NEW CLASSES OF $k$-UNIFORMLY CONVEX AND STARLIKE FUNCTIONS WITH RESPECT TO OTHER POINTS

C. SELVARAJ and K. A. SELVAKUMARAN

Abstract. In this paper we introduce new subclasses of $k$-uniformly convex and starlike functions with respect to other points. We provide necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for these classes. We also obtain integral means inequalities with the extremal functions for these classes.

1. Introduction, Definitions and Preliminaries

Let $A$ denote the class of functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are regular in the unit disc $D = \{ z : |z| < 1 \}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let $S$ be the subclass of $A$ consisting of functions that are regular and univalent in $D$. Let $S^*$ be the subclass of $S$ consisting of functions starlike in $D$. It is known that $f \in S^*$ if and only if $\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0$, $z \in D$.

In [6], Sakaguchi defined the class of starlike functions with respect to symmetric points as follows:

Let $f \in S$. Then $f$ is said to be starlike with respect to symmetric points in $D$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in D.$$

We denote this class by $S^*_s$. Obviously, it forms a subclass of close-to-convex functions and hence univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [6]. EL-Ashwah and Thomas in [2] introduced two other classes, namely the class $S^*_c$ consisting of functions starlike with respect to conjugate points and $S^*_sc$ consisting of functions starlike with respect to symmetric conjugate points.
Motivated by $S_s^*$, many authors discussed the following class $C_s^*$ of functions convex with respect to symmetric points and its subclasses (See [4, 5, 7, 11]).

Let $f \in S$. Then $f$ is said to be convex with respect to symmetric points in $D$ if and only if

\[
\text{Re}\left\{ \frac{(zf'(z))'}{f'(z) + f'(-z)} \right\} > 0, \quad z \in D.
\]

Let $T$ denote the class consisting of functions $f$ of the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n,
\]

where $a_n$ is a non-negative real number.

Silverman [8] introduced and investigated the following subclasses of $T$:

\[
T_s^*(\alpha) := S_s^*(\alpha) \cap T \quad \text{and} \quad C_s^*(\alpha) := K_s(\alpha) \cap T \quad (0 \leq \alpha < 1).
\]

In this paper we introduce the class $S_s(\lambda, k, \beta)$ of functions regular in $D$ given by (1) and defined as follows

**Definition 1.1.** A function $f(z) \in A$ is said to be in the class $S_s(\lambda, k, \beta)$ if for all $z \in D$,

\[
\text{Re}\left[ \frac{2zf'(z) + 2\lambda z^2f''(z)}{(1-\lambda)(f(z) - f(-z)) + \lambda(f'(z) + f'(-z))} \right] > k \left| \frac{2zf'(z) + 2\lambda z^2f''(z)}{(1-\lambda)(f(z) - f(-z)) + \lambda(f'(z) + f'(-z))} - 1 \right| + \beta,
\]

for some $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$.

For suitable values of $\lambda, k, \beta$ the class of functions $S_s(\lambda, k, \beta)$ reduces to various new classes of regular functions. We also observe that

\[
S_s(0, 0, 0) \equiv S_s^* \quad \text{and} \quad S_s(1, 0, 0) \equiv C_s^*.
\]

We now let $T_s(\lambda, k, \beta) = S_s(\lambda, k, \beta) \cap T$.

In the present investigation of the function class $T_s(\lambda, k, \beta)$ we obtain necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points, radii of close-to-convexity, starlikeness and convexity. We also obtain integral means inequality for the functions belonging to this class. Analogous results are also obtained for the class of functions $f \in T$ and $k$-uniformly convex and starlike with respect to conjugate points. The class is defined below:

**Definition 1.2.** A function $f(z) \in A$ is said to be in the class $S_s(\lambda, k, \beta)$ if for all $z \in D$,

\[
\text{Re}\left[ \frac{2zf'(z) + 2\lambda z^2f''(z)}{(1-\lambda)(f(z) + f(\bar{z})) + \lambda(f'(z) + f'(\bar{z}))} \right] > k \left| \frac{2zf'(z) + 2\lambda z^2f''(z)}{(1-\lambda)(f(z) + f(\bar{z})) + \lambda(f'(z) + f'(\bar{z}))} - 1 \right| + \beta,
\]

for some $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$. 
Here we let $T_{S_c}(\lambda, k, \beta) = S_c(\lambda, k, \beta) \cap T$.
We now state two lemmas which we may need to establish our results in the sequel.

**Lemma 1.3.** If $\beta$ is a real number and $w$ is a complex number, then
\[
\Re(w) \geq \beta \Leftrightarrow |w + (1 - \beta)| - |w - (1 + \beta)| \geq 0.
\]

**Lemma 1.4.** If $w$ is a complex number and $\beta, k$ are real numbers, then
\[
\Re(w) \geq k|w - 1| + \beta \Leftrightarrow \Re\{w(1 + ke^{i\theta}) - ke^{i\theta}\} \geq \beta, \quad -\pi \leq \theta \leq \pi.
\]

2. **Coefficient Inequalities**

We employ the technique adopted by Aqlan et al. [1] to find the coefficient estimates for the function class $T_{S_c}(\lambda, k, \beta)$.

**Theorem 2.1.** A function $f \in T_{S_c}(\lambda, k, \beta)$ if and only if
\[
\sum_{n=2}^{\infty} \left[2(1 + k)n - (k + \beta)(1 - (-1)^n)\right](1 - \lambda + \lambda n)a_n \leq 2(1 - \beta)
\]
for $0 \leq \lambda \leq 1, \ 0 \leq \beta < 1$ and $k \geq 0$.

**Proof.** Let a function $f(z)$ of the form (2) in $T$ satisfy the condition (5). We will show that (3) is satisfied and so $f \in T_{S_c}(\lambda, k, \beta)$. Using Lemma 1.4 it is enough to show that
\[
\Re\left\{2zf'(z) + 2\lambda z^2f''(z)\left(1 - \lambda + \lambda n\right)a_n \right\} \geq \beta,
\]
\[-\pi \leq \theta \leq \pi.
\]
That is, \(\Re\left\{\frac{A(z)}{B(z)}\right\} \geq \beta\), where
\[
A(z) := [2zf'(z) + 2\lambda z^2f''(z)](1 + k e^{i\theta}) - ke^{i\theta}\left(1 - \lambda\right)(f(z) - f(-z)) + \lambda z(f'(z) + f'(-z)),
\]
\[
B(z) := (1 - \lambda)(f(z) - f(-z)) + \lambda z(f'(z) + f'(-z)).
\]
In view of Lemma 1.3, we only need to prove that
\[
|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)| \geq 0.
\]
For $A(z)$ and $B(z)$ as above, we have
\[
|A(z) + (1 - \beta)B(z)|
\]
\[
= \left| (4 - 2\beta)z - \sum_{n=2}^{\infty} [2n + (1 - \beta)(1 - (-1)^n)](1 - \lambda + \lambda n)a_n z^n - ke^{i\theta} \sum_{n=2}^{\infty} [2n - (1 - (-1)^n)](1 - \lambda + \lambda n)a_n z^n \right|
\]
\[ C. SELLVARAJ \text{ and K. A. SELVAKUMARAN} \]

\[\geq (4 - 2\beta)|z| - \sum_{n=2}^{\infty} [2n + (1 - \beta)(1 - (-1)^n)](1 - \lambda + \lambda n)a_n |z|^n \]
\[-k \sum_{n=2}^{\infty} [2n - (1 - (-1)^n)](1 - \lambda + \lambda n)a_n |z|^n.\]

Similarly, we obtain
\[|A(z) - (1 + \beta)B(z)| \leq 2\beta|z| + \sum_{n=2}^{\infty} [2n - (1 + \beta)(1 - (-1)^n)](1 - \lambda + \lambda n)a_n |z|^n \]
\[+ k \sum_{n=2}^{\infty} [2n - (1 - (-1)^n)](1 - \lambda + \lambda n)a_n |z|^n.\]

Therefore, we have
\[|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)| \geq 4(1 - \beta)|z| - 2 \sum_{n=2}^{\infty} [2(1 + k)n - (k + \beta)(1 - (-1)^n)](1 - \lambda + \lambda n)a_n |z|^n \]
\[\geq 0,\]
by the given condition (5). Conversely, suppose \(f \in TS_{s}(\lambda, k, \beta)\). Then by Lemma 1.4 we have (6). Choosing the values of \(z\) on the positive real axis the inequality (6) reduces to
\[\text{Re}\left\{ \frac{2(1 - \beta)}{2 - \sum_{n=2}^{\infty} (1 - \lambda + \lambda n)(1 - (-1)^n)a_n z^{n-1}} \right\} \geq 0.\]

In view of the elementary identity \(\text{Re}(e^{i\theta}) \geq -|e^{i\theta}| = -1\), the above inequality becomes
\[\text{Re}\left\{ \frac{2(1 - \beta)}{2 - \sum_{n=2}^{\infty} (1 - \lambda + \lambda n)(1 - (-1)^n)a_n r^{n-1}} \right\} \geq 0.\]

Letting \(r \to 1^{-}\) we get the desired inequality (5).

The following coefficient estimate for \(f \in TS_{s}(\lambda, k, \beta)\) is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** If \(f \in TS_{s}(\lambda, k, \beta)\), then
\[a_n \leq \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)}, \quad n \geq 2\]
where $\Phi(\lambda, k, \beta, n) = (1 - \lambda + \lambda n)[2(1 + k)n - (k + \beta)(1 - (-1)^n)]$.

The equality holds for the function

$$f(z) = z - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)} z^n.$$  

We now state coefficient properties for the functions belonging to the class $\mathcal{TS}_c(\lambda, k, \beta)$. Method of proving Theorem 2.3 is similar to that of Theorem 2.1.

**Theorem 2.3.** A function $f \in \mathcal{TS}_c(\lambda, k, \beta)$ if and only if

$$\sum_{n=2}^{\infty} [1 + k - (k + \beta)](1 - \lambda + \lambda n)a_n \leq (1 - \beta)$$

for $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$.

**Theorem 2.4.** If $f \in \mathcal{TS}_c(\lambda, k, \beta)$, then

$$a_n \leq \frac{(1 - \beta)}{\Theta(\lambda, k, \beta, n)}$$

for $n \geq 2$, where $\Theta(\lambda, k, \beta, n) = (1 - \lambda + \lambda n)[1 + k - (k + \beta)]$.

The equality holds for the function

$$f(z) = z - \frac{(1 - \beta)}{\Theta(\lambda, k, \beta, n)} z^n.$$

### 3. Distortion and Covering Theorems

**Theorem 3.1.** Let $f$ be defined by (2). If $f \in \mathcal{TS}_s(\lambda, k, \beta)$ and $|z| = r < 1$, then we have the sharp bounds

$$r - \frac{1 - \beta}{2(1 + k)(1 + \lambda)} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{2(1 + k)(1 + \lambda)} r^2$$

and

$$1 - \frac{1 - \beta}{(1 + k)(1 + \lambda)} r \leq |f'(z)| \leq 1 + \frac{1 - \beta}{(1 + k)(1 + \lambda)} r.$$

**Proof.** We only prove the right side inequality in (8), since the other inequalities can be justified using similar arguments.

First, it is obvious that

$$4(1 + k)(1 + \lambda) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} [2(1 + k)n - (k + \beta)(1 - (-1)^n)](1 - \lambda + \lambda n)a_n$$

and as $f \in \mathcal{TS}_s(\lambda, k, \beta)$, the inequality (5) yields

$$\sum_{n=2}^{\infty} a_n \leq \frac{1 - \beta}{2(1 + k)(1 + \lambda)}.$$
From (2) with \(|z| = r (r < 1)\), we have

\[ |f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \frac{1 - \beta}{2(1 + k)(1 + \lambda)} r^2. \]

The distortion bounds in Theorem 3.1 are sharp for 

\[ f(z) = z - \frac{1 - \beta}{2(1 + k)(1 + \lambda)} z^2, \quad z = \pm r. \]

\[ (9) \]

\[ \square \]

**Theorem 3.2.** If \( f \in TS_s(\lambda, k, \beta) \), then \( f \in T^*(\delta) \), where

\[ \delta = 1 - \frac{1 - \beta}{2(1 + k)(1 + \lambda) - (1 - \beta)} \]

The result is sharp for the function given by (9).

**Proof.** It is sufficient to show that (5) implies

\[ \sum_{n=2}^{\infty} (n - \delta) a_n \leq 1 - \delta \]

that is

\[ \frac{n - \delta}{1 - \delta} \leq \frac{[2(1 + k)n - (k + \beta)(1 - (1)^n)](1 - \lambda + \lambda n)}{2(1 - \beta)}, \quad n \geq 2. \]

(10)

Since, (10) is equivalent to

\[ \delta \leq 1 - \frac{2(n - 1)(1 - \beta)}{[2(1 + k)n - (k + \beta)(1 - (-1)^n)](1 - \lambda + \lambda n) - 2(1 - \beta)} = \psi(n), \quad n \geq 2 \]

and \( \psi(n) \leq \psi(2) \), (10) holds true for any \( n \geq 2, k \geq 0 \) and \( 0 \leq \beta < 1 \). This completes the proof of Theorem 3.2. \( \square \)

For completeness, we now state the following results with regards to the class \( TS_c(\lambda, k, \beta) \).

**Theorem 3.3.** Let \( f \) be defined by (2) and \( f \in TS_c(\lambda, k, \beta) \). Then for \( \{ z : 0 < |z| = r < 1 \} \) we have the sharp bounds

\[ r - \frac{1 - \beta}{(2 + k - \beta)(1 + \lambda)} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{(2 + k - \beta)(1 + \lambda)} r^2 \]

and

\[ 1 - \frac{2(1 - \beta)}{(2 + k - \beta)(1 + \lambda)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \beta)}{(2 + k - \beta)(1 + \lambda)} r. \]

The result in (11) is sharp for the function

\[ f(z) = z - \frac{1 - \beta}{(2 + k - \beta)(1 + \lambda)} z^2, \quad z = \pm r. \]

(12)
Theorem 3.4. If \( f \in TS_c(\lambda, k, \beta) \), then \( f \in T^*(\delta) \), where
\[
\delta = 1 - \frac{1 - \beta}{(2 + k - \beta)(1 + \lambda) - (1 - \beta)}.
\]
The result is sharp for the function given by (12).

4. Extreme Points

Theorem 4.1. Let \( f_1(z) = z \) and
\[
f_n(z) = z - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)} z^n \quad (n \geq 2),
\]
where \( \Phi(\lambda, k, \beta, n) \) is defined in Theorem 2.2. Then \( f(z) \) is in \( TS_s(\lambda, k, \beta) \) if and only if it can be expressed in the form \( f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \) where \( \lambda_n \geq 0 \) and \( \sum_{n=1}^{\infty} \lambda_n = 1 \).

Proof. Adopting the same technique used by Silverman [8], we first assume that
\[
f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \lambda_n \left[ \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)} z^n \right].
\]
\[
\sum_{n=2}^{\infty} \lambda_n \left\{ \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)} \right\} \cdot \left[ \frac{\Phi(\lambda, k, \beta, n)}{2(1 - \beta)} \right] = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1.
\]
Therefore by Theorem 2.1, \( f \in TS_s(\lambda, k, \beta) \).
Conversely, suppose \( f \in TS_s(\lambda, k, \beta) \). Then by Theorem 2.2
\[
a_n \leq \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, n)}, \quad n \geq 2.
\]
Now, by letting
\[
\lambda_n = \left\{ \frac{\Phi(\lambda, k, \beta, n)}{2(1 - \beta)} \right\} a_n, \quad n \geq 2
\]
and \( \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \) we conclude the theorem, since
\[
f(z) = \sum_{n=1}^{\infty} \lambda_n f_n = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z).
\]

\( \Box \)

Now, we give extreme points for functions belonging to \( TS_c(\lambda, k, \beta) \). We omit the proof of Theorem 4.2 as it is similar to that of Theorem 4.1.

Theorem 4.2. Let \( f_1(z) = z \) and
\[
f_n(z) = z - \frac{(1 - \beta)}{\Theta(\lambda, k, \beta, n)} z^n \quad (n \geq 2),
\]
where \( \Theta(\lambda, k, \beta, n) \) is defined in Theorem 2.4. Then \( f(z) \) is in \( TS_c(\lambda, k, \beta) \) if and only if it can be expressed in the form \( f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \) where \( \lambda_n \geq 0 \) and \( \sum_{n=1}^{\infty} \lambda_n = 1 \).
5. Radii of Close-To-Convexity, Starlikeness and Convexity

**Theorem 5.1.** If \( f(z) \in TS_s(\lambda, k, \beta) \), then \( f \) is close-to-convex of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_1(\lambda, k, \beta, \gamma) \), where

\[
r_1(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2n(1-\beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2
\]

and \( \Phi(\lambda, k, \beta, n) \) is defined in Theorem 2.2.

**Proof.** By a computation, we have

\[
|f'(z) - 1| = \left| -\sum_{n=2}^{\infty} na_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.
\]

Now, \( f \) is close-to-convex of order \( \gamma \) if we have the condition

\[
\sum_{n=2}^{\infty} \left( \frac{n}{1-\gamma} \right) a_n |z|^{n-1} \leq 1.
\]

Considering the coefficient conditions required by Theorem 2.1, the above inequality (14) is true if

\[
\left( \frac{n}{1-\gamma} \right) |z|^{n-1} \leq \frac{\Phi(\lambda, k, \beta, n)}{2(1-\beta)},
\]

or if

\[
|z| \leq \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2(n-\gamma)(1-\beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2.
\]

This last expression yields the bound required by the above theorem.

**Theorem 5.2.** If \( f(z) \in TS_s(\lambda, k, \beta) \), then \( f \) is starlike of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_2(\lambda, k, \beta, \gamma) \), where

\[
r_2(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1-\gamma)\Phi(\lambda, k, \beta, n)}{2(n-\gamma)(1-\beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2
\]

and \( \Phi(\lambda, k, \beta, n) \) is defined in Theorem 2.2.

**Proof.** By a computation, we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| -\sum_{n=2}^{\infty} \frac{(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leq \sum_{n=2}^{\infty} \frac{(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.
\]
Now, $f$ is starlike of order $\gamma$ if we have the condition
\begin{equation}
\sum_{n=2}^{\infty} \left( \frac{n - \gamma}{1 - \gamma} \right) a_n |z|^{n-1} \leq 1.
\end{equation}

Considering the coefficient conditions required by Theorem 2.1, the above inequality (16) is true if
\begin{equation}
\frac{n - \gamma}{1 - \gamma} |z|^{n-1} \leq \frac{\Phi(\lambda, k, \beta, n)}{2(1 - \beta)}
\end{equation}
or if
\begin{equation}
|z| \leq \left\{ \frac{(1 - \gamma)\Phi(\lambda, k, \beta, n)}{2(n - \gamma)(1 - \beta)} \right\}^{\frac{1}{1 - \gamma}}, \quad n \geq 2.
\end{equation}

This last expression yields the bound required by the above theorem.

\begin{flushright}
$\Box$
\end{flushright}

Theorem 5.3. If $f(z) \in TS_s(\lambda, k, \beta)$, then $f$ is convex of order $\gamma$ ($0 \leq \gamma < 1$) in $|z| < r_3(\lambda, k, \beta, \gamma)$, where
\begin{equation}
r_3(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)\Phi(\lambda, k, \beta, n)}{2n(n - \gamma)(1 - \beta)} \right\}^{\frac{1}{1 - \gamma}}, \quad n \geq 2
\end{equation}
and $\Phi(\lambda, k, \beta, n)$ is defined in Theorem 2.2.

\begin{flushright}
Proof. By a computation, we have
\end{flushright}

\begin{equation}
\frac{|zf''(z)|}{f'(z)} \leq \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}} \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}}.
\end{equation}

Now, $f$ is convex of order $\gamma$ if we have the condition
\begin{equation}
\sum_{n=2}^{\infty} \frac{n(n - \gamma)}{1 - \gamma} a_n |z|^{n-1} \leq 1.
\end{equation}

Considering the coefficient conditions required by Theorem 2.1, the above inequality (18) is true if
\begin{equation}
\frac{n(n - \gamma)}{1 - \gamma} |z|^{n-1} \leq \frac{\Phi(\lambda, k, \beta, n)}{2(1 - \beta)}
\end{equation}
or if
\begin{equation}
|z| \leq \left\{ \frac{(1 - \gamma)\Phi(\lambda, k, \beta, n)}{2n(n - \gamma)(1 - \beta)} \right\}^{\frac{1}{1 - \gamma}}, \quad n \geq 2.
\end{equation}

This last expression yields the bound required by the above theorem. \hfill $\Box$

For completeness, we give, without proof, theorem concerning the radii of close-to-convexity, starlikeness and convexity for the class $TS_c(\lambda, k, \beta)$. 
Theorem 5.4. If \( f(z) \in TS_c(\lambda, k, \beta) \), then \( f \) is close-to-convex of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_4(\lambda, k, \beta, \gamma) \), where

\[
(19) \quad r_4(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)\Theta(\lambda, k, \beta, n)}{n(1 - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2
\]

and \( \Theta(\lambda, k, \beta, n) \) is defined in Theorem 2.4.

Theorem 5.5. If \( f(z) \in TS_c(\lambda, k, \beta) \), then \( f \) is starlike of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_5(\lambda, k, \beta, \gamma) \), where

\[
(20) \quad r_5(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)\Theta(\lambda, k, \beta, n)}{(n - \gamma)(1 - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2
\]

and \( \Theta(\lambda, k, \beta, n) \) is defined in Theorem 2.4.

Theorem 5.6. If \( f(z) \in TS_c(\lambda, k, \beta) \), then \( f \) is convex of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in \( |z| < r_6(\lambda, k, \beta, \gamma) \), where

\[
(21) \quad r_6(\lambda, k, \beta, \gamma) = \inf_n \left\{ \frac{(1 - \gamma)\Theta(\lambda, k, \beta, n)}{n(n - \gamma)(1 - \beta)} \right\}^{\frac{1}{n-1}}, \quad n \geq 2
\]

and \( \Theta(\lambda, k, \beta, n) \) is defined in Theorem 2.4.

6. Integral Means Inequalities

In [8], Silverman found that the function \( f_2(z) = z - \frac{z^2}{\pi} \) is often extremal over the family \( T \). He applied this function to resolve his integral means inequality, conjectured in [9] and settled in [10], that

\[
\int_0^{2\pi} |f(re^{i\theta})|^n d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^n d\theta,
\]

for all \( f \in T, n > 0 \) and \( 0 < r < 1 \). In [10], he also proved his conjecture for the subclasses \( T^*(\alpha) \) and \( C(\alpha) \) of \( T \).

Now, we prove Silverman’s conjecture for the class of functions \( TS_c(\lambda, k, \beta) \). An analogous result is also obtained for the family of functions \( TS_c(\lambda, k, \beta) \).

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [3].

Two given functions \( f \) and \( g \), which are analytic in \( D \), the function \( f \) is said to be subordinate to \( g \) in \( D \) if there exists a function \( w \) analytic in \( D \) with

\[
w(0) = 0, \quad |w(z)| < 1 \quad (z \in D),
\]

such that

\[
f(z) = g(w(z)) \quad (z \in D).
\]

We denote this subordination by \( f(z) \prec g(z) \).
Lemma 6.1. If the functions $f$ and $g$ are analytic in $D$ with $f(z) \prec g(z)$, then for $\eta > 0$ and $z = r e^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |g(r e^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(r e^{i\theta})|^\eta d\theta.$$ 

Now, we discuss the integral means inequalities for functions $f$ in $TS_s(\lambda, k, \beta)$.

Theorem 6.2. Let $f \in TS_s(\lambda, k, \beta)$, $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $k \geq 0$ and $f_2(z)$ be defined by

$$f_2(z) = z - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, 2)} z^2,$$

where $\Phi(k, \beta, \lambda, n)$ is defined in Theorem 2.2. Then for $z = r e^{i\theta}$, $0 < r < 1$, we have

(22) $$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta.$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (22) is equivalent to

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^{\eta} d\theta \leq \int_0^{2\pi} \left| 1 - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, 2)} z \right|^{\eta} d\theta.$$

By Lemma 6.1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, 2)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{2(1 - \beta)}{\Phi(\lambda, k, \beta, 2)} w(z),$$

and using (5), we obtain

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, k, \beta, 2)}{2(1 - \beta)} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\lambda, k, \beta, n)}{2(1 - \beta)} a_n \leq |z|.$$ 

This completes the proof by Theorem 2.1.

For completeness, we now give the integral means inequality for the class $TS_c(\lambda, k, \beta)$. The method of proving Theorem 6.3 is similar as that of Theorem 6.2.
Theorem 6.3. Let \( f \in TS_c(\lambda, k, \beta) \), \( 0 \leq \lambda \leq 1 \), \( 0 \leq \beta < 1 \), \( k \geq 0 \) and \( f_2(z) \) be defined by
\[
f_2(z) = z - \frac{(1 - \beta)}{\Theta(\lambda, k, \beta, 2)} z^2,
\]
where \( \Theta(\lambda, k, \beta, n) \) is defined in Theorem 2.4. Then for \( z = r e^{i\theta} \), \( 0 < r < 1 \), we have
\[
\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta.
\]

References
9. __________, A survey with open problems on univalent functions whose coefficients are negative, Rocky Mountain J. Math. 21(3) (1991), 1099–1125.

C. Selvaraj, Department of Mathematics, Presidency College (Autonomous), Chennai-600 005, India, e-mail: pamc9439@yahoo.co.in

K. A. Selvakumaran, Department of Mathematics, R.M.K. Engg. College, Kavaraipettai-601 206, India, e-mail: selvaa1826@gmail.com