CURVES WHOSE SECANT DEGREE IS ONE
IN POSITIVE CHARACTERISTIC

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Abstract. Here we study (in positive characteristic) integral curves \( X \subset \mathbb{P}^r \) with secant degree one, i.e., for which a general \( P \in \text{Sec}^{k-1}(X) \) is in a unique \( k \)-secant \((k-1)\)-dimensional linear subspace.

1. Introduction

Let \( \mathbb{K} \) be an algebraically closed base field. Let \( X \subset \mathbb{P}^r \) be an integral and non-degenerate closed subvariety. For each \( x \in \{0, \ldots, r\} \), let \( G(x, r) \) denote the Grassmannian of all \( x \)-dimensional linear subspaces of \( \mathbb{P}^r \). For each integer \( k \geq 1 \) let \( \sigma_k(X) \) denote the closure in \( \mathbb{P}^r \) of the union of all \( A \in G(k-1, r) \) spanned by \( k \) points of \( X \) (the variety \( \sigma_k(X) \) is sometimes called the \((k-1)\)-secant variety of \( X \) and written \( \text{Sec}^{k-1}(X) \), but we prefer to call it the \( k \)-secant variety of \( X \)).

The integral variety \( \sigma_k(X) \) may be obtained in the following way. Assume that \( X \) is non-degenerate. For any closed subscheme \( E \subseteq \mathbb{P}^r \) let \( \langle E \rangle \) denote its linear span. Let \( V(X, k) \subseteq G(k-1, r) \) denote the closure in \( G(k-1, r) \) of the set of all \( A \in G(k-1, r) \) spanned by \( k \)-points of \( X \). Set

\[
S[X, k] := \{(P, A) \in \mathbb{P}^r \times G(k-1, r) : P \in A, A \in V(X, k)\}.
\]

Let \( p_1 : \mathbb{P}^r \times G(k-1, r) \to \mathbb{P}^r \) denote the projection onto the first factor. We have \( \sigma_k(X) = p_1(S[X, k]) \). Set \( m_{X, k} := p_1|_{S[X, k]} \). If \( \sigma_k(X) \) has the expected dimension \( k \cdot (\dim(X) + 1) - 1 \) (i.e., if \( m_{X, k} \) is generically finite), then we write \( i_k(X) \) for the inseparable degree of \( m_{X, k} \) and \( s_k(X) \) for its separable degree. For any \( P \in X_{\text{reg}} \), let \( T_P X \subset \mathbb{P}^r \) denote the tangent space to \( X \) at \( P \). If \( k \geq 2 \), we say that \( X \) is \( k \)-unconstrained if

\[
\dim((T_{P_1} X \cup \cdots \cup T_{P_k} X)) = \dim(\sigma_k(X))
\]

for a general \((P_1, \ldots, P_k) \in X^k \). Terracini’s lemma says that

\[
\dim((T_{P_1} X \cup \cdots \cup T_{P_k} X)) \leq \dim(\sigma_k(X))
\]
and that in characteristic zero equality always holds ([1, §1] or [3, §2]). The case \(k = 2\) of this notion was introduced in [3]. A non-degenerate curve \(Y \subset \mathbb{P}^r\) is 2-unconstrained if and only if either \(r = 2\) or \(Y\) is not strange [3, Example (e1) at page 333]. From now on we assume \(\dim(X) = 1\). We first prove the following result.

**Theorem 1.** Fix integers \(r \geq 2k \geq 4\). Let \(X \subset \mathbb{P}^r\) be an integral, non-degenerate and \(k\)-unconstrained curve. Then \(s_k(X) = 1\).

For each integer \(i\) such that \(2 \leq 2i \leq r\) we define the integer \(e_i(X)\) in the following way. Fix a general \((P_1, \ldots, P_i) \in X^i\). Thus \(P_j \in X_{\text{reg}}\) for all \(j\). Set \(V := (T_{P_1}X \cup \cdots \cup T_{P_i}X)\). Notice that \((V \cap X)_{\text{red}} \supseteq \{P_1, \ldots, P_i\}\) and the scheme \(V \cap X\) is zero-dimensional. Varying \((P_1, \ldots, P_i)\) in \(X^i\) we see that each \(P_j\) appears with the same multiplicity in the zero-dimensional scheme \(V \cap X\). We call \(e_i(X)\) this multiplicity. In characteristic zero we always have \(e_i(X) = 2\). The integer \(e_1(X)\) is the intersection multiplicity of \(X\) with its tangent line at its contact point. Hence if \(\text{char}(\mathbb{K})\) is odd the curve \(X\) is reflexive if and only if \(e_1(X) = 2\) ([4, 3.5]). In the general case we have \(e_1(X) \geq 2\) and \(e_i(X) \leq e_{i+1}(X)\). For any \(P \in X_{\text{reg}}\) and any integer \(t \in \{1, \ldots, r\}\), let \(O(X, P, t) \in G(t, r)\) denote the \(t\)-dimensional osculating plane of \(X\) at \(P\). Thus \(O(X, P, 1) = T_{P}X\). Fix integers \(i \geq 1\) and \(j_h \geq 0\), \(1 \leq h \leq i\). We only need the case \(2i + \sum_{h=1}^{i} j_h \leq r\). Fix a general \((P_1, \ldots, P_i) \in X^i\) and set \(V := (\cup_{h=1}^{i} O(X, P_h, 1 + j_h))\). For any \(h \in \{1, \ldots, i\}\), let \(E(X; i; j_1, \ldots, j_i; h)\) be the multiplicity of \(P_h\) in the scheme \(V \cap X\). We will only use the case \(j_1 = 1\) and \(j_h = 0\) for all \(h \neq 1\). If either \(\text{char}(\mathbb{K}) = 0\) or \(\text{char}(\mathbb{K}) > \deg(X)\), then \(E(X; i; j_1, \ldots, j_i; h) = 2 + j_h\) (Lemma 9). Here we prove the following result.

**Theorem 2.** Let \(X \subset \mathbb{P}^{2k-1}\), \(k \geq 2\), be an integral, non-degenerate and \(k\)-unconstrained curve. Set \(j_k := 1\) and \(j_h := 0\) for all \(h \in \{2, \ldots, k-1\}\).

(a) If \(s_k(X) = 1\) and \(E(X; k - 1; j_1, \ldots, j_{k-1}; 1) = e_{k-1}(X) + 1\), then \(X\) is smooth and rational and \(\deg(X) = (k - 1)e_{k-1}(X) + 1\).

(b) \(X\) is a rational normal curve if and only if \(s_k(X) = 1\), \(e_{k-1}(X) = 2\) and \(E(X; k - 1; j_1, \ldots, j_{k-1}; 1) = 3\).

We do not know if in the statement of Theorem 2 we may drop the conditions “\(e_{k-1}(X) = 2\)” and “\(E(X; k - 1; j_1, \ldots, j_{k-1}; 1) = 3\)”. We are able to prove that we may drop the first one in the case \(k = 2\), i.e., we prove the following result.

**Proposition 1.** Let \(X \subset \mathbb{P}^3\) be an integral and non-degenerate curve. The following conditions are equivalent:

(a) \(X\) is not strange, \(s_2(X) = 1\) and \(E(X; 1; 1; 1) = e_1(X) + 1\);

(b) \(i_2(X) = s_2(X) = 1\) and \(E(X; 1; 1; 1) = e_1(X) + 1\);

(c) \(X\) is a rational normal curve.

The picture is very easy if \(\text{char}(\mathbb{K}) > \deg(X)\). As a byproduct of Theorem 2 we give the following result.

**Theorem 3.** Let \(X \subset \mathbb{P}^{2k-1}\) be an integral and non-degenerate curve. Assume \(\text{char}(\mathbb{K}) > \deg(X)\). \(X\) is a rational normal curve if and only if \(s_k(X) = 1\).
2. THE PROOFS

Remark 1. Assume $X$ of arbitrary dimension and that
\[ \dim(\sigma_k(X)) = k(\dim(X) + 1) - 1. \]
As in [3] (the case $k = 2$) $X$ is $k$-unconstrained if and only if $i_k(X) = 1$.

Lemma 1. Fix integers $c > 0$, $s > y \geq 2$ and $r \geq s(c + 1) - 1$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate $c$-dimensional subvariety such that $\dim(\sigma_s(X)) = s(c + 1) - 1$. If $X$ is $s$-unconstrained, then $X$ is $y$-unconstrained.

Proof. Since $\dim(\sigma_s(X)) = s(c + 1) - 1$ and $X$ is $s$-unconstrained, we have
\[ \dim(\langle T_{P_i}X \cup \cdots \cup T_{P_y}X \rangle) = s(c + 1) - 1 \]
for a general $(P_1, \ldots, P_y) \in X^s$. Hence $\dim(\langle T_{P_i}X \cup \cdots \cup T_{P_y}X \rangle) = y(c + 1) - 1$. Hence $X$ is $y$-unconstrained. \(\square\)

We recall the following very useful result ([1, §1]).

Lemma 2. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Then $X$ is non-defective, i.e., $\dim(\sigma_a(X)) = \min\{r, 2a - 1\}$ for all integers $a \geq 2$.

From Lemmas 1 and 2 we get the following result.

Lemma 3. Fix integers $s > y \geq 2$ and $r \geq 2s - 1$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. If $X$ is $s$-unconstrained, then $X$ is $y$-unconstrained and not strange.

We recall that a finite set $S \subset \mathbb{P}^r$ is said to be in linearly general position if \( \dim(\langle S' \rangle) = \min\{x, 2(\dim(S') - 1)\} \) for every $S' \subset S$. The general hyperplane section of a non-degenerate curve $X \subset \mathbb{P}^r$ is in linearly general position if $X$ is not strange ([6, Lemma 1.1]). Hence Lemma 3 implies the following result.

Lemma 4. Fix integers $r, s$ such that $r \geq 2s - 1 \geq 3$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume that $X$ is $s$-unconstrained. Then $X$ is not strange and a general hyperplane section of $X$ is in linearly general position.

Proof of Theorem 1. We extend the proof of the case $k = 2$ given in [3, §4]. By Lemma 4 a general $(k - 1)$-dimensional $k$-secant plane of $X$ meets $X$ at exactly $k$ points. Fix a general $(P_1, \ldots, P_k) \in X^k$ and set $V := \langle T_{P_1}X \cup \cdots \cup T_{P_k}X \rangle$. Since $X$ is $k$-unconstrained, we have $\dim(V) = 2k - 1$. Since $2k - 1 < r$ and $X$ is non-degenerate, the set $S := (V \cap X)_{\text{red}}$ is finite. Fix a general $P \in \langle \{P_1, \ldots, P_k\} \rangle$. Assume $s_k(X) \geq 2$. Since a general hyperplane section of $X$ is in linearly general position (Lemma 4), the integer $s_k(X)$ is the number of different $k$-ples of points of $X$ such that a general point of $\sigma_k(X)$ is in their linear span. Since $P$ may be considered as a general point of $\sigma_k(X)$ and $s_k(X) \geq 2$, there is $(Q_1, \ldots, Q_k) \in X^k$ such that $P \in \langle \{Q_1, \ldots, Q_k\} \rangle$ and $\{P_1, \ldots, P_k\} \neq \{Q_1, \ldots, Q_k\}$. For general $P$ we may also assume that $(Q_1, \ldots, Q_k)$ is general in $X^k$. Hence each $P_i$ and each $Q_j$ is a smooth point of $X$. Terracini’s lemma gives $\langle T_{P_1}X \cup \cdots \cup T_{P_k}X \rangle \subset T_P\sigma_k(X)$ and $\langle T_{Q_1}X \cup \cdots \cup T_{Q_k}X \rangle \subset T_P\sigma_k(X)$. Since $X$ is $k$-unconstrained and both $(P_1, \ldots, P_k)$
and \((Q_1, \ldots, Q_k)\) are general in \(X^k\), we have \(\langle T_{P_1}X \cup \cdots \cup T_{P_k} \rangle = T_P \sigma_k(X)\) and \(\langle T_{Q_1}X \cup \cdots \cup T_{Q_\ell}X \rangle = T_P \sigma_k(X)\). Hence \(\{Q_1, \ldots, Q_\ell\} \subseteq S\). Since \(S\) is finite, the union of the linear spans of all \(S' \subseteq S\) with \(\dim(S') = k\) is a finite number of linear subspaces of dimension at most \(k - 1\) and \(\langle S' \rangle = \langle \{P_1, \ldots, P_\ell\} \rangle\) if and only if \(S' = \{P_1, \ldots, P_\ell\}\), because \(\langle \{P_1, \ldots, P_\ell\} \rangle \cap X = \{P_1, \ldots, P_\ell\}\).

Hence \(\dim(\langle S' \rangle \cap \langle \{P_1, \ldots, P_\ell\} \rangle) \leq k - 2\) for all \(S' \neq \{P_1, \ldots, P_\ell\}\). Varying \(P \in \{\{P_1, \ldots, P_\ell\}\} \cong \mathbb{P}^{k-1}\), we get a contradiction. \[\square\]

**Lemma 5.** Let \(X \subseteq \mathbb{P}^r\), \(r \geq 2k - 1 \geq 5\), be an integral, non-degenerate and \(k\)-unconstrained curve. Fix an integer \(s\) such that \(1 \leq s \leq k - 2\). Fix a general \((A_1, \ldots, A_\ell) \in X^s\) and set \(W := \langle T_{A_1}X \cup \cdots \cup T_{A_\ell}X \rangle\). Then \(\dim(W) = 2s - 1\).

Let \(\ell_W : \mathbb{P}^r \setminus \mathbb{P}^{r-2s} \rightarrow \mathbb{P}^{r-2s}\) denote the linear projection from \(W\). Let \(Y \subseteq \mathbb{P}^{r-2s}\) denote the closure of \(\ell_W(Y) \setminus W\). Then \(Y\) is \((k-s)\)-unconstrained and it is not strange.

**Proof.** Fix a general \(A_{s+1}, \ldots, A_k \in X^{k-s}\). Notice that \(\langle \ell_W(A_{s+1}), \ldots, \ell_W(A_k) \rangle\) is general in \(Y^{k-s}\) and

\[
\ell_W((W \cup T_{A_{s+1}}X \cup \cdots \cup T_{A_k}X) \setminus W) = \langle T_{\ell_W(A_{s+1})}Y \cup \cdots \cup T_{\ell_W(A_k)}Y \rangle.
\]

Hence the latter space has dimension \(2k-2s-1\). Hence \(Y\) is \((k-s)\)-unconstrained. Since \(k - s \geq 2\), \(Y\) is not strange. \[\square\]

**Lemma 6.** Fix integers \(c > 0\), \(k \geq 2\) and \(r \geq (c + 1)k - 1\). Let \(X \subseteq \mathbb{P}^r\) be a \(k\)-unconstrained \(c\)-dimensional variety such that \(\dim(\sigma_k(X)) = (c + 1)k - 1\). Fix an integer \(s \in \{1, \ldots, k - 1\}\) and a general \((P_1, \ldots, P_s) \in X^s\). Set \(V := \langle T_{P_1}X \cup \cdots \cup T_{P_s}X \rangle\). Then \(\dim(V) = (c + 1)s - 1\) and the restriction to \(X\) of the linear projection \(\ell_V : \mathbb{P}^r \setminus \mathbb{P}^{r-(c+1)s} \rightarrow \mathbb{P}^{r-(c+1)s}\) is a generically finite separable morphism.

**Proof.** Since \(s + 1 \leq k\) and \(\dim(\sigma_k(X)) = (c + 1)k - 1\), we have \(\dim(\sigma_s(X)) = (c + 1)s - 1\). Lemma 1 gives that \(X\) is \(s\)-unconstrained. Since \(X\) is \((s+1)\)-unconstrained and \(\dim(\sigma_{s+1}(X)) = (c + 1)(s + 1) - 1\), we have

\[
\dim(\langle V \cup T_{P_s}X \rangle) = \dim(V) + \dim(T_{P_s}X) + 1
\]

for a general \(P_s \in X\), i.e., \(V \cap T_{P_s}X = \emptyset\) for a general \(P_s \in X\). Thus \(\ell_V(X) - V\) has differential with rank \(c\), i.e., it is separable and generically finite. \[\square\]

**Proof of Theorem 2.** If \(X\) is a rational normal curve, then it is \(k\)-unconstrained, \(s_k(X) = 1\) ([2, First 4 lines of page 128]) and \(i_k(X) = 1\) (Remark 1).

Now assume \(s_k(X) = 1\). In step (c) we will use the assumption \(E(X; k - 1; 1, 0, \ldots, 0; 1) = e_{k-1}(X) + 1\). We need to adapt a part of the characteristic zero proof given in [2] to the positive characteristic case. We will follow [2] as much as possible. Fix a general \((P_1, \ldots, P_{k-1}) \in X^{k-1}\) and set \(V := \langle T_{P_1}X \cup \cdots \cup T_{P_{k-1}}X \rangle\).

Since \(X\) is \(k\)-unconstrained, we have \(\dim(V) = 2k - 3\). Since \(X\) is non-degenerate, the set \(S := (V \cap X)_{\text{red}}\) is finite.

(a) Here we check that \(S \subseteq X_{\text{reg}}\). If \(k = 2\), then for a general \(P_1\) we have \(T_{P_1}X \cap \text{Sing}(X) = \emptyset\), because \(X\) is not strange by [3, Example (e1) at page 333]. Now assume \(k \geq 3\). Since \(X\) is not strange (use Lemma 1), for general \(P_1 \in X\), we have \(T_{P_1}X \cap \text{Sing}(X) = \emptyset\). Then by induction on \(i\) we check using a linear
projection from $T_{P_i}X$ as in Lemma 5 that $(T_{P_i}X \cup \cdots \cup T_{P_i}X) \cap \text{Sing}(X) = \emptyset$ (more precisely, for any finite set $\Sigma \subset X$ we check by induction on $i$ that $(T_{P_i}X \cup \cdots \cup T_{P_i}X) \cap \Sigma = \emptyset$ for a general $(P_1, \ldots, P_i) \in X^i$). For $i = k-1$ we get $S \subset X_{\text{reg}}$.

(b) Obviously $\{P_1, \ldots, P_{k-1}\} \subseteq S$. Here we check that $S = \{P_1, \ldots, P_{k-1}\}$. Assume for the moment the existence of $z \in S \setminus \{P_1, \ldots, P_{k-1}\}$. Since $X$ is not strange, it is not very strange, i.e., a general hyperplane section of $X$ is in linearly general position ([6, Lemma 1.1]). Since $(P_1, \ldots, P_{k-1})$ is general in $X^{k-1}$, we get $\langle \{P_1, \ldots, P_{k-1}\} \rangle \cap X = \{P_1, \ldots, P_{k-1}\}$. Thus $\dim(\langle \{P_1, \ldots, P_{k-1}, Q\} \rangle) = k-1$. Fix a general $z \in \langle \{P_1, \ldots, P_{k-1}, Q\} \rangle$. We have

$$
P^{2k-1} = T_z\sigma_k(X) \supseteq \langle T_{P_i}X \cup \cdots \cup T_{P_{k-1}}X \cup T_QX \rangle
$$

(Terracini’s lemma ([3, §2] or [1, Proposition 1.9])). The additive map giving Terracini’s lemma for joins in the proof of [1, Proposition 1.9], shows that the map $m_{X,k}$ has non-invertible differential over the point $z$. Since $P^{2k-1}$ is smooth and $m_{X,k}$ is separable, we get that $m_{X,k}$ is not finite of degree 1 near $z$. Since $s_k(X) = 1$, $m_{X,k}$ contracts a curve over $z$. Since $z$ lies in infinitely many $(k-1)$-dimensional $k$-secant subspaces, we get that $\dim(\sigma_k(X)) \leq 2k-2$, contradicting Lemma 2. The contradiction proves $S = \{P_1, \ldots, P_{k-1}\}$.

(c) Step (b) means that $\{P_1, \ldots, P_{k-1}\}$ is the reduction of the scheme-theoretically intersection $X \cap V$. Let $Z_i$ denote the connected component of the scheme $X \cap V$ supported by $P_i$. Set $e := \deg(Z_1)$. Since $T_{P_i}X \subseteq V$, we have $e \geq 2$. Varying $(P_1, \ldots, P_{k-1})$ in $X^{k-1}$ we get $\deg(Z_i) = e$ for all $i$. The definition of the integer $e_k-1(X)$ gives $e = e_k-1(X)$. Set $\phi := \ell_V(X \setminus V \cap X)$. Since $X \cap V \subset X_{\text{reg}}$, $\phi$ is dominant and $X_{\text{reg}}$ is a smooth curve, $\phi$ induces a finite morphism $\psi: X \to P^1$. Bezout’s theorem gives $\deg(X) = (k-1)e + \deg(\psi)$. Lemma 6 gives that $\psi$ is separable. Hence $\deg(\psi)$ is the separable degree of $\psi$. Assume $\deg(\psi) \geq 2$. Since $P^1$ is algebraically simply connected, there is $Q \in X$ at which $\psi$ ramifies.

First assume $Q \in X_{\text{reg}}$. Since $E(X; k-1; 1, 0, \ldots, 0; 1) = e_k-1(X) + 1$, $\psi$ is not ramified at $P_1$. Moving $P_1, \ldots, P_{k-1}$ we get $Q \notin \{P_1, \ldots, P_{k-1}\}$. The definition of $\phi$ gives $\dim(V \cup T_QX) \leq \dim(V) + 1$. Hence the additive map giving Terracini’s lemma for joins in the proof of [1, Proposition 1.9], shows that the map $m_{X,k}$ has non-invertible differential over the general point $z \in \langle \{P_1, \ldots, P_{k-1}, Q\} \rangle$. As in step (b) we get a contradiction.

Now assume $Q \in \text{Sing}(X)$. Let $u: C \to X$ denote the normalization map. Since we assumed $\deg(\psi) \geq 2$, we have $\deg(\psi \circ u) \geq 2$. Since $P^1$ is algebraically simply connected, there is $Q' \in C$ such that $\psi \circ u$ is ramified at $Q'$. We repeat the construction of joins and secant variety starting from the non-embedded curve $C$ and get a contradiction using $Q'$ instead of $Q$. Thus $\deg(\psi) = 1$, i.e.

$$
\deg(X) = (k-1)e_k-1(X) + 1,
$$

and $X$ is rational.

$X$ is a rational normal curve if and only if $\deg(X) = 2k - 1$, i.e., if and only if $e = 2$. Take any $P \in \text{Sing}(X)$ (if any). Set $H := \langle \{P\} \cup V \rangle$. Since $X$ is singular
at $P$, we have $\deg(H \cap X) \geq 2 + (k - 1)e > \deg(X)$, that is contradiction. Thus $X$ is smooth. \qed

**Proof of Proposition 1.** We have $i_2(X) = 1$ if and only if $X$ is 2-unconstrained ([13] or Remark 1). Obviously $X$ is 2-unconstrained. Hence it is sufficient to prove that if $X$ is 2-unconstrained, $s_2(X) = 1$, and $E(X; 1; 1; 1) = e_1(X) + 1$, then $X$ is a rational normal curve. Theorem 2 says that $X$ is smooth and rational and $\deg(X) = e_1(X) + 1$. Thus it is sufficient to prove $e_1(X) = 2$. Assume $e_1(X) \geq 3$. Since $\deg(X) = e_1(X) + 1$, Bezout’s theorem says that any two different tangent lines are disjoint. Let $P$ be an ordinary double point of $X$. Take instead of a general $P \in \mathbb{P}^3$ a general $P / \{ \{ P_1, P_2 \} \}$ contained in the plane $\ell(\mathbb{P}^3)$. Hence it is sufficient to prove $e_1(X) = 2$. Assume $e_1(X) = 2$. Since $\deg(X) = e_1(X) + 1$, Bezout’s theorem says that any two different tangent lines are disjoint. Let $TX \subset \mathbb{P}^3$ denote the tangent developable of $X$. Fix a general $P \in \mathbb{P}^3$ and let $\ell_P : \mathbb{P}^3 \setminus \{ P \} \to \mathbb{P}^2$ be the linear projection from $P$. Set $\ell := \ell_P | X$. Since $P \notin TX$, $\ell$ is unramified. Since $X$ is smooth, $s_2(X) = 1$ and $P$ is general, the map $\ell$ is birational onto its image and the curve $\ell(X)$ has a unique singular point (the point $\ell(P_1) = \ell(P_2)$ with $P \in \langle \{ P_1, P_2 \} \rangle$ and $(P_1, P_2) \in X^2$). We have $p_a(\ell(X)) = e_1(X)(e_1(X) - 1)/2 \geq 2$. Since $P \notin TX$, we have $P \notin T_{P_i}X$, $i = 1, 2$. Since $T_{P_1}X \cap T_{P_2}(X) = \emptyset$, the line $T_{P_2}X$ is not contained in the plane $\langle \{ P \} \cup T_{P_1}X \rangle$. Thus $\ell_P(T_{P_1}X) \neq \ell_P(T_{P_2}X)$. Thus $\ell(P_1)$ is an ordinary double point of $X$. Hence $\ell(X)$ has geometric genus $p_a(X) - 1 > 0$, that is contradiction. \qed

**Lemma 7.** Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume $\text{char}(\mathbb{K}) > \deg(X)$. Then $e_i(X) = 2$ for all positive integers $i$ such that $2i \leq r$.

**Proof.** We have $e_1(X) = 2$, because in large characteristic the Hermite sequence of $X$ at its general point is the classical one ([5, Theorem 15]). The case $i \geq 2$ is obtained by induction on $i$ taking instead of $X$ its image by the linear projection from $T_{P_i}X$, $P_i$ general in $X$. \qed

**Lemma 8.** Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume $\text{char}(\mathbb{K}) > \deg(X)$. Then $X$ is $i$-unconstrained for all integers $i \geq 2$.

**Proof.** Fix a linear subspace $V \subset \mathbb{P}^r$ such that $v := \dim(V) \leq r - 2$. Let $\ell_V : \mathbb{P}^r \setminus V \to \mathbb{P}^{r - v - 1}$ denote the linear projection from $V$. Since $\text{char}(\mathbb{K}) > \deg(X)$, the restriction of $\ell_V$ to $X$ is separable. Hence $T_{P_i}X \cap V = \emptyset$ for a general $P_i \in X$. Take $V = \langle T_{P_1}X \cup \cdots \cup T_{P_{i - 1}}X \rangle$ with $(P_i, \ldots, P_{i - 1})$ general in $X^{i - 1}$ and use induction on $i$. \qed

**Lemma 9.** Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. Assume $\text{char}(\mathbb{K}) > \deg(X)$. Then $E(X; i; j_1, \ldots, j_i; h) = 2 + j_h$ for all $i, j_1, \ldots, j_i$ such that

$$2i + \sum_{j=1}^{i} j_x \leq r$$

and for a general $(P_1, \ldots, P_i) \in X^i$, the linear span of the osculating spaces

$$O(X, P_x, 1 + j_x), 1 \leq x \leq i,$$ has dimension $2i - 1 + \sum_{x=1}^{i} j_x$. 

Proof. The case $i = 1$ is true by [5, Theorem 15]. Hence we may assume $i \geq 2$. Fix an index $c \in \{1, \ldots, i\} \setminus \{h\}$. For a general $P_c \in X$, the point $P_c$ appears with multiplicity exactly $j_c + 2$ in the scheme $O(X, P_c, j_c + 1)$ ([5, Theorem 15]). Since $\text{char}(K) > \deg(X)$, the rational map $\ell$ obtained restricting to $X$ the linear projection from $O(X, P_c, 1 + j_c)$ is separable. Call $Y$ the closure in $\mathbb{P}^{r-j_c-2}$ of $\ell(X \setminus O(X, P, 1 + j_c) \cap X)$. Take $P_x, x \neq c$, such that $(P_1, \ldots, P_i)$ is general in $X^c$ and write $Q_x := \ell(P_x)$ for all $x \neq c$. Let $V$ be the linear span of the osculating spaces $O(X, P_x, 1 + j_x), 1 \leq x \leq i$, $U$ the linear span of the osculating spaces $O(X, P_x, 1 + j_x), x \neq c$, and $W$ the linear span of the osculating spaces $O(Y, Q_x, 1 + j_x), x \neq c$. By the inductive assumption $U$ and $W$ have dimension $2i - 3 + \sum_{x \neq c} j_x$. Hence $\ell(U) = W$ and $\dim(V) = 2i - 1 + \sum_{x=1}^{i} j_x$. Since the points $Q_x$ are general and $\ell$ is separable, the scheme $\ell^{-1}((2 + j_x)Q_x))$, $x \neq c$, is a divisor of $X$ whose connected component supported by $P_x$ has degree $2 + j_x$. Use the inductive assumption on $Y$ to get $E(X; i; j_1, \ldots, j_i; h) = 2 + j_h$. □

Proof of Theorem 3. Apply Theorem 2 and Lemmas 7, 8 and 9. □

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References


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