REMARKS ON ŠEDA THEOREM

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Abstract. We found sufficient conditions on a sequences \((\lambda_n)\) and \((b_n)\) when the equation \(f'' + a_0 f = 0\) has an entire solution \(f\) such that \(f(\lambda_n) = b_n\).

In [10] V. Šeda proved that for any sequence \((\lambda_n)\) of distinct complex numbers with no finite limit points there exists an entire function \(A_0\) such that the equation
\[ f'' + A_0 f = 0 \] (1)
has an entire solution \(f\) with zeros only at points \(\lambda_n\). On the other hand ([3, p. 201], [7, p. 300–301]), for every sequence \((\lambda_n)\) of distinct complex numbers with no finite limit points and for every sequence \((b_n)\) of complex numbers there exists an entire function \(f\) such that
\[ f(\lambda_n) = b_n. \] (2)

This result was extended to the case of functions holomorphic in open subsets of the complex plane \(\mathbb{C}\) by C. Berenstein and B. Taylor [2]. In particular, we generalize the above-mentioned results from [10] and [3].

**Theorem 1.** For any sequence \((\lambda_n)\) of distinct complex numbers in the domain \(D \subset \mathbb{C}\) with no limit points in \(D\) and every sequence \((b_n)\) of complex numbers there exists a holomorphic in \(D\) function \(A_0\) such that the equation (1) has a holomorphic solution \(f\) satisfying (2).

Šeda result was developed in papers [1, 4, 5, 8, 9]. For meromorphic function \(A_0\) it was extended in [11]. Bank [1] obtained a necessary condition for a sequence with a finite exponent of convergence to be the zero-sequence of a solution of the equation (1). In [1] there is also proved the following proposition.

**Theorem A** ([1, p. 3]). Let \(K > 1\) be a real number and let \((\lambda_n)\) be any sequence of non-zero complex points satisfying \(|\lambda_{n+1}| \geq K|\lambda_n|\) for \(n \in \mathbb{N}\). Then there exists an entire transcendental function \(A(z)\) of order zero such that the equation (1) possesses a solution whose zero-sequence is \((\lambda_n)\).

In [8] Sauer obtain a more general sufficient condition.

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Theorem B ([8, p.1144]). Let \((\lambda_n)\) be a sequence with finite exponent of convergence, \(p\) be its genus and
\[
\mu_k := \prod_{m \neq k} \left(1 - \frac{\lambda_k}{\lambda_m}\right)^{-1} e_p \left(\frac{\lambda_k}{\lambda_m}\right)^{-1},
\]
where \(e_p(z)\) denotes the Weierstrass convergence factor. If there exists a real number \(b > 0\) and a positive integer \(k_0\) such that
\[
|\mu_k| \leq \exp\left(|\lambda_k|^b\right)
\]
for all \(k \geq k_0\), then \((\lambda_n)\) is the zero-sequence of a solution of an equation (1) with entire transcendental function \(A_0(z)\) of finite order.

In [4] J. Heittokangas and I. Laine improved the above results and, in particular, proved the following statement.

Theorem C ([4, p. 300]). Let \((\lambda_n)\) be an infinite sequence of non-zero complex points having a finite exponent of convergence \(\lambda\), a finite genus \(p\) and no finite limit points. Let \(L\) be the canonical product associated with \((\lambda_n)\),
\[
\inf \left\{|\lambda_k|^q |L'(\lambda_k)| \right\} > 0
\]
for some \(q \geq 0\) and arbitrary \(\varepsilon > 0\). Then \((\lambda_n)\) is the zero-sequence of a solution of an equation (1) with entire transcendental function \(A_0(z)\) such that
\[
\rho_{A_0} \leq \max\{\lambda + \varepsilon; q\}.
\]

From estimates in [4] it is possible to get the following result.

Corollary 1. Let \(\rho \in (0; +\infty)\), \(L\) be the canonical product associated with the sequence \((\lambda_n)\) of distinct complex numbers and the conditions
\[
\lambda := \lim_{j \to \infty} \frac{\log j}{\log |\lambda_j|} \leq \rho,
\]
\[
\lim_{j \to \infty} \frac{\log^+ \log^+ |1/L'(\lambda_j)|}{\log |\lambda_j|} \leq \rho
\]
be satisfied. Then there exists an entire function \(A_0\) of order \(\rho_{A_0} \leq \rho\) such that the equation (1) has an entire solution \(f\) for which \((\lambda_n)\) is the zero-sequence.

This corollary also follows from the following theorem. The Theorem 2 is our second main result.

Theorem 2. Let \(\rho \in (0; +\infty)\), \((b_n)\) be an arbitrary sequence of complex numbers and \(L\) be the canonical product associated with the sequence \((\lambda_n)\) of distinct complex numbers. If the conditions (3), (4) and
\[
\lim_{j \to \infty} \frac{\log^+ \log^+ |b_j|}{\log |\lambda_j|} \leq \rho
\]
hold, then there exists an entire function \(A_0\) of order \(\rho_{A_0} \leq \rho\) such that the equation (1) has an entire solution \(f\) satisfying (2).
To prove Theorem 1 we need the following lemma.

**Lemma 1** ([2, p. 118]). Let \((a_{j,1})\) and \((a_{j,2})\) be sequences of complex numbers, \((\lambda_j)\) be a sequence of distinct complex numbers in domain \(D \subset \mathbb{C}\) with no limit points in \(D\). Then there exists a holomorphic in \(D\) function \(g\) such that

\[
g(\lambda_j) = a_{j,1}, \quad g'(\lambda_j) = a_{j,2}
\]

for all \(j \in \mathbb{N}\).

**Proof of Theorem 1.** Let \(\{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : b_n = 0\}\) and \(\{m_k : k \in \mathbb{N}\} = \mathbb{N}\setminus\{n_k : k \in \mathbb{N}\}\). Then \(\{\lambda_{n_k}\} \cup \{\lambda_{m_k}\} = \{\lambda_n\}\). Let \(\log u = \log |u| + i \varphi, \varphi = \arg u \in [-\pi; \pi),\) and \(Q\) be a holomorphic function in \(D\) with simple zeros at the points \(\lambda_{n_k}\) and \(Q(\lambda_{m_k}) \neq 0\) for all \(k\). Denote

\[
a_{j,1} = \begin{cases} \log \frac{b_j}{Q(\lambda_j)}, & j \in \{m_k\}, \\ 0, & j \notin \{m_k\}, \end{cases} \quad a_{j,2} = \begin{cases} 0, & j \notin \{n_k\}, \\ -\frac{Q''(\lambda_j)}{2Q'(\lambda_j)}, & j \in \{n_k\}. \end{cases}
\]

By Lemma 1 it follows that there exists a holomorphic function \(g\) in \(D\) such that (6) is valid. Hence the function

\[
A_0 = -\frac{Q'' + 2Q'g'}{Q} - g'' - g'^2
\]

is holomorphic in \(D\) and the function \(f = Qe^g\) is a solution of the equation (1) and satisfies the condition (2). \(\square\)

To prove Theorem 2 we need the following statement.

**Lemma 2** ([6, p. 146–147]). Let \(\rho \in (0; +\infty)\) and \((\lambda_n)\) be a sequence of distinct complex numbers. For any sequences \((a_{j,1})\) and \((a_{j,2})\) of complex numbers such that

\[
\lim_{j \to \infty} \frac{\log^+|a_{j,s}|}{\log |\lambda_j|} \leq \rho, \quad s \in \{1; 2\},
\]

there exists at least one entire function \(g\) of order \(\rho\) satisfying (6) if and only if the condition (3) and

\[
\lim_{j \to \infty} \frac{\log^+|\gamma_{j,s}|}{\log |\lambda_j|} \leq \rho, \quad s \in \{1; 2\},
\]

hold, where \(F = L^2,\)

\[
\gamma_{j,1} = \left( \frac{(z - \lambda_j)^2}{F(z)} \right)_{\|z=\lambda_j\}, \quad \gamma_{j,2} = \left( \frac{(z - \lambda_j)^2}{F(z)} \right)'_{\|z=\lambda_j},
\]

\[
L(z) = \prod_{j=1}^{\infty} (1 - z/\lambda_j) \exp \left( \sum_{i} \frac{1}{i} \left( \frac{z}{\lambda_j} \right)^i \right)
\]
and $p$ is the smallest integer for which the series
\[ \sum_j \frac{1}{|\lambda_j|^{p+1}} \]
converges.

**Proof of Theorem 2.** Let \( \{n_k : k \in \mathbb{N}\} = \{n \in \mathbb{N} : b_n = 0\} \) and \( \{m_k : k \in \mathbb{N}\} = \mathbb{N}\setminus\{n_k : k \in \mathbb{N}\} \). Then \( \{\lambda_{n_k}\} \cup \{\lambda_{m_k}\} = \{\lambda_n\} \). Denote
\[
Q(z) = \prod_{j=1, j \in \{n_k\}}^\infty (1 - z/\lambda_j) \exp \left( \sum_1^p \frac{1}{i} \left( \frac{z}{\lambda_j} \right)^i \right),
\]
\[
G(z) = \prod_{j=1, j \in \{m_k\}}^\infty (1 - z/\lambda_j) \exp \left( \sum_1^p \frac{1}{i} \left( \frac{z}{\lambda_j} \right)^i \right)
\]
and
\[
a_{j,1} = \begin{cases} 
\log \frac{b_j}{Q(\lambda_j)}, & j \in \{m_k\}, \\
0, & j \notin \{n_k\},
\end{cases} \quad a_{j,2} = \begin{cases} 
0, & j \notin \{n_k\}, \\
-\frac{Q''(\lambda_j)}{2Q'(\lambda_j)}, & j \in \{n_k\}.
\end{cases}
\]
Since \( L(z) = Q(z)G(z) \), \( L'(z) = Q'(z)G(z) + Q(z)G'(z) \), we see that \( 1/Q(\lambda_{m_k}) = G'(\lambda_{m_k})/L'(\lambda_{m_k}) \) and \( 1/Q'(\lambda_{n_k}) = G(\lambda_{n_k})/L'(\lambda_{n_k}) \). Using (3)–(5), we get that the sequences \((a_{j,1})\) and \((a_{j,2})\) satisfy the condition (7). Since
\[
F(z) = \sum_{j=0}^m \frac{F^{(j)}(\lambda_j)}{j!} (z - \lambda_j)^j + o((z - \lambda_j)^m), \quad z \to \lambda_j
\]
for each \( m \in \mathbb{Z}_+ \), we have
\[
\gamma_{j,1} = \frac{2}{F'''(\lambda_j)}, \quad \gamma_{j,2} = -\frac{2}{3} \frac{F'''(\lambda_j)}{(F'(\lambda_j))^2}.
\]
Since
\[
F'''(\lambda_j) = 2(L'(\lambda_j))^2, \quad F'''(\lambda_j) = -2L''(\lambda_j)/L'(\lambda_j),
\]
then
\[
\gamma_{j,1} = \frac{1}{(L'(\lambda_j))^2}, \quad \gamma_{j,2} = \frac{L''(\lambda_j)}{3(L'(\lambda_j))^3}.
\]
Taking into account (3) and (4), we obtain (8). From Lemma 2 it follows that there exists an entire function \( g \) such that the condition (6) holds. Moreover \( \rho_g \leq \rho \). Then \( f = Qe^g \) is a solution of the equation (1), where
\[
A_0 = -\frac{Q'' + 2Q'g'}{Q} - g'' - g'^2.
\]
By standard methods we obtain \( \rho A_0 \leq \rho \). \( \square \)

A question of sharpness of the condition (7) remains open.
REFERENCES

10. Šeda V., *On some properties of solutions of the differential equation $y'' = Q(z)y$, where $Q(z) \not\equiv 0$ is an entire function*, Acta F.R.N. Univ. Comen. Mathem. 4 (1959), 223–253. (in Slovak)

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