ON SOME PROPERTIES OF A FUNCTION CONNECTING WITH AN INFINITE SERIES

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Abstract. An attempt has been made in this paper to investigate some set theoretic properties of a function suitably defined on the space of all sequences of non-negative real numbers endowed with Fréchet metric.

0. Introduction

Inspiration for this paper arises from the papers [1], [2] where the authors proved several interesting theorems in relation to Borel and Baire classifications of functions defined by the exponent of convergence of the family of all non-decreasing sequences of real numbers, the first term of which is at least $\gamma$ where $\gamma$ is a positive real number, endowed with Fréchet metric. Our approach in this paper is somewhat different. Instead of taking the family of all non-decreasing sequences $x = \{\xi_k\}_{k=1}^\infty$ of real numbers with $\xi_1 > 0$, we consider the set of all sequences of non-negative real numbers with Fréchet metric and after defining a function suitably different from [1], [2] we study the behaviour of the function from various aspects.

Let $X$ be the set of all real sequences $\{x_n\}$ with Fréchet metric $d(x, y)$ given by

$$d(x, y) = \sum_{k=1}^\infty \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where $x = \{x_k\}$, $y = \{y_k\} \in X$.

The metric space $(X, d)$ is complete. Let $S$ denote the set of all sequences $\{x_n\}$ of non-negative real numbers with Fréchet metric. The convergence in this space is considered to be the point-wise convergence.

Let $x \in S$ and $r > 0$. We denote by $B(x, r)$, the open sphere with $x$ as the center and $r$ as the radius. It follows easily that if $x_n = y_n$ for $n = 1, 2, 3, \ldots, N$, then $y \in B(x, \frac{1}{2N})$. If $x, y$ etc. are points of $S$, we shall represent them generally by $x = \{x_k\}$, $y = \{y_k\}$ etc. Also $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers. On the space $S$ we shall define a real function

Received July 16, 2009; revised October 11, 2009.
2000 Mathematics Subject Classification. Primary 40A05.
Key words and phrases. Borel classification of sets; first category; residual sets; first Baire class of sets and Darboux property.
\[ \phi : S \to [1, \infty) \]

as follows

\[ \phi(x) = \inf \left\{ p > 1 : \sum_{n=1}^{\infty} p^{-x_n} < \infty \right\}, \text{ for } x = \{x_n\} \in S. \]

It may happen that \( \phi(x) = +\infty. \) We shall study some properties of \( \phi : S \to [1, \infty). \)

The interval \([1, \infty)\) is considered with usual topology.

**Proposition 0.1.** Let \( \{a_n\} \in S, \) \( a_n > 0 \) be such that \( \sum_{n=1}^{\infty} 1/a_n < \infty \) and \( \sup a_n^{1/x_n} > 0, \) where \( \{x_n\} \in S, \) \( x_n > 0. \) Then there exists \( a > 0 \) such that \( \sum_{n=1}^{\infty} a^{-x_n} < \infty. \)

**Proof.** Take \( a = \sup a_n^{1/x_n}. \) Then \( a > 0. \) Since \( \sup a_n^{1/x_n} = a, \) we have \( a_n^{1/x_n} \leq a, \) for all \( n \in \mathbb{N}. \) Therefore \( \sum_{n=1}^{\infty} a^{-x_n} \leq \sum_{n=1}^{\infty} 1/a_n < \infty. \) Hence the result. \( \square \)

In support of the proposition we present an example.

**Example.** Let \( x_n = \log n, n > 1 \) and \( a_n = n^2. \) Then \( a_n^{1/x_n} = (n^2)^{1/\log n} = (e^{2\log n})^{1/\log n} = e^2, \) for each \( n > 1. \) Take \( a = e^2. \)

**Proposition 0.2.** \((S, d)\) is complete and has the power of continuum.

**Proof.** Let \( \{x^{(r)}_n\}_r \in S \) be any sequence converging to \( x = \{x_n\}. \) Since the convergence in \( S \) is the point-wise convergence in the sense of Fréchet metric, it follows that \( x \in S \) and \( S \) becomes a closed set. Let \( x = \{x_n\} \in S. \) Then we have a sequence \( x^{(r)} = \{x^{(r)}_n\}_n \in S \) such that \( \lim_{r \to \infty} x^{(r)} = x \) where

\[ x^{(r)}_k = x_k, \quad \text{for } k = 1, 2, \ldots, r \]

and \( x^{(r)}_k = x_k + 1, \quad \text{for } k > r; \quad r \in \mathbb{N}. \)

So, \( S \) becomes a perfect set and therefore \( S \) has the cardinal number \( c \) where \( c \) is the power of continuum and hence \((S, d)\) is complete. \( \square \)

1. SOME SET THEORETIC PROPERTIES OF THE FUNCTION \( \phi \)

**Theorem 1.1.** The function \( \phi : S \to (1, \infty) \) is onto but not one-to-one.

**Proof.** Let \( A = \{a_n\}_{n=1}^{\infty} \) be a monotonic increasing sequence with \( a_n \to \infty \) as \( n \to \infty. \) It is well known ([4, p. 40]) that there exists a unique \( \lambda = \lambda(A), \) \( 0 \leq \lambda(A) \leq \infty \) such that

\[ \sum_{n=1}^{\infty} a_n^{-\sigma} = +\infty, \quad \text{for each } \sigma \in \mathbb{R}, \ \sigma > 0, \ \sigma < \lambda \]

and \( \sum_{n=1}^{\infty} a_n^{-\sigma} < +\infty, \) \( \text{for each } \sigma \in \mathbb{R}, \ \sigma > 0, \ \sigma > \lambda, \)

i.e.

\[ \lambda(A) = \inf \{\sigma > 0 : \sum_{n=1}^{\infty} a_n^{-\sigma} < +\infty\}. \]

Now, we can choose such a sequence \( A = \{a_n\}_{n=1}^{\infty} \) with \( \lambda(A) = +\infty. \)
We know [5] that the function \( \lambda : (0, 1] \to [0, \infty) \) is onto. Then for \( 1 < a < \infty \), there exists a subsequence \( \{a_{n_k}\}_{k=1}^\infty \) of \( \{a_n\}_{n=1}^\infty \) such that
\[
a = \inf \left\{ \sigma > 0 : \sum_{k=1}^\infty a_{n_k}^{-\sigma} < +\infty \right\}.
\]

Now, we show that there exists \( y \in S \) such that \( \phi(y) = a \).

Let \( t = \frac{a}{\log a} \) and choose \( y = \{y_n\}_{n=1}^\infty \in S \) such that \( y_k = \log a_{n_k}^t \). Now, for any real number \( b > a \),
\[
\sum_{k=1}^\infty (b)^{-\log a_{n_k}^t} = \sum_{k=1}^\infty (e)^{(-\log b) \log a_{n_k}^t} = \sum_{k=1}^\infty a_{n_k}^{-t \log b} < +\infty,
\]
since \( t \log b > a \).

Again if \( c \) is a real number such that \( 1 < c < a \), then
\[
\sum_{k=1}^\infty (c)^{-\log a_{n_k}^t} = \sum_{k=1}^\infty (e)^{(-\log c) \log a_{n_k}^t} = \sum_{k=1}^\infty a_{n_k}^{-t \log c} = +\infty,
\]
since \( t \log c < a \).

Therefore
\[
\inf \left\{ p > 1 : \sum_{k=1}^\infty p^{-\log a_{n_k}^t} < \infty \right\} = a,
\]
i.e. \( \phi(y) = a \).

We now show that \( \phi \) is not one-to-one.

Let \( a \in (1, \infty) \). Then there exists \( x = \{x_n\}_{n=1}^\infty \in S \) such that \( \phi(x) = a \), i.e.
\[
a = \inf \left\{ p > 1 : \sum_{n=1}^\infty p^{x_n} < \infty \right\}.
\]

Let \( y_n = x_{n+1} \), for \( n = 1, 2, 3, \ldots \). Then \( y = \{y_n\}_{n=1}^\infty \in S \). Clearly
\[
\inf \left\{ p > 1 : \sum_{n=1}^\infty p^{-y_n} < \infty \right\} = a,
\]
i.e. \( \phi(y) = a \). So \( \phi(x) = \phi(y) \) when \( x \neq y \). Therefore, \( \phi \) is not one-to-one. \( \square \)

**Theorem 1.2.** The sets \( H^t = \{ x \in S : \phi(x) < t \} \) and \( H_t = \{ x \in S : \phi(x) > t \} \) belong to the third additive Borel class for every \( t \in (-\infty, \infty) \).

**Proof.** If \( t \leq 1 \), then \( H^t = \phi \) and the theorem is true.

Let \( t > 1 \). Then,
\[
H^t = \{ x \in S : \phi(x) < t \}
= \{ x = \{x_i\}_{i=1}^\infty \in S : \sum_{i=1}^\infty (a)^{-x_i} < \infty \}, \text{ for some } a > 1 \text{ and } 1 < a < t,
= \{ x \in S : \sum_{i=1}^\infty \left( \frac{1}{t} \right)^{-x_i} < \infty \},
\]
for \( k \geq k_0 \) and \( k_0 \) is the least positive integer such that \( a = t - 1/k > 1 \).

We consider \( F(k) = \{ x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} a^{-x_i} < \infty \} \), for some \( a > 1 \) and \( 1 < a < t, \ k = k_0, k_0 + 1, k_0 + 2, \ldots \). Then

\[
F(k) = \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ x : a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p} \right\}.
\]

Set

\[
F(k, p, q, m, n) = \left\{ x : a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p} \right\}.
\]

Let \( x^{(r)} = \{x_i^{(r)}\}_{i=1}^{\infty} \in F(k, p, q, m, n) \) and \( \lim_{r \to \infty} x^{(r)} = x \). So \( \lim_{r \to \infty} a^{-x^{(r)}} = a^{-x} \)

for each \( n = q + m, q + m + 1, q + m + 2, \ldots, q + m + n \), whence \( x \in F(k, p, q, m, n) \).

Consequently, each of the set \( F(k, p, q, m, n) \) is closed. This proves that \( H^t \) is an \( F_{\sigma\delta} \) set. Hence, the set \( \{ x \in S : \phi(x) < t \} \) belongs to the third additive Borel class.

We now investigate the set \( H_t \).

If \( t < 1 \), then \( H_t = S \) and the theorem is true.

If \( t \geq 1 \), then

\[
H_t = \{ x \in S : \phi(x) > t \}
\]

\[
= \bigcup_{k=1}^{\infty} \left\{ x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} (t + \frac{1}{k})^{-x_i} = \infty \right\}.
\]

Consider the set \( G(k) = \{ x = \{x_i\}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} (a)^{-x_i} = \infty \} \), where \( a = t + 1/k, \ k = 1, 2, 3, \ldots \). Then,

\[
G(k) = \bigcap_{p=1}^{\infty} \bigcap_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ x \in S : \sum_{i=1}^{q+m} (a)^{-x_i} \geq p \right\}, \quad k = 1, 2, \ldots.
\]

It is clear that each of the sets \( G(k, p, q, m) = \{ x \in S : \sum_{i=1}^{q+m} (a)^{-x_i} \geq p \} \) is closed. Therefore, the set

\[
\{ x \in S : \phi(x) > t \} = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcup_{m=1}^{\infty} G(k, p, q, m)
\]

is an \( F_{\sigma\delta} \) set, i.e. \( H_t \) belongs to the third additive Borel class. \( \Box \)

**Theorem 1.3.** The set \( H^t = \{ x \in S : \phi(x) < t \} \) is of first category for every \( t \in (-\infty, \infty) \).

**Proof.** It follows from the previous theorem that

\[
H^t = \bigcup_{k=k_0}^{\infty} \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} F(k, p, q, m, n) = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} F(k, p),
\]

where

\[
F(k, p) = \left\{ x \in S : \sum_{q=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(a^{-x_{q+m}} + a^{-x_{q+m+1}} + \ldots + a^{-x_{q+m+n}} \leq \frac{1}{p} \right) \right\}.
\]
In order to show that each of the set \( F(k,p) \) is of first category in \( S \), it is sufficient to show that \( F(k,p) \) is an \( F_\sigma \) set and its complement is dense in \( S \).

Let \( \varepsilon > 0 \). Let \( u = \{ u_n \}_n \) and \( B(u, \varepsilon) \) be an open sphere with \( u \) as the center and \( \varepsilon \) as the radius. Let \( r \) be the smallest positive integer such that \( \sum_{i=r+1}^{\infty} 1/2^i < \varepsilon \).

Define a sequence \( x = \{ x_n \}_n \) in \( S \) as follows: \( x_i = u_i \) for \( i = 1, 2, \ldots, r \).

If \( x_r \leq r + 1 \), take \( x_h = \frac{1}{h} \) for \( h = r + 1, r + 2, \ldots \).

If \( x_r > r + 1 \), set \( x_j = u_r \), for \( j = r + 1, r + 2, \ldots, l - 1 \), where \( l \) is the smallest positive integer for which \( l \geq x_r \) and \( x_h = \frac{1}{h} \) for \( h = l, l + 1, l + 2, \ldots \).

Therefore, we can find an integer \( q \) such that \( x_i = 1/i \) for \( i = q, q + 1, q + 2, \ldots \). Clearly \( x = \{ x_n \}_n \in B(u, \varepsilon) \). For every integer \( q \), there exist integers \( m \) and \( n \) such that

\[
a^{-1/(q+m+1)} + a^{-1/(q+m+2)} + \ldots + a^{-1/(q+m+n)} = \sum_{\alpha=q+m+1}^{q+m+n} a^{-1/\alpha} > \frac{1}{p}
\]

since the series \( \sum_{n=1}^{\infty} a^{-1/n} \) is divergent. Thus, the complement of \( F(k,p) \) is dense in \( S \). Also each of the set \( F(k,p,q,m,n) \) is closed and hence

\[
F(k,p) = \bigcup_{q=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} F(k,p,q,m,n)
\]

is an \( F_\sigma \) set. Then \( F(k,p) \) is of first category in \( S \). But

\[
F(k) = \{ x = \{ x_i \}_{i=1}^{\infty} \in S : \sum_{i=1}^{\infty} a^{-x_i} < \infty \} \quad \text{for some } a > 1, \quad 1 < a < t
\]

\[
= \{ x \in S : \phi(x) < t \} = H^1
\]

Hence, \( H^1 = \bigcup_{k=k_0}^{\infty} \bigcap_{p=1}^{\infty} F(k,p) \) is of first category in \( S \).

**Theorem 1.4.** The set \( \{ x \in S : \phi(x) = \infty \} \) is residual in \( S \).

**Proof.** By Theorem 1.3, the set

\[
\{ x \in S : \phi(x) < \infty \} = \bigcup_{n=1}^{\infty} \{ x \in S : \phi(x) < n \}
\]

is of first category in \( S \) and also the space \( S \) is complete. Hence, the set \( \{ x \in S : \phi(x) = \infty \} \) is residual in \( S \).

**Theorem 1.5.** The function \( \phi \) is discontinuous everywhere in \( S \).

**Proof.** Let \( x = \{ x_k \}_k \in S \). We choose a sequence \( y = \{ y_k \}_k \in S \) such that \( \phi(x) \neq \phi(y) \). Let \( \delta > 0 \). It is sufficient to show that there exists a sequence \( z = \{ z_k \}_k \) in the neighborhood \( B(x, \delta) \) such that \( \phi(z) = \phi(y) \). For \( \delta > 0 \), let \( l \)
be the smallest positive integer such that \( \sum_{i=l+1}^{\infty} 1/2^i < \delta \). Now, we consider the sequence \( \{z_k\}_{k=1}^{\infty} \) as follows:
\[
z_k = \begin{cases} x_k, & \text{for } k \leq l \\ y_k, & \text{for } k > l \end{cases}
\]
It is clear that \( z \in B(x, \delta) \) and \( \phi(z) = \inf \left\{ p > 1 : \sum_{k=1}^{\infty} p^{-z_k} < \infty \right\} \). Now, we consider the sequence \( \{z_k\}_{k=1}^{\infty} \) as follows:
\[
z_k = \begin{cases} x_k, & \text{for } k \leq l \\ y_k, & \text{for } k > l \end{cases}
\]
Hence \( \phi \) is discontinuous everywhere in \( S \).

**Lemma 1.7.** For \( a \in (1, \infty) \), we consider the set
\[
D_a = \{ y(t) = \{y_k\} \in S : y_k = t \cdot x_k, \text{ for } k \leq i, \quad \text{and} \quad y_k = x_k, \text{ for } k > i, \quad 0 < t \leq 1 \}
\]
where \( i \in \mathbb{N} \) and \( \phi(x) = a \), for some \( x = \{x_k\}_{k=1}^{\infty} \in S \). Then \( D_a = \bigcup_{i \in \mathbb{N}} D_a^i \) is connected and \( \phi(D_a) = a \).

**Proof.** Since \( \{x_n\} \in D_a \), \( D_a \) is nonempty. It is clear that \( \phi(D_a) = a \). Now our goal is to show that \( D_a \) is connected. For this purpose we define a function \( f : (0, 1] \to S \) by
\[
f(t) = y(t), \text{ for } t \in (0, 1] \text{ and } y(t) \in D_a^i.
\]
It is clear that \( f \) is continuous in \( t \) on \( (0, 1] \). So, \( f(0, 1] = D_a^i \) is a connected set in \( S \). Again \( f(1) = \{x_n\} \in D_a^i \) for each \( i \in \mathbb{N} \) and hence \( \bigcap_{i \in \mathbb{N}} D_a^i \neq \phi \). Thus \( \bigcup_{i \in \mathbb{N}} D_a^i = D_a \) is connected.

**Theorem 1.8.** Let \( B \) be an arbitrary nontrivial subset of \( (1, \infty) \). Then there exists a connected set \( D \subseteq S \) such that \( \phi(D) = B \).
Proof. Let $a \in B$. Since $\phi$ is onto, there exists $x = \{x_n\} \in S$ such that $\phi(x) = a$. Define the set $D_a = \bigcup_{i \in \mathbb{N}} D_i^a$, where

$$\begin{align*}
D_i^a &= \{ y(t) = \{y_k\} \in S : y_k = t \cdot x_k, \quad \text{for} \quad k \leq i, \quad \text{and} \quad y_k = x_k, \quad \text{for} \quad k > i, \quad 0 < t \leq 1 \} \\
\end{align*}$$

where $i \in \mathbb{N}$. Let $D = \bigcup_{a \in B} D_a$. Then by the previous lemma, $\phi(D_a) = a$. Therefore $\phi(D) = B$. We are to show that $D$ is connected. Let $a_1, a_2 \in B$ be such that $a_1 \neq a_2$. Then there exist $x^{(1)} = \{x_n^{(1)}\}_{n=1}^{\infty}$ and $x^{(2)} = \{x_n^{(2)}\}_{n=1}^{\infty} \in S$ such that $\phi(x^{(1)}) = a_1$ and $\phi(x^{(2)}) = a_2$. Let $y = \{y_n\} \in D_{a_1}$ and $\varepsilon > 0$. Since $\{y_n\} \in D_{a_1}$, there exists $i \in \mathbb{N}$ such that

$$y_n = \begin{cases} 
  t \cdot x_n^{(1)}, & \text{for} \quad n \leq i, \\
  x_n^{(1)}, & \text{for} \quad k > i, \quad 0 < t \leq 1 
\end{cases}$$

We choose $j \in \mathbb{N}$ such that $\sum_{k=j+1}^{\infty} 1/2^k < \varepsilon$. We construct a sequence $z = \{z_k\} \in S$ as follows

$$z_k = \begin{cases} 
  y_k, & \text{for} \quad k \leq j, \\
  x_k^{(2)}, & \text{for} \quad k > j; \quad k \in \mathbb{N}. 
\end{cases}$$

Then $z \in D_{a_2}$ and $d(y, z) < \varepsilon$. This shows that every $\varepsilon$-ball of $y$ contains a member of $D_{a_2}$. So $y \in \overline{D_{a_2}}$, where the symbol ‘bar’ indicates the closure of the set. Hence $D_{a_1} \subseteq \overline{D_{a_2}}$. Similarly $D_{a_2} \subseteq \overline{D_{a_1}}$. Therefore, $D_{a_1}$ and $D_{a_2}$ are not separated. This implies that no two of the sets $\{D_a, a_i \in B\}$ are separated. Thus $D$ is connected. This completes the proof. \hfill $\square$

Corollary 1.9. The function $\phi : S \to (1, \infty)$ is not Darboux.

References


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