CERTAIN CLASSES OF $p$-VALENT FUNCTIONS ASSOCIATED WITH WRIGHT’S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. The Wright’s generalized hypergeometric function is used here to introduce a new class of $p$-valent functions $WT_p(\lambda, \alpha, \beta)$ defined in the open unit disc and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity of functions belonging to the class $WT_p(\lambda, \alpha, \beta)$.

1. Introduction

Let $A(p)$ denote the class of functions of the form

\begin{equation}
    f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n, \quad p < k; \quad p, k \in \mathbb{N} = \{1, 2, 3, \ldots\}
\end{equation}

which are analytic in the open disc $U = \{z : z \in \mathbb{C}; \ |z| < 1\}$. For functions $f \in A(p)$ given by (1.1) and $g \in A(p)$ given by

\[ g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\} \]

we define the Hadamard product (or convolution) of $f$ and $g$ by

\begin{equation}
    f(z) \ast g(z) = (f \ast g)(z) = z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \quad z \in U.
\end{equation}

For positive real parameters $\alpha_1, A_1, \ldots, \alpha_l, A_l$ and $\beta_1, B_1, \ldots, \beta_m, B_m$ ($l, m \in \mathbb{N} = 1, 2, 3, \ldots$) such that

\[ 1 + \sum_{n=k}^{m} B_n - \sum_{n=k}^{l} A_n \geq 0, \quad z \in U, \]

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the Wright’s generalized hypergeometric function [11]
\[
\hat{\Psi}_{m}[(\alpha_{1}, A_{1}), \ldots, (\alpha_{l}, A_{l}); (\beta_{1}, B_{1}), \ldots, (\beta_{m}, B_{m}); z] = \hat{\Psi}_{m}[(\alpha_{j}, A_{j})_{1,l}(\beta_{j}, B_{j})_{1,m}; z]
\]
is defined by
\[
\hat{\Psi}_{m}[(\alpha_{j}, A_{j})_{1,l}(\beta_{i}, B_{i})_{1,m}; z] = \sum_{n=k}^{\infty} \left( \prod_{j=0}^{l} \Gamma(\alpha_{j} + nA_{j}) \right) \left( \prod_{j=0}^{m} \Gamma(\beta_{j} + nB_{j}) \right)^{-1} \frac{z^{n}}{n!}, \quad z \in U.
\]
If \( A_{j} = 1(j = 1, 2, \ldots, l) \) and \( B_{j} = 1(j = 1, 2, \ldots, m) \), we have the relationship:
\[
\Omega \hat{\Psi}_{m}[(\alpha_{j}, 1)_{1,l}(\beta_{j}, 1)_{1,m}; z] = \hat{\Gamma}_{m}(\alpha_{1}, \ldots, \alpha_{l}; \beta_{1}, \ldots, \beta_{m}; z)
\]
(1.3)
\[
\Omega = \left( \prod_{j=0}^{l} \Gamma(\alpha_{j}) \right)^{-1} \left( \prod_{j=0}^{m} \Gamma(\beta_{j}) \right)
\]
(1.4)

\[
\hat{\phi}_{m}[(\alpha_{j}, A_{j})_{1,l}; (\beta_{j}, B_{j})_{1,m}; z] = \Omega z^{p} \hat{\Psi}_{m}[(\alpha_{j}, A_{j})_{1,l}(\beta_{j}, B_{j})_{1,m}; z].
\]

Let \( \Theta[(\alpha_{j}, A_{j})_{1,l}; (\beta_{j}, B_{j})_{1,m}] : A(p) \rightarrow A(p) \) be a linear operator defined by
\[
\Theta[(\alpha_{j}, A_{j})_{1,l}; (\beta_{j}, B_{j})_{1,m}]f(z) := z^{p} \hat{\phi}_{m}[(\alpha_{j}, A_{j})_{1,l}; (\beta_{j}, B_{j})_{1,m}; z] \ast f(z)
\]

We observe that, for \( f(z) \) of the form (1.1), we have
\[
\Theta[(\alpha_{j}, A_{j})_{1,l}; (\beta_{j}, B_{j})_{1,m}]f(z) = z^{p} + \sum_{n=k}^{\infty} \sigma_{n} a_{n} z^{n}
\]
(1.5)

where \( \sigma_{n} \) is defined by
\[
\sigma_{n} = \frac{\Omega \Gamma(\alpha_{1} + A_{1}(n - p)) \ldots \Gamma(\alpha_{l} + A_{l}(n - p))}{(n - p)! \Gamma(\beta_{1} + B_{1}(n - p)) \ldots \Gamma(\beta_{m} + B_{m}(n - p))}.
\]
(1.6)
For convenience, we write

\begin{equation}
\Theta[\alpha_1]f(z) = \Theta[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z)
\end{equation}

Indeed, by setting $A_j = 1(j = 1, \ldots, l)$, $B_j = 1(j = 1, \ldots, m)$ and $p = 1$ the linear operator $\Theta[\alpha_1]$, leads immediately to the Dziok-Srivastava operator \[2\] which contains, as its further special cases, such other linear operators of Geometric Function Theory as the Hohlov operator, the Carlson-Shaffer operator \[1\], the Ruscheweyh derivative operator \[6\], the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator \[8\]. See also \[2\] and \[3\] in which comprehensive details of various other operators are given.

Motivated by the earlier works of \[2, 4, 5, 7, 9, 10\] we introduce a new subclass of $p$-valent functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $W_p(\lambda, \alpha, \beta)$ be the subclass of $A(p)$ consisting of functions of the form (1.1) and satisfying the inequality

\begin{equation}
\left| \frac{J_\lambda(z) - 1}{J_\lambda(z) + (1 - 2\alpha)} \right| < \beta \quad (z \in U)
\end{equation}

where

\begin{equation}
J_\lambda(z) = (1 - \lambda) \frac{\Theta[\alpha_1]f(z)}{z^p} + \lambda \frac{(\Theta[\alpha_1]f(z))'}{pz^{p-1}},
\end{equation}

$\Theta[\alpha_1]f(z)$ is given by (1.7). Further let $WT_p(\lambda, \alpha, \beta) = W_p(\lambda, \alpha, \beta) \cap T(p)$, where

\begin{equation}
T(p) := \left\{ f \in A(p) : f(z) = z^p - \sum_{n=k}^{\infty} a_n z^n, \quad a_n \geq 0; \quad z \in U \right\}.
\end{equation}

The purpose of the present paper is to investigate the coefficient estimates, extreme points, distortion theorems and the radii of convexity and starlikeness of the class $WT_p(\lambda, \alpha, \beta)$.

\section{Coefficient Bounds}

In this section we obtain coefficient estimates and extreme points of the class $WT_p(\lambda, \alpha, \beta)$.

**Theorem 2.1.** Let the function $f$ be defined by (1.10). Then $f \in WT_p(\lambda, \alpha, \beta)$ if and only if

\begin{equation}
\sum_{n=k}^{\infty} (p + n\lambda - p\lambda)(1 + \beta)\sigma_n a_n \leq 2p\beta(1 - \alpha).
\end{equation}
Proof. Suppose \( f \) satisfies (2.1). Then for \( z \in U \) we have
\[
|J_\lambda(z) - 1| - \beta |J_\lambda(z) + (1 - 2\alpha)| = \left| - \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} (1 + \beta)\sigma_n a_n z^{n-p} \right| - \beta \left| 2(1 - \alpha) - \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \sigma_n a_n z^{n-p} \right|
\]
\[
\leq \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \sigma_n a_n - 2\beta(1 - \alpha) + \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \beta \sigma_n a_n
\]
\[
= \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \left[ 1 + \beta \sigma_n a_n - 2\beta(1 - \alpha) \right] \leq 0.
\]
Hence, by maximum modulus theorem and (1.8), \( f \in WT_p(\lambda, \alpha, \beta) \). To prove the converse assume that
\[
\left| \frac{J_\lambda(z) - 1}{J_\lambda(z) + (1 - 2\alpha)} \right| = \left| - \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \sigma_n a_n z^{n-p} \right| / \left| 2(1 - \alpha) - \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \sigma_n a_n z^{n-p} \right| \leq \beta, \quad z \in U.
\]
Thus
\[
\text{Re} \left\{ \frac{\sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} a_n \sigma_n z^{n-p}}{2(1 - \alpha) - \sum_{n=k}^\infty \frac{(p + n\lambda - p\lambda)}{p} \sigma_n a_n z^{n-p}} \right\} < \beta,
\]
since \( \text{Re}(z) \leq |z| \) for all \( z \). Choose values of \( z \) on the real axis such that \( J_\lambda(z) \) is real. Upon clearing the denominator in (2.2) and letting \( z \to 1^- \) through real values, we obtain the desired inequality (2.1). \( \square \)

Corollary 2.1. If \( f(z) \) of the form (1.10) is in \( WT_p(\lambda, \alpha, \beta) \), then
\[
a_n \leq \frac{2p\beta(1 - \alpha)}{(p + n\lambda - p\lambda)(1 + \beta)\sigma_n}, \quad n = k, k+1, \ldots,
\]
with the equality only for the function
\[
f(z) = z^p - \frac{2p\beta(1 - \alpha)}{(p + n\lambda - p\lambda)(1 + \beta)\sigma_n} z^n, \quad n = k, k+1, \ldots.
\]

Theorem 2.2 (Extreme Points). Let
\[
f_p(z) = z^p \quad \text{and} \quad f_n(z) = z^p - \frac{2p\beta(1 - \alpha)}{(p + n\lambda - p\lambda)(1 + \beta)\sigma_n} z^n, \quad n = k, k+1, \ldots.
\]
Then \( f(z) \) is in the class \( WT_p(\lambda, \alpha, \beta) \) if and only if it can be expressed in the form
\[
f(z) = \mu_p z^p + \sum_{n=k}^\infty \mu_n f_n(z),
\]
where \( \mu_n \geq 0 \) and \( \mu_p + \sum_{n=k}^\infty \mu_n = 1 \).
Proof. Suppose \( f(z) \) can be written as in (2.6). Then
\[
f(z) = \mu_p z^p - \sum_{n=k}^{\infty} \mu_n \left[ z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n \right]
\]
\[
= z^p - \sum_{n=k}^{\infty} \mu_n \frac{2p\beta(1-\alpha)}{(p+n\lambda-p\lambda)[1+\beta]\sigma_n} z^n.
\]

Now,
\[
\sum_{n=k}^{\infty} \frac{(p+n\lambda-p\lambda)[1+\beta]\sigma_n}{2p\beta(1-\alpha)} \mu_n \leq 1 - \mu_p \leq 1.
\]

Thus \( f \in \mathcal{W}T_p(\lambda, \alpha, \beta) \). Conversely, let us have \( f \in \mathcal{W}T_p(\lambda, \alpha, \beta) \). Then by using (2.3), we set
\[
\mu_n = \left( \frac{p+n\lambda-p\lambda}{2p\beta(1-\alpha)} \right) \sigma_n, \quad n \geq k
\]
and \( \mu_p = 1 - \sum_{n=k}^{\infty} \mu_n \). Then we have (2.6) and hence this completes the proof of Theorem 2.2. \( \square \)

3. Distortion Bounds

In this section we obtain distortion bounds for the class \( \mathcal{W}T_p(\lambda, \alpha, \beta) \).

**Theorem 3.1.** Let \( f \) be in the class \( \mathcal{W}T_p(\lambda, \alpha, \beta) \), \( |z| = r < 1 \) and \( c_n = (p+n\lambda-p\lambda)\sigma_n \). If the sequence \( \{c_n\} \) is nondecreasing for \( n > k \), then
\[
r^p - \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^k \leq |f(z)| \leq r^p + \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^k \leq |f'(z)| \leq r^{p-1} + \frac{2pk\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} r^{k-1}.
\]

The bounds in (3.1) and (3.2) are sharp since the equalities are attained by the function
\[
f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k} z^k.
\]

**Proof.** In the view of Theorem 2.1, we have
\[
\sum_{n=k}^{\infty} a_n \leq \frac{2p\beta(1-\alpha)}{(p+k\lambda-p\lambda)[1+\beta]\sigma_k}
\]
Using (1.10) and (3.4), we obtain

\[ |z|^p - |z|^k \sum_{n=k}^{\infty} a_n \leq |f(z)| \leq |z|^p + |z|^k \sum_{n=k}^{\infty} a_n \]

(3.5)

\[ r^p - r^k \leq |f(z)| \leq r^p + r^k \frac{2p\beta(1-\alpha)}{(p + k\lambda - p\lambda)[1 + \beta]\sigma_k}. \]

Hence (3.1) follows from (3.5). Also,

\[ |f'(z)| \leq pr^{p-1} + r^{k-1} \sum_{n=k}^{\infty} n a_n \leq pr^{p-1} + r^{k-1} \frac{2pk\beta(1-\alpha)}{(p + k\lambda - p\lambda)[1 + \beta]\sigma_k}. \]

Similarly, we can prove the left hand inequality given in (3.2) which completes the proof of the theorem.

□

4. Radius of Starlikeness and Convexity

The radii of close-to-convexity, starlikeness and convexity for the class \( WT_p(\lambda, \alpha, \beta) \) are given in this section.

**Theorem 4.1.** Let the function \( f(z) \) defined by (1.10) belong to the class \( WT_p(\lambda, \alpha, \beta) \). Then \( f(z) \) is \( p \)-valently close-to-convex of order \( \delta \) \((0 \leq \delta < p)\) in the disc \( |z| < r_1 \), where

\[ r_1 := \inf_{n \geq k} \left[ \frac{(p-\delta)(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2p\beta(1-\alpha)} \right]^{\frac{1}{p-\delta}}. \]

**Proof.** The function \( f \in T(p) \) is close-to-convex of order \( \delta \), if

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta. \]

(4.2)

For the left-hand side of (4.2) we have

\[ \left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{n=k}^{\infty} na_n|z|^{n-p}. \]

The last expression is less than \( p - \delta \) if

\[ \sum_{n=k}^{\infty} \frac{n}{p-\delta} a_n|z|^{n-p} < 1. \]

Using the fact that \( f \in WT_p(\lambda, \alpha, \beta) \) if and only if

\[ \sum_{n=k}^{\infty} \frac{(p + n\lambda - p\lambda)[1 + \beta]\sigma_n a_n}{2p\beta(1-\alpha)} \leq 1, \]

we can say (4.2) is true if

\[ \frac{n}{p-\delta}|z|^{n-p} \leq \frac{(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2p\beta(1-\alpha)}. \]
Or, equivalently,

\[ |z|^{n-p} = \left[ \frac{(p - \delta)(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2pn\beta(1 - \alpha)} \right] \]

which completes the proof. \( \square \)

**Theorem 4.2.** Let \( f \in WT_p(\lambda, \alpha, \beta) \). Then

1. \( f \) is \( p \)-valently starlike of order \( \delta \) (0 \( \leq \) \( \delta \) \( < \) \( p \)) in the disc \( |z| < r_2 \); that is,
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \delta, \quad (|z| < r_2)
\]

where
\[
r_2 = \inf_{n \geq k} \left\{ \frac{(p - \delta)(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2p\beta(1 - \alpha)(k + p - \delta)} \right\}^{\frac{1}{n}}.
\]

2. \( f \) is \( p \)-valently convex of order \( \delta \) (0 \( \leq \) \( \delta \) \( < \) \( p \)) in the disc \( |z| < r_3 \), that is
\[
\text{Re} \left\{ 1 + \frac{z''f(z)}{f'(z)} \right\} > \delta, \quad (|z| < r_3)
\]

where
\[
r_3 = \inf_{n \geq p+1} \left\{ \frac{(p - \delta)(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2n\beta(1 - \alpha)(n - \delta)} \right\}^{\frac{1}{n}}.
\]

**Proof.** (1) The function \( f \in T(p) \) is \( p \)-valently starlike of order \( \delta \), if

\[
|zf'(z) - p| \leq \sum_{n=k}^{\infty} (n - p)a_n |z|^n.
\]

The last expression is less than \( p - \delta \) if

\[
\sum_{n=k}^{\infty} \frac{n - \delta}{p - \delta} a_n |z|^n < 1.
\]

Using the fact that \( f \in WT_p(\lambda, \alpha, \beta) \) if and only if

\[
\sum_{n=k}^{\infty} \frac{(p + n\lambda - p\lambda)[1 + \beta]\sigma_n a_n}{2p\beta(1 - \alpha)} < 1,
\]

we can say (4.3) is true if

\[
\frac{n - \delta}{p - \delta} |z|^n < \frac{(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2p\beta(1 - \alpha)}.
\]

Or, equivalently,

\[
|z|^n < \frac{(p - \delta)(p + n\lambda - p\lambda)[1 + \beta]\sigma_n}{2p\beta(1 - \alpha)(n - \delta)}
\]

which yields the starlikeness of the family.

(2) Using the fact that \( f \) is convex if and only if \( zf' \) is starlike, we can prove (2), on lines similar to the proof of (1). \( \square \)
Remark. In view of the relationship (1.3) the linear operator (1.5) and by setting $A_j = 1 \ (j = 1, \ldots, l)$ and $B_j = 1 (j = 1, \ldots, m)$ and specific choices of parameters $l, m, \alpha_1, \beta_1$ the various results presented in this paper would provide interesting extensions and generalizations of $p$-valent function classes. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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References


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