \in \text{Ces}_p^q \left(\Lambda^2_f\right)\). Therefore \(\left(\text{Ces}_p^q \left(\chi^2_f\right)\right)^* = \text{Ces}_p^q \left(\Lambda^2_f\right)\). □

References


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