THE DUAL SPACE OF THE SEQUENCE SPACE $bv_p$ ($1 \leq p < \infty$)

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Abstract. The sequence space $bv_p$ consists of all sequences $(x_k)$ such that $(x_k - x_{k-1})$ belongs to the space $l_p$. The continuous dual of the sequence space $bv_p$ has recently been introduced by Akhmedov and Basar [Acta Math. Sin. Eng. Ser., 23(10), 2007, 1757–1768]. In this paper, we show a counterexample for case $p = 1$ and introduce a new sequence space $d_\infty$ instead of $d_1$ and show that $bv_1^* = d_\infty$.

Also we have modified the proof for case $p > 1$. Our notations improve the presentation and are confirmed by last notations $l_1^* = l_\infty$ and $l_p^* = l_q$.

1. Priliminaries, background and notation

Let $\omega$ denote the space of all complex-valued sequences, i.e., $\omega = \mathbb{C}^\mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Any vector subspace of $\omega$ which contains $\phi$, the set of all finitely non-zero sequences, is called a sequence space. The continuous dual of a sequence space $\lambda$ which is denoted by $\lambda^*$ is the set of all bounded linear functionals on $\lambda$.

The space $bv_p$ is the set of all sequences of $p$-bounded variation and is defined by

$$bv_p = \left\{ x = (x_k) \in \omega : \left( \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p \right)^{\frac{1}{p}} < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$bv_\infty = \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |x_k - x_{k-1}| < \infty \right\}$$

where $x_{-1} = 0$.

Now, let

$$\|x\|_{bv_p} = \left( \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_{bv_\infty} = \sup_{k \in \mathbb{N}} |x_k - x_{k-1}|.$$
\[ b^{(k)} = (b^{(k)}_n)_{n=0}^\infty \] of elements of the space \( bv_p \) for every fixed \( k \in \mathbb{N} \) by
\[
b^{(k)}_n = \begin{cases} 
0, & \text{if } n < k \\
1, & \text{if } n \geq k
\end{cases}
\]
then the sequence \((b^{(k)})_{k=0}^\infty\) is a Schauder basis for \( bv_p \) and any \( x \in bv_p \) has a unique representation of the form
\[
x = \sum_{k=0}^\infty \lambda_k b^{(k)}
\]
where \( \lambda_k = (x_k - x_{k-1}) \) for all \( k \in \mathbb{N} \).

2. A COUNTEREXAMPLE

In [1, Theorem 2.3] for case \( p = 1 \) suppose \( f = (3, -1, 0, 0, 0, \ldots) \), i.e.,
\[
f_0 = f(e^0) = 3, \quad f_1 = f(e^1) = -1, \quad f_k = f(e^k) = 0 \quad \text{for all } k \geq 2.
\]
Trivially \( f \in bv_1^* \) and
\[
f(x) = f \left( \sum_{k=0}^\infty (\Delta x)_k b^{(k)} \right) = 2(\Delta x)_0 - (\Delta x)_1.
\]
So
\[
\|f\| = \sup_{\|x\|_{bv_1} = 1} |f(x)| = \sup_{\sum_{i=0}^{\infty} |(\Delta x)_i| = 1} |2(\Delta x)_0 - (\Delta x)_1| = 2.
\]

Now inequality (2.5) in [1, Theorem 2.3] asserts that \( \|f\| \geq \sup_{k,n \in \mathbb{N}} |\sum_{j=k}^n f_j| = 3 \) which is a contradiction.

3. THE SPACES \( d_\infty \) AND \( d_q \) (\( 1 < q < \infty \))

In this section, we introduce two sequence spaces and show that they are Banach spaces and then we give the main theorem of the paper. Let
\[
d_\infty = \left\{ a = (a_k)_{k=0}^\infty \in \omega : \|a\|_{d_\infty} = \sup_{k \in \mathbb{N}} \left| \sum_{j=k}^\infty a_j \right| < \infty \right\}
\]
and
\[
d_q = \left\{ a = (a_k)_{k=0}^\infty \in \omega : \|a\|_{d_q} = \left( \sum_{k=0}^\infty \left| \sum_{j=k}^\infty a_j \right|^q \right)^{\frac{1}{q}} < \infty \right\}, \quad (1 < q < \infty).
\]

**Theorem 3.1.** \( d_\infty \) is a sequence space with usual coordinatewise addition and scalar multiplication and \( \|\cdot\|_{d_\infty} \) is a norm on \( d_\infty \).
Proof. We only show that $\|\cdot\|_{d_{\infty}}$ is a norm on $d_{\infty}$. Let

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

Then

$$Da = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} a_j \\ \sum_{j=1}^{\infty} a_j \\ \sum_{j=2}^{\infty} a_j \\ \sum_{j=3}^{\infty} a_j \\ \vdots \end{bmatrix}.$$ 

So $\|a\|_{d_{\infty}} = \sup_{k \in \mathbb{N}} \sum_{j=k}^{\infty} a_j = \sup_{k \in \mathbb{N}} \| (Da)_k \| = \| Da \|_{l_{\infty}}$. Now, if $a \in d_{\infty}$ then $\| Da \|_{l_{\infty}} = \|a\|_{d_{\infty}} < \infty$ hence $Da \in l_{\infty}$. Also if $Da \in l_{\infty}$, then $\|a\|_{d_{\infty}} = \| Da \|_{l_{\infty}} < \infty$ hence $a \in d_{\infty}$. So $a \in d_{\infty}$ if and only if $Da \in l_{\infty}$. Now since

(I) $0 \leq \| Da \|_{l_{\infty}} = \|a\|_{d_{\infty}} < \infty$

(II) $\|a + b\|_{d_{\infty}} = \| Da + Db \|_{l_{\infty}} \leq \| Da \|_{l_{\infty}} + \| Db \|_{l_{\infty}} = \|a\|_{d_{\infty}} + \|b\|_{d_{\infty}}$

(III) $\alpha \cdot \| a \|_{d_{\infty}} = \| \alpha \cdot Da \|_{l_{\infty}} = |\alpha| \cdot \| Da \|_{l_{\infty}} = |\alpha| \cdot \| a \|_{d_{\infty}}$

$\|\cdot\|_{d_{\infty}}$ is a norm on $d_{\infty}$.

**Theorem 3.2.** $d_{\infty}$ is a Banach space.

Proof. Let $(a^{(n)})_{n=0}^{\infty}$ is a Cauchy sequence in $d_{\infty}$. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for all $n, m \geq N$

$$\|a^{(n)} - a^{(m)}\|_{d_{\infty}} < \varepsilon.$$ 

So

$$\| Da^{(n)} - Da^{(m)}\|_{l_{\infty}} = \|a^{(n)} - a^{(m)}\|_{d_{\infty}} < \varepsilon.$$ 

So the sequence $(Da^{(n)})_{n=0}^{\infty}$ is Cauchy in $l_{\infty}$. So there exists $a \in l_{\infty}$ such that $Da^{(n)} \to a$ in $l_{\infty}$. So $\| Da^{(n)} - DD^{-1}a \|_{l_{\infty}} \to 0$ and $\|a^{(n)} - D^{-1}a\|_{d_{\infty}} \to 0$

Furthermore, $D^{-1}a \in d_{\infty}$ since $DD^{-1}a = a \in l_{\infty}$.

**Theorem 3.3.** $bv_1^*$ is isometrically isomorphic to $d_{\infty}$.

Proof. Define $T : bv_1^* \to d_{\infty}$ and $T f = (f(e^{(0)}), f(e^{(1)}), f(e^{(2)}), \ldots)$ where $e^{(k)} = (0, \ldots, 0, \underbrace{1}_{k^{\text{th\ term}}}, 0, \ldots)$. Trivially, $T$ is linear and injective since

$$T f = 0 \Rightarrow f = 0.$$
$T$ is surjective since if $\tilde{g} = (g_0, g_1, g_2, g_3, \ldots) \in d_\infty$ then if we define $f : bv_1 \to \mathbb{C}$ by

$$f(x) = \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} g_j.$$ 

Then $f \in bv_*$. Trivially, since $f$ is linear and

$$|f(x)| = \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} g_j \right| \leq \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \sum_{j=k}^{\infty} |g_j| = \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \|\tilde{g}\|_{d_\infty}$$

and $Tf = \tilde{g}$, so $T$ is surjective. Now we show that $T$ is norm preserving, we have

$$\|f(x)\| = \left| f\left( \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} e^{(j)} \right) \right| = \left| \sum_{k=0}^{\infty} (\Delta x)_k \sum_{j=k}^{\infty} f(e^{(j)}) \right|$$

$$\leq \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \sum_{j=k}^{\infty} |f(e^{(j)})| \leq \sum_{k=0}^{\infty} |(\Delta x)_k| \cdot \sup_{k \in \mathbb{N}} \sum_{j=k}^{\infty} |f(e^{(j)})|$$

$$\leq \|x\|_{bv_1} \cdot \|Tf\|_{d_\infty}.$$ 

So

(*)

$$\|f\| \leq \|Tf\|_{d_\infty}$$

On the other hand, $|\sum_{j=k}^{\infty} f(e^{(j)})| = \left| f(b^{(k)}) \right| \leq \|f\| \cdot \|b^{(k)}\|_{bv_1} = \|f\|$. So

$$\sum_{j=k}^{\infty} f(e^{(j)}) \leq \|f\|$$

for all $k \in \mathbb{N}$. So

So

$$\sup_{k \in \mathbb{N}} |\sum_{j=k}^{\infty} f(e^{(j)})| \leq \|f\|,$$

i.e.,

$$(\dagger) \quad \|Tf\|_{d_\infty} \leq \|f\|$$

by (*) and (†) we are done.

Theorem 3.4. $d_q$ $(1 \leq q < \infty)$ is a sequence space with usual coordinatewise addition and scalar multiplication and $\|\|_{d_q}$ is a norm on $d_q$.

Proof. With notations of Theorem 3.1, $\|a\|_{d_q} = \|Da\|_{l_q}$ and $a \in d_q \Leftrightarrow Da \in l_q$. The continuation of the proof is similar to Theorem 3.1. □

Theorem 3.5. $d_q$ $(1 \leq q < \infty)$ is a Banach space.

Proof. The proof is similar to proof of Theorem 3.2 and we omit it. □
Theorem 3.6. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $\text{bv}_p^*$ is isometrically isomorphic to $d_q$.

Proof. Define $A : \text{bv}_p^* \rightarrow d_q$ by $f \mapsto Af = (f(e^{(1)})_k, f(e^{(2)})_k, \ldots)$. Trivially $A$ is linear. Additionally, since $Af = 0 = (0, 0, 0, \ldots)$ implies $f = 0$, $A$ is injective. $A$ is surjective since if $a = (a_k) \in d_q$, then $\exists f \in \text{bv}_p^*$ such that $f(x) = \sum_{k=0}^\infty (\Delta x)_k \sum_{j=k}^\infty a_j$.

Then $f$ is linear. Since

$$|f(x)| \leq \sum_{k=0}^\infty |(\Delta x)_k| \sum_{j=k}^\infty |a_j| \leq \left[ \sum_{k=0}^\infty |(\Delta x)_k|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{j=0}^\infty |a_j|^q \right]^{\frac{1}{q}} = \|x\|_{\text{bv}_p^*} \cdot \|a\|_{d_q},$$

it yields to $\|f\| \leq \|a\|_{d_q} < \infty$. So $f \in \text{bv}_p^*$ and $Af = a$.

Now, we show that $A$ is norm preserving. Let $f \in \text{bv}_p^*$ and $x = (x_k)_{k=0}^\infty \in \text{bv}_p$, then

$$|f(x)| = \left| \sum_{k=0}^\infty (\Delta x)_k \sum_{j=k}^\infty f(e^{(j)}) \right| \leq \sum_{k=0}^\infty |(\Delta x)_k| \sum_{j=k}^\infty |f(e^{(j)})| \leq \left[ \sum_{k=0}^\infty |(\Delta x)_k|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{j=0}^\infty \sum_{k=0}^\infty |f(e^{(j)})|^q \right]^{\frac{1}{q}} = \|x\|_{\text{bv}_p^*} \cdot \|Af\|_{d_q}.$$

So

$$(*) \quad \|f\| \leq \|Af\|_{d_q}.$$  

On the other hand, suppose $f \in \text{bv}_p^*$ and $x^{(n)} = (x^{(n)}_k)_{k=0}^\infty$ are such that

$$(\Delta x^{(n)})_k = \begin{cases} \sum_{j=k}^\infty f(e^{(j)}) \left| \begin{array}{c} q-1 \\ \frac{1}{p} \end{array} \right| \text{sgn} \left( \sum_{j=k}^\infty f(e^{(j)}) \right), & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n. \end{cases}$$

We note that $\sum_{j=k}^\infty f(e^{(j)}) = f(b^{(k)})$. So $x^{(n)} \in \text{bv}_p$ since $\Delta x^{(n)} \in l_p$.

Then it is clear that

$$\Delta x^{(n)} = \left( \sum_{j=0}^\infty f(e^{(j)}) \left| \begin{array}{c} q-1 \\ \frac{1}{p} \end{array} \right| \text{sgn} \left( \sum_{j=0}^\infty f(e^{(j)}) \right), \ldots, \sum_{j=n}^\infty f(e^{(j)}) \left| \begin{array}{c} q-1 \\ \frac{1}{p} \end{array} \right| \text{sgn} \left( \sum_{j=n}^\infty f(e^{(j)}) \right), 0, 0, \ldots \right).$$
So

\[ x^{(n)} = \left( \sum_{j=0}^{\infty} f(e^{(j)}) \right)^{q-1} \left( \sum_{j=0}^{\infty} f(e^{(j)}) \right) + b_0 + \left( \sum_{j=1}^{\infty} f(e^{(j)}) \right)^{q-1} \left( \sum_{j=1}^{\infty} f(e^{(j)}) \right) \]

So if we let \( f_k = f(e^{(k)}) \), then

\[ f(x^{(n)}) = b_0 f_0 + b_0 f_1 + b_1 f_1 + \ldots + b_0 f_n + b_1 f_n + \ldots + b_0 f_{n+1} + b_1 f_{n+1} + \ldots + b_0 f_{n+3} + b_1 f_{n+3} + \ldots \]

\[ = \sum_{k=0}^{n} \sum_{j=k}^{\infty} f_j^{q} \]

So

\[ \sum_{k=0}^{n} \sum_{j=0}^{\infty} f_j = f(x^{(n)}) = |f(x^{(n)})| \leq \|f\| \cdot \|x^{(n)}\|_{bp} = \|f\| \cdot \left[ \sum_{k=0}^{n} \sum_{j=0}^{\infty} f_j^{q} \right]^\frac{1}{p} \]

Since

\[ \|x^{(n)}\|_{bp} = \|\Delta x^{(n)}\|_{p} = \left[ \sum_{k=0}^{\infty} |\Delta x_k^{(n)}|^p \right]^\frac{1}{p} = \left[ \sum_{k=0}^{n} |\Delta x_k^{(n)}|^p \right]^\frac{1}{p} \]

\[ = \left[ \sum_{k=0}^{n} \left( \sum_{j=k}^{\infty} f_j \right)^{q-1} \right]^{\frac{1}{p}} \]

\[ = \left[ \sum_{k=0}^{n} \left( \sum_{j=k}^{\infty} f_j \right)^{q} \right]^{\frac{1}{p}} \]
The dual space of the sequence space $bv_p (1 \leq p < \infty)$

So

$$\left[ \sum_{k=0}^{n} \left| \sum_{j=k}^{\infty} f_j \right| \right]^{\frac{1}{q}} \leq \|f\| \cdot \left[ \sum_{k=0}^{n} \left| \sum_{j=k}^{\infty} f_j \right| \right]^{\frac{1}{p}}.$$ 

So

$$\|f\| \geq \left[ \sum_{k=0}^{n} \left| \sum_{j=k}^{\infty} f_j \right| \right]^{\frac{1}{2}} = \|Af\|_{d_q}.$$

Therefore, by combining the results (*) and (†), $A$ is norm preserving. Hence $bv_p^\ast$ is isometrically isomorphic to $d_q$. □

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