Abstract. Let $R$ be a ring. A right ideal $I$ of $R$ is called small in $R$ if $I + K \neq R$ for every proper right ideal $K$ of $R$. A ring $R$ is called right small finitely injective (briefly, SF-injective) (resp., right small principally injective (briefly, SP-injective)) if every homomorphism from a small and finitely generated right ideal (resp., a small and principally right ideal) to $R_R$ can be extended to an endomorphism of $R$. The class of right SF-injective and SP-injective rings are broader than that of right small injective rings (in [15]). Properties of right SF-injective rings and SP-injective rings are studied and we give some characterizations of a QF-ring via right SF-injectivity with ACC on right annihilators. Furthermore, we answer a question of Chen and Ding.

1. Introduction

Throughout the paper $R$ represents an associative ring with identity $1 \neq 0$ and all modules are unitary $R$-module. We write $M_R$ (resp. $R_M$) to indicate that $M$ is a right (resp. left) $R$-module. We use $J$ (resp. $Z_r$, $S_r$) for the Jacobson radical (resp. the right singular ideal, the right socle of $R$) and $E(M_R)$ for the injective hull of $M_R$. If $X$ is a subset of $R$, the right (resp. left) annihilator of $X$ in $R$ is denoted by $r_R(X)$ (resp. $l_R(X)$) or simply $r(X)$ (resp. $l(X)$) if no confusion appears. If $N$ is a submodule of $M$ (resp. proper submodule) we denote by $N \leq M$ (resp. $N < M$). Moreover, we write $N \leq^* M$, $N \ll M$, $N \leq^\oplus M$ and $N \leq^{\text{max}} M$ to indicate that $N$ is an essential submodule, a small submodule, a direct summand and a maximal submodule of $M$, respectively. A module $M$ is called uniform if $M \neq 0$ and every non-zero submodule of $M$ is essential in $M$. $M$ is finite dimensional (or has finite rank) if $E(M)$ is a finite direct sum of indecomposable submodules; or equivalently, if $M$ has an essential submodule which is a finite direct sum of uniform submodules.

A module $M_R$ is called $F$-injective (resp., $P$-injective) if every right homomorphism from a finitely generated (resp., principal) right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right $F$-injective (resp.,
right P-injective) if $R_R$ is F-injective (resp., P-injective). $R$ is called right min-injective if every right $R$-homomorphism from a minimal right ideal to $R$ can be extended to an endomorphism of $R_R$. A ring $R$ is said to be a right PF-ring if the right $R_R$ is an injective cogenerator in the category of right $R$-modules. A ring $R$ is called QF-ring if it is right (or left) Artinian and right (or left) self-injective.

In [15], a module $M_R$ is called small injective if every homomorphism from a small right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right small injective if $R_R$ is small injective. Under small injective condition, Shen and Chen ([15]) gave some new characterizations of QF rings and right PF rings. In [18], authors showed some characterizations of Jacobson radical $J$ via small injectivity. They proved that $J$ is Noetherian as a right $R$-module if and only if every direct sum of small injective right $R$-modules is small injective if and only if $E^{(n)}$ is small injective for every small injective module $E_R$.

In 1966, Faith proved that $R$ is QF if and only if $R$ is right self-injective and satisfies ACC on right annihilators. Then around 1970, Björk proved that $R$ is QF if and only if $R$ is right F-injective and satisfies ACC on right annihilators. In this paper, we show that $R$ is QF if and only if $R$ is a semiregular and right SF-injective ring with ACC on right annihilators if and only if $R$ is a semilocal and right SF-injective ring with ACC on right annihilators if and only if $R$ is a right SF-injective ring with ACC on right annihilators in which $S_r \leq R_R$. We also give some characterizations of rings whose $R$-homomorphism from a small, finitely generated right ideal to $R$ with a simple image, can be extended to an endomorphism of $R_R$. Furthermore, we prove that if $R$ is a right perfect, right simple-injective and left pseudo-coherent ring, then $R$ is QF. Some known results are obtained as corollaries.

A general background material can be found in [1], [7], [19].

2. ON SP(SF)-INJECTIVE RINGS

**Definition 2.1.** A module $M_R$ is called small principally injective (briefly, SP-injective) if every homomorphism from a small and principal right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A module $M_R$ is called small finitely injective (briefly, SF-injective) if every homomorphism from a small and finitely generated right ideal to $M_R$ can be extended to an $R$-homomorphism from $R_R$ to $M_R$. A ring $R$ is called right SP-injective (resp., right SF-injective) if $R_R$ is SP-injective (resp., SF-injective).

The following implications are obvious:

\[ \text{small injective} \quad \iff \quad \text{injection} \quad \iff \quad \text{SF - injective} \quad \implies \quad \text{SP - injective} \]

\[ \text{injection} \quad \iff \quad \text{F - injective} \]
Lemma 2.2. The following conditions are equivalent for a ring $R$:

1. $R$ is right SP-injective.
2. $lr(a) = Ra$ for all $a \in J$.
3. $r(a) \leq r(b)$, where $a \in J$, $b \in R$, implies $Rb \leq Ra$.
4. $l(bR \cap r(a)) = l(b) + Ra$ for all $a \in J$ and $b \in R$.
5. If $\gamma : aR \to R$, $a \in J$, is an $R$-homomorphism, then $\gamma(a) \in Ra$.

Proof. A similar proving to [10, Lemma 5.1].

We also have:

Lemma 2.3. A ring $R$ is right SF-injective if and only if it satisfies the following two conditions:

1. $l(T \cap T') = l(T) + l(T')$ for all small, finitely generated right ideals $T$ and $T'$.
2. $R$ is right SP-injective.

Proof. ($\Rightarrow$): Assume that $R$ is right SF-injective. If $T$ and $T'$ are small, finitely generated right ideals, then $T + T'$ is a small finitely generated right ideal. Let $b \in l(T \cap T')$ and then we define $\alpha : T + T' \to R$ via $\alpha(t + t') = bt$, for all $t \in T$ and $t' \in T'$, so $\alpha = a_\ast$, for some $a \in R$ by hypothesis. Then $b - a \in l(T)$ and $a \in l(T')$. Hence $b \in l(T) + l(T')$. Thus (1) holds. (2) is clear.

($\Leftarrow$): We can prove it by induction on the number of generators of $T$ and $T'$. □

Corollary 2.4. Let $R$ be a right SP-injective ring such that $l(T \cap T') = l(T) + l(T')$ for all right ideals $T$ and $T'$ of $R$ where $T$ is small, finitely generated. Then every $R$-homomorphism $\varphi : I \to R$ extends to $R \to R$ where $I$ is a small right ideal and the image $\varphi(I)$ is finitely generated.

Proposition 2.5. A direct product $R = \prod_{i \in I} R_i$ of rings $R_i$ is right SF-injective (resp., right SP-injective) if and only if $R_i$ is right SF-injective (resp., right SP-injective) for each $i \in I$.

Proof. Assume that $R = \prod_{i \in I} R_i$ is right SF-injective. For each $i \in I$, we take any $a_i \in J(R_i)$ and $b_i \in R_i$ such that $r_{R_i}(a_i) \leq r_{R_i}(b_i)$. Let $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I}$, where $a_j = 0, b_j = 0$, $\forall j \neq i$ and $a_j = a_i, b_j = b_i$ if $j = i$. Then $a \in J(R), b \in R$ and $r_R(a) \leq r_R(b)$. So $b \in Ra$ since $R$ is right SF-injective. Therefore $b_i \in R_i a_i$. Thus $R_i$ is right SP-injective. On the other hand, for all small, finitely generated right ideals $T_i$ and $T'_i$ of $R_i$, $\iota_i(T_i), \iota_i(T'_i)$ are small, finitely generated right ideals of $R$, where $\iota_i : R_i \hookrightarrow R$ is the inclusion for each $i \in I$. By hypothesis, $l_R(\iota_i(T_i) \cap \iota_i(T'_i)) = l_R(\iota_i(T_i)) + l_R(\iota_i(T'_i))$. This implies that $l_R(T_i \cap T'_i) = l_{R_i}(T_i) + l_{R_i}(T'_i)$. Thus $R_i$ is right SF-injective by Lemma 2.3.

Conversely, $R = \prod_{i \in I} R_i$, where $R_i$ is right SF-injective. For each $a = (a_i)_{i \in I} \in J(R)$ and $b = (b_i)_{i \in I} \in R$ such that $r_R(a) \leq r_R(b)$, then for each $i \in I$, $a_i \in J(R_i)$ and $r_{R_i}(a_i) \leq r_{R_i}(b_i)$. Since $R_i$ is right SF-injective, $b_i \in R_i a_i$. Hence $b \in Ra$. If $T$ and $T'$ are small, finitely generated right ideals of $R$, then we can prove that $l_R(T \cap T') = l_R(T) + l_R(T')$. Thus $R$ is right SF-injective. □
A ring $R$ is called left minannihilator if $lr(K) = K$ for every minimal left ideal $K$ of $R$.

**Proposition 2.6.** Let $R$ be a right SP-injective ring. Then:

1. $R$ is right mininjective and left minannihilator.
2. $J \leq Z_r$.

**Proof.** (1) Since every minimal one-sided ideal of $R$ is either nilpotent or a one-sided direct summand of $R$, each right SP-injective ring is right mininjective and left minannihilator.

(2) If $a \in J$ we will show that $r(a) \leq^e R_R$. In fact, let $b \in R$ such that $bR \cap r(a) = 0$. By Lemma 2.2, $R = l(b) + Ra$, so $l(b) = R$ because $a \in J$. Hence $b = 0$. This proves that $a \in Z_r$.

A ring $R$ is called right Kasch if every simple right $R$-module embeds in $R_R$.

**Proposition 2.7.** Let $R$ be a right Kasch ring. Then:

1. If $R$ is right SP-injective, then:
   a) The map $\psi : T \mapsto l(T)$ from the set of maximal right ideals $T$ of $R$ to the set of minimal left ideals of $R$ is a bijection. And the inverse map is given by $K \mapsto r(K)$, where $K$ is a minimal left ideal of $R$.
   b) For $k \in R$, $R_k$ is minimal iff $kR$ is minimal, in particular $S_e = S_1$.
2. If $R$ is right SF-injective, then $rl(I) = I$ for every small, finitely generated right ideal $I$ of $R$. In particular, $R$ is left SP-injective.

**Proof.** (1) a): By Proposition 2.6 (1) and [10, Theorem 2.32]. For b), if $Rk$ is minimal, then $r(k)$ is maximal by a). This means $kR$ is minimal. Conversely, by [10, Theorem 2.21].

(2): Firstly, we have $J = rl(J)$ because $R$ is right Kasch. Let $T$ be a right small, finitely generated ideal of $R$. Therefore, $T \leq rl(T) \leq rl(J) = J$. If $b \in rl(T) \backslash T$, take $I$ such that $T \leq I \leq^{\text{max}} (bR + T)$. Since $R$ is right Kasch, we can find a monomorphism $\sigma : (bR + T)/I \rightarrow R$, and then define $\gamma : bR + T \rightarrow R$ via $\gamma(x) = \sigma(x + I)$. Since $bR + I$ is a small, right finitely generated ideal of $R$ and $R$ is right SF-injective, it follows that $\gamma = c$, where $c \in R$. Hence $cb = \sigma(b + I) \neq 0$ because $b \notin I$. But if $t \in T$, then $ct = \sigma(t + I) = 0$ because $T \leq I$, so $c \in l(I)$. Since $b \in rl(T)$ this gives $cb = 0$, a contradiction. Thus $T = rl(T)$. It is clear that $R$ is left SP-injective.

Recall that a ring $R$ is called semiregular if $R/J$ is von Neumann regular and idempotents can be lifted modulo $J$. Note that if $R$ is semiregular, then for every finitely generated right ideal $I$ of $R$, $R_H = H \oplus K$, where $H \leq I$ and $I \cap K \ll R$.

Motivated by [15, Lemma 3.1] we have the following result.

**Lemma 2.8.** If $R$ is a semiregular ring and $I$ is a right ideal of $R$, then the following conditions are equivalent:

1. Every homomorphism from a finitely generated right ideal to $I$ can be extended to an endomorphism of $R_H$.
(2) Every homomorphism from a small, finitely generated right ideal to $I$ can be extended to an endomorphism of $R_R$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Let $f : K \to I$ be an $R$-homomorphism, where $K$ is a finitely generated right ideal. Since $R$ is semiregular, then $R = H \oplus L$, where $H \leq K$ and $K \cap L \ll R$. Hence $R = K + L$ and $K = H \oplus (K \cap L)$, $K \cap L$ is a small, finitely generated right ideal of $R$. Thus there exists an endomorphism $g$ of $R_R$ such that $g(x) = f(x)$ for all $x \in K \cap L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(r) = f(k) + g(l)$ for any $r = k + l$, $k \in K$, $l \in L$. Now we show that $\varphi$ is well defined. Indeed, if $k_1 + l_1 = k_2 + l_2$, where $k_i \in K$, $l_i \in L$, $i = 1, 2$, then $k_1 - k_2 = l_1 - l_2 \in K \cap L$. Hence $f(k_1 - k_2) = g(l_1 - l_2)$, which implies that $\varphi(k_1 + l_1) = \varphi(k_2 + l_2)$. Thus $\varphi$ is an endomorphism of $R_R$ such that $\varphi|_K = f$. □

Let $I$ be an ideal of $R$. A ring $R$ is called right $I$-semiregular if for every $a \in I$, $aR = eR \oplus T$, where $e^2 = e$ and $T \leq I_R$.

**Corollary 2.9.** Let $R$ be a right $Z_r$-semiregular ring. Then $R$ is right SF-injective if and only if $R$ is right F-injective.

It is well-known if $R$ is semiperfect and right small injective with $S_r \leq^e R_R$, then $R$ is right self-injective. This result is proved by Yousif and Zhou (see [20, Theorem 2.11]). In [15, Theorem 3.4], they showed that a semilocal (or semiregular) ring $R$ is right self-injective if and only if $R$ is right small injective. From Lemma 2.8 we also have a similar result.

**Theorem 2.10.** Let $R$ be a semiregular ring. Then

(1) $R$ is right $P$-injective if and only if $R$ is right SP-injective.

(2) $R$ is right $F$-injective if and only if $R$ is right SF-injective.

Because a semiperfect ring is semiregular, we have:

**Corollary 2.11.** Let $R$ be a semiperfect ring. Then

(1) $R$ is right $P$-injective if and only if $R$ is right SP-injective.

(2) $R$ is right $F$-injective if and only if $R$ is right SF-injective.

Next we obtain some characterizations of QF-ring via right SF-injectivity with ACC on right annihilators. The following theorem extends [15, Theorem 3.8].

**Theorem 2.12.** For a ring $R$, the following conditions are equivalent:

(1) $R$ is QF.

(2) $R$ is a semiregular and right SF-injective ring with ACC on right annihilators.

(3) $R$ is a semilocal and right SF-injective ring with ACC on right annihilators.

(4) $R$ is a right SF-injective ring with ACC on right annihilators in which $S_r \leq^e R_R$. 


Proof. It is obvious that (1) ⇒ (2), (3), (4).
(2) ⇒ (1): By Theorem 2.10, R is right F-injective. Thus R is QF by [3, Theorem 4.1].
(3) ⇒ (1): Since R satisfies ACC on right annihilators, Z_r is nilpotent and so Z_r ⊆ J. Therefore, J = Z_r is nilpotent by Proposition 2.6. Hence R is semiprimary.
(4) ⇒ (1): By [13, Theorem 2.1] or [14, Lemma 2.11], R is semiprimary. □

Corollary 2.13. Let R be a ring. Then R is QF if and only if R is a semilocal, left and right SP-injective ring with ACC on right annihilators.

Remark. The condition “semilocal” in Theorem 2.12 can not be omitted, since the ring of integers Z is SP-injective, Noetherian, but Z is not QF.

The following result extends [11, Theorem 2.2].

Proposition 2.14. If R is right SP-injective and R/Soc(RR) has ACC on right annihilators, then J is nilpotent.

Proof. Here we use a similar argument to that one in [2, Theorem 3]. Suppose that R/Soc(RR) has ACC on right annihilators. Let S = Soc(RR) and S = R/S. For any a_1, a_2, . . ., in J, since
\[ r_R(\bar{a}_1) \leq r_R(\bar{a}_2 \bar{a}_1) \leq \ldots, \]
by hypothesis there exists a positive integer m such that
\[ r_R(\bar{a}_m \ldots \bar{a}_2 \bar{a}_1) = r_R(\bar{a}_{m+k} \ldots \bar{a}_2 \bar{a}_1) \]
for k = 0, 1, 2, . . . . Now for any positive integer n, since a_{n+1}a_n . . . a_1 ∈ J ≤ Z_r, r(a_{n+1}a_n . . . a_1) ≤ S R. Hence S ≤ r(a_{n+1}a_n . . . a_1). We claim that
\[ r_R(\bar{a}_n \ldots \bar{a}_2 \bar{a}_1) \leq r(a_{n+1}a_n \ldots a_1)/S \leq r_R(\bar{a}_{n+1} \ldots \bar{a}_2 \bar{a}_1). \]
In fact, assume b + S ∈ r_R(\bar{a}_n \ldots \bar{a}_2 \bar{a}_1). Then we have a_n . . . a_1 b ∈ S. But since S ≤ r(a_{n+1}), we get a_{n+1}a_n . . . a_1 b = 0. Thus b ∈ r(a_{n+1}a_n . . . a_1), and so b + S ∈ r(a_{n+1}a_n . . . a_1)/S. Now the other inclusion r(a_n . . . a_1)/S ≤ r_R(\bar{a}_{n+1} \ldots \bar{a}_2 \bar{a}_1) is obvious.

By this fact, it follows that
\[ r(a_{m+1}a_m \ldots a_1)/S = r(a_{m+2}a_{m+1} \ldots a_1)/S \]
because r_R(\bar{a}_m \ldots \bar{a}_2 \bar{a}_1) = r_R(\bar{a}_{m+2} \ldots \bar{a}_2 \bar{a}_1). Therefore
\[ r(a_{m+1}a_m \ldots a_1) = r(a_{m+2}a_{m+1}a_m \ldots a_1), \]
and hence (a_{m+1}a_m . . . a_1) R ∩ r(a_{m+2}) = 0. But r(a_{m+2}) is an essential right ideal of R, and so a_{m+1}a_m . . . a_1 = 0. Hence J is right T-nilpotent and the ideal (J + S)/S of the ring R = R/S is also right T-nilpotent. By [1, Proposition 29.1], (J + S)/S is nilpotent, and so there is a positive integer t such that J^t ≤ S. Hence J^{t+1} ≤ SJ. Thus J is nilpotent. □

Theorem 2.15. If R is a semilocal and right SF-injective ring such that R/S_r is right Goldie, then R is QF.
Proof. By Proposition 2.14, \( J \) is nilpotent, and hence \( R \) is semiprimary. Hence \( R \) is right \( F \)-injective by Theorem 2.10. This implies that \( R \) is right GPF (i.e., \( R \) is semiperfect, right \( P \)-injective with \( S_r \leq R_R \)) and so \( R \) is right Kasch by [11, Corollary 2.3]. Therefore \( R \) is left \( P \)-injective by [3, Proposition 4.1]. Thus \( R \) is QF by [10, Theorem 3.38]. \( \Box \)

**Corollary 2.16.** If \( R \) is a semilocal and right SF-injective ring satisfying ACC on essential right ideals, then \( R \) is QF.

Now we consider rings whose small and finitely generated right ideals are projective. We have the following result.

**Theorem 2.17.** For a ring \( R \) the following conditions are equivalent:

1. Every small and finitely generated right ideal of \( R \) is projective.
2. Every quotient module of a SF-injective module is SF-injective.
3. Every quotient module of a \( F \)-injective module is SF-injective.
4. Every quotient module of a small injective module is SF-injective.
5. Every quotient module of an injective module is SF-injective.

Proof. (2) \( \Rightarrow \) (3) \( \Rightarrow \) (5) and (2) \( \Rightarrow \) (4) \( \Rightarrow \) (5) are obvious.

(1) \( \Rightarrow \) (2): Assume that \( E_R \) is SF-injective and \( \pi : E \to B \) is an epimorphism. Let \( f : I \to B \) be an \( R \)-homomorphism, where \( I \) is a small and finitely generated right ideal of \( R \).

\[
\begin{array}{c}
0 \\
E \\
\end{array} \xleftarrow{\iota} \begin{array}{c} I \\
\downarrow f \end{array} \xrightarrow{\iota} \begin{array}{c} R \\
\downarrow \pi \\
B \\
\end{array} \xrightarrow{\psi} 0
\]

where \( \iota \) is the inclusion.

By (1), \( I \) is projective. Therefore there exists an \( R \)-homomorphism \( h : I \to E \) such that \( \pi h = f \). Now since \( E \) is SF-injective, there is an \( R \)-homomorphism \( h' : R \to E \) such that \( h' \iota = h \). Let \( h'' = \pi h' : R \to B \), then \( h'' \iota = f \). This means \( B_R \) is SF-injective.

(5) \( \Rightarrow \) (1): For every small and finitely generated right ideal \( I \) of \( R \), we consider the epimorphism \( h : A \to B \) and \( R \)-homomorphism \( \alpha : I \to B \).

Since \( B = h(A) \cong A/\text{Ker } h \xleftarrow{\iota_1} E(A)/\text{Ker } h \), where \( \iota_1 \) is the inclusion and \( \psi(h(a)) = a + \text{Ker } h \), for all \( a \in A \). Then let \( j = \iota_1 \psi \). We consider the following diagram:

\[
\begin{array}{ccc}
I & \xrightarrow{\iota} & R \\
\downarrow \varphi & \downarrow \alpha & \\
E & \xrightarrow{h} & B \\
\downarrow j & & \downarrow p \\
E(A) & \xrightarrow{\psi} & E(A)/\text{Ker } h \xrightarrow{0}
\end{array}
\]

where \( \iota \) is the inclusion and \( p \) is the natural epimorphism.
By (5), $E(A)/\ker h$ is SF-injective and then there exists an $R$-homomorphism $\alpha' : R \to E(A)/\ker h$ such that $\alpha' \circ j = h$. Since $R_R$ is projective, there is an $R$-homomorphism $\alpha'' : R \to E(A)$ such that $p\alpha'' = \alpha'$. Let $h' = \alpha'' \circ j : I \to E(A)$.

It is easy to see that $h'(I) \subseteq A$, so there exists an $R$-homomorphism $\varphi : I \to A$ such that $\varphi(x) = h'(x)$, for all $x \in I$.

Now we claim that $h\varphi = \alpha$. In fact, for each $x \in I$ we have

$$j(\alpha(x)) = \alpha'(j(x)) = p(\alpha''(x)) = p(h'(x)) = p(\varphi(x)).$$

Since $\alpha$ is the epimorphism, $\alpha(x) = h(a)$ for some $a \in A$. Therefore $j(\alpha(x)) = j(h(a)) = a + \ker h$, and so $a + \ker h = \varphi(x) + \ker h$, $h(a - \varphi(x)) = 0$. Hence $h\varphi(x) = h(a) = \alpha(x)$. Thus $I$ is projective.

\begin{proof}
Example 2.18. i) Let $R = F[x_1, x_2, \ldots]$, where $F$ is a field and $x_i$ are commuting indeterminants satisfying the relations: $x_i^n = 0$ for all $i$, $x_ix_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all $i$ and $j$. Then $R$ is a commutative, semiprimary $F$-injective ring. But $R$ is not a self-injective ring (see [10, Example 5.45]). Thus $R$ is SF-injective, but $R$ is not a small injective ring. Because if $R$ is small injective, then $R$ is self-injective by [15, Theorem 3.4], a contradiction.

ii) Let $F$ be a field and assume that $a \mapsto a$ is an isomorphism $F \to \overline{F} \subseteq F$, where the subfield $\overline{F} \neq F$. Let $R$ denote the left vector space on basis $\{1, t\}$, and make $R$ into an $F$-algebra by defining $t^2 = 0$ and $ta = at$ for all $a \in F$ (see [10, Example 2.5]). Then $R$ is a right SP-injective (since $R$ is right P-injective) and semiprimary ring but not a right SF-injective ring. If $R$ is a right SF-injective ring, then $R$ is right $F$-injective by Theorem 2.10. This is a contradiction by [10, Example 5.22]. Moreover, $R$ is not left SP-injective since $R$ is not left mininjective.

iii) The ring of integers $\mathbb{Z}$ is a commutative ring with $J = 0$. So $R$ is small injective, but $R$ is not $P$-injective.

3. On simple-FJ-injective rings

Definition 3.1. A ring $R$ is called right simple-FJ-injective if every right $R$-homomorphism from a small, finitely generated right ideal to $R$ with a simple image, can be extended to an endomorphism of $R_R$.

We have the implications simple-injective $\Rightarrow$ simple-$J$-injective $\Rightarrow$ simple-FJ-injective. But the converses in general are not true. By Example 2.18(i), $R$ is commutative, semiprimary and simple-FJ-injective. But $R$ is not simple-$J$-injective. In fact, if $R$ is simple-$J$-injective then $R$ is simple-injective by [15, Corollary 3.6]. Hence $R$ is self-injective by [10, Theorem 6.47]. This is a contradiction.

Lemma 3.2. If $R$ is right simple-FJ-injective, then $R$ is right mininjective and a left min annihilator.

Proof. We can prove it as in Proposition 2.6. \qed
Lemma 3.3. A ring $R$ is right simple-FJ-injective a ring if and only if every $R$-homomorphism $f : I \to R$ extends to $R_R \to R$, where $I$ is a small, finitely generated right ideal and $f(I)$ is finitely generated, semisimple.

Proof. Write $f(I) = \bigoplus_{i=1}^{n} S_i$, where $S_i$ is a simple right ideal. Let $\pi_i : \bigoplus_{i=1}^{n} S_i \to S_i$ be the projection for each $i$. Since $R$ is right simple-FJ-injective, $\pi_i f = c_i$, for some $c_i \in R$ and for each $i$. Thus $f = c$, with $c = c_1 + \ldots + c_n$. \hfill $\square$

Proposition 3.4. Let $R$ be a right simple-FJ-injective and right Kasch ring. Then

(1) $\text{rl}(I) = I$ for every small, finitely generated right ideal $I$ of $R$.

(2) $S_r = S_l$.

Proof. By Proposition 2.7. \hfill $\square$

In [20], a ring $R$ is called right $(I - K) - m$-injective if for any $m$-generated right ideal $U \leq I$ and any $R$-homomorphism $f : U_R \to K_R$, $f = c_r$, for some $c_r \in R$, where $I, K$ are two right ideals of $R$ and $m \geq 1$.

Lemma 3.5 ([20], Lemma 2.5). If $R$ is a right $(J, S_r) - 1$-injective, right Kasch and semiregular ring, then $\text{Im}(I)$ is an essential left ideal of $R_R$.

Lemma 3.6. Let $R$ be a right simple-FJ-injective and semiregular ring. Then every $R$-homomorphism $f : K \to R$ extends to $R_R \to R_R$ where $K$ is a finitely generated right ideal and $f(K)$ is simple.

Proof. Let $f : K \to I$ be an $R$-homomorphism, where $K$ is a finitely generated right ideal and $f(K)$ is simple. Since $R$ is semiregular, then $K = eR \oplus L$, where $e^2 = e \in R$ and $L \leq J$. So $L$ is a small, finitely generated right ideal of $R$. It is easy to see that $K = eR \oplus (1 - e)L$. Therefore $(1 - e)L$ is a small, finitely generated right ideal of $R$. By hypothesis, there exists an endomorphism $g$ of $R_R$ such that $g(x) = f(x)$ for all $x \in (1 - e)L$. We construct a homomorphism $\varphi : R_R \to R_R$ defined by $\varphi(x) = f(eRx) + g((1 - e)x)$ for any $x \in R$. Then $\varphi|K = f$. \hfill $\square$

Proposition 3.7. Let $R$ be a right simple-FJ-injective ring. Then

(1) If $R$ is semiregular and $e$ is a local idempotent of $R$, then $\text{Soc}(eR)$ is either $0$ or simple and essential in $eR_R$.

(2) If $R$ is semiperfect, then the following conditions are equivalent

a) $\text{Soc}(eR) \neq 0$ for each local idempotent $e$.

b) $S_r$ is finitely generated and essential in $R_R$.

Proof. (1) Suppose that $\text{Soc}(eR) \neq 0$ and let $aR$ be a simple right ideal of $eR$. If $0 \neq b \in eR$ such that $aR \cap bR = 0$, then we construct an $R$-homomorphism $\gamma : aR \oplus bR \to eR$ by $\gamma(ax + by) = ax$, for all $x, y \in R$. Therefore $\text{Im}(\gamma) = aR$ is simple. By Lemma 3.6, $\gamma = e$. for some $e \in R$. Let $e' = ece \in eRe$. So $(e - e')a = ea - eca = 0$. On the other hand, $\text{End}(eR_R) \cong eRe$ is local. It implies that $e'$ is invertible in $eRe$, but $e'b = eceb = ec \neq 0$ and so $b = 0$, which is a contradiction. Hence $aR \cap bR \neq 0$, $aR \leq bR$ since $aR$ is simple. Thus $\text{Soc}(eR)$ is simple and essential in $eR_R$. 
(2) If \(1 = e_1 + \ldots + e_n\), where the \(e_i\) are orthogonal local idempotents, then \(S_r = \bigoplus_{i=1}^n \text{Soc}(e_i R)\) and \(a \Rightarrow b\) follows from (1). The converse is clear. □

**Proposition 3.8.** Let \(R\) be a semiperfect, right simple-FJ-injective ring with \(\text{Soc}(e R) \neq 0\) for each local idempotent \(e \in R\). Then:

1. \(\text{rl}(I) = I\) for every finitely generated right ideal \(I\) of \(R\), so \(R\) is left \(P\)-injective.
2. \(R\) is left and right Kasch.
3. \(S_r = S_l = S_r = S_l = r(J) = l(J)\) is essential in \(R\) and in \(R_R\).
4. \(J = Z_r = Z_l = r(S) = l(S)\), with \(S_r = S_l = S\).
5. \(R\) is left and right finitely cogenerated.

**Proof.** (2): by [12, Theorem 3.7] and (1) by Proposition 3.4 and [20, Lemma 1.4].

(3): \(S_r = S_l = S\) is essential in \(R_R\) and in \(R_R\) by Proposition 3.4, Lemma 3.5 and Proposition 3.7. \(S = r(J) = l(J)\) because \(R\) is left and right Kasch.

(4): follows from (2) and (3).

(5): follows from Proposition 3.7 and [10, Theorem 5.31]. □

**Remark.** There exists a semiprimary and right simple-FJ-injective ring, but it can not be right simple-injective. On the other hand, there is a ring \(R\) that is right simple-FJ-injective but not right \(SP\)-injective (see [20, Example 1.7]).

From the above proposition, we have the following result.

**Proposition 3.9.** If \(R\) is a right simple-FJ-injective ring with \(\text{ACC}\) on right annihilators in which \(S_r \leq^e R_R\), then \(R\) is \(QF\).

**Proof.** By [13, Theorem 2.1] or [14, Lemma 2.11], \(R\) is semiprimary. Hence \(R\) is left and right mininjective by Proposition 3.8. Thus \(R\) is \(QF\). □

**Corollary 3.10** ([14, Theorem 2.15]). If \(R\) is a right simple-injective ring with \(\text{ACC}\) on right annihilators in which \(S_r \leq^e R_R\) then \(R\) is \(QF\).

Recall that a ring \(R\) is called right pseudo-coherent if \(r(S)\) is finitely generated for every finite subset \(S\) of \(R\) (see [3]). Chen and Ding [5] proved that if \(R\) is a left perfect, right simple-injective and right (or left) pseudo-coherent ring, then \(R\) is \(QF\). They gave a question: If \(R\) is a right simple-injective ring which is also right perfect and right (or left) pseudo-coherent, is \(R\) a \(QF\) ring? The following results are motivated by this question.

Firstly, we have the following result

**Lemma 3.11** (Osofsky’s Lemma). If \(R\) is a left perfect ring in which \(J/J^2\) is right finitely generated, then \(R\) is right Artinian.

**Theorem 3.12.** Assume that \(R\) is left perfect, right simple-FJ-injective. If \(R\) is right (or left) pseudo-coherent ring, then \(R\) is \(QF\).

**Proof.** Since \(R\) is left perfect, \(\text{Soc}(e R) \neq 0\) for each local idempotent \(e \in R\). Thus by Proposition 3.8, \(J = r(S) = l(S)\) with \(S = S_r = S_l = r(J) = l(J)\) is a finitely generated left and right ideal. Hence by hypothesis, \(R\) is left (or right)
pseudo-coherent, and so \( J \) is a finitely generated left (or right) ideal. If \( J \) is a finitely generated right \( R \)-module, then \( J/J^2 \) is too. Consequently, \( R \) is right Artinian by Lemma 3.11. If \( J \) is a finitely generated left \( R \)-module, then \( J \) is nilpotent by [10, Lemma 5.64], and so \( R \) is semiprimary. Hence \( R \) is left Artinian by Lemma 3.11. Thus \( R \) is QF. \( \square \)

**Corollary 3.13** ([5], Theorem 2.6). Assume that \( R \) is left perfect, right simple-injective. If \( R \) a is right (or left) pseudo-coherent ring, then \( R \) is QF.

We consider a ring which is right simple-FJ-injective and left pseudo-coherent.

**Theorem 3.14.** If \( R \) is a right perfect, right simple-FJ-injective and left pseudo-coherent ring then \( R \) is QF.

**Proof.** Since \( R \) is right perfect and left pseudo-coherent, \( R \) satisfies DCC on finitely generated left ideals. Hence if \( A \subseteq R \), \( l(A) = l(A_0) \) for some finite subset \( A_0 \) of \( A \). It follows that \( R \) satisfies DCC on left annihilators, and hence \( R \) has ACC on right annihilators. Therefore \( R \) is semiprimary by [6, Proposition 1]. Thus \( R \) is QF by Theorem 3.12. \( \square \)

**Corollary 3.15.** If \( R \) is a right perfect, right simple-injective and left pseudo-coherent ring, then \( R \) is QF.

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**References**


Le van Thuyet, Department of Mathematics, Hue University, Vietnam, *e-mail*: lvthuyethue@gmail.com

Truong Cong Quynh, Department of Mathematics, Da Nang University, Vietnam, *e-mail*: matht2q2004@hotmail.com