TRANSVERSALS OF RECTANGULAR ARRAYS

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Abstract. The paper deals with \( m \times n \) rectangular arrays whose \( mn \) cells are filled with symbols. A section of the array consists of \( m \) cells, one from each row and no two from the same column. The paper focuses on the existence of sections that do contain symbols with high multiplicity.

1. Introduction

An \( n \times n \) array of cells filled with symbols 1, 2, \ldots, \( n \) such that each symbol appears in each row and each column exactly once is called a Latin square. A section is a set of \( n \) cells, one from each row such that no two cells are in the same column. A section is called a transversal if each of its symbols is distinct. H. J. Ryser [5] conjectured that every \( n \times n \) Latin square has a transversal for odd \( n \). P. W. Shor [6] proved that an \( n \times n \) Latin square has a section with \( n - 5.53(\ln n)^2 \) distinct symbols. S. K. Stein [7] showed that if an \( n \times n \) array is filled with symbols 1, 2, \ldots, \( n \) such that each symbol appears exactly \( n \) times then there is a section with 0.634 \( n \) distinct symbols. P. Erdős and J. H. Spencer [4] proved that if an \( n \times n \) array is filled with symbols such that each symbol appears at most \((n - 1)/16 \) times, then the array has a transversal. In this paper we will use the Erdős-Spencer technique to show that \( m \times n \) arrays have sections in which no symbol appears with high multiplicity.

2. The graph \( G \)

Consider an \( m \times n \) table filled with symbols 1, 2, \ldots such that each symbol appears at most \( k \) times. In order to avoid trivial cases we assume that \( 2 \leq m \leq n \). For a given value of \( m \) and \( n \) there is a large number of such tables. We will work with a fixed table. The symbol in the \( a \)-th row and the \( b \)-th column is denoted by \( f(a, b) \). The \( s \) cells

\[
[x_1, y_1], \ldots, [x_s, y_s]
\]

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in the table is called an $s$-clique if

(1) $x_1, \ldots, x_s$ are distinct numbers,
(2) $y_1, \ldots, y_s$ are distinct numbers,
(3) $f(x_1, y_1) = \cdots = f(x_s, y_s)$.

Again to avoid non-desired cases we assume that $2 \leq s \leq m \leq n$. Let $T$ be the set of all $s$-cliques in the table. We define a graph $G$ in the following way. Let the elements of $T$ be the vertices of $G$. Two distinct vertices

\[
\{[x_1, y_1], \ldots, [x_s, y_s]\} \text{ and } \{[x'_1, y'_1], \ldots, [x'_s, y'_s]\}
\]

are connected if

\[
\{x_1, \ldots, x_s\} \cap \{x'_1, \ldots, x'_s\} \neq \emptyset
\]

or

\[
\{y_1, \ldots, y_s\} \cap \{y'_1, \ldots, y'_s\} \neq \emptyset.
\]

Note that the degree of a vertex of $G$ is at most

\[
[s(m - s) + s(n - s) + s^2] \binom{k - 1}{s - 1}.
\]

The reason is the following. Choose an $s$-clique $C$. Then consider the $s$ rows and $s$ columns of the table that contain a cell from $C$. These $s$ rows and $s$ columns occupy $s(m - s) + s(n - s) + s^2$ cells of the table. Let us call this the shaded area of the table. Another $s$-clique $C'$ is connected to $C$ if and only if $C'$ has a cell from the shaded area. There are at most $s(m - s) + s(n - s) + s^2$ choices for such a cell. The common cell contains a symbol. This symbol appears at most $k$ times in the table. So there are at most \( \binom{k-1}{s-1} \) choices for the remaining $s - 1$ cells of the clique $C'$.

3. The Probability Space $\Omega$

Let $\omega$ be an injective map from $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. The set of cells

\[
[i, \omega(i)], \ 1 \leq i \leq m
\]

is called a section of the table. Intuitively a section consists of $m$ cells of the table such that no two cells are in the same row and no two cells are in the same column.

Let $\Omega$ be the probability space consisting of all sections of the table. Clearly,

\[
|\Omega| = n(n - 1) \cdots (n - m + 1).
\]

We assign the same probability to each element of $\Omega$. For an element $\{[x_1, y_1], \ldots, [x_s, y_s]\}$ of $T$ we define $A([x_1, y_1], \ldots, [x_s, y_s])$ to be the subset of $\Omega$ which contains all $\omega$ with $\omega(x_1) = y_1, \ldots, \omega(x_s) = y_s$. Intuitively, $A([x_1, y_1], \ldots, [x_s, y_s])$ is the set of all sections that contain the cells $[x_1, y_1], \ldots, [x_s, y_s]$. For notational convenience we number the elements of $T$ by $1, 2, \ldots, \mu$ and identify the elements of $T$ by their numbers. If the vertex $\{[x_1, y_1], \ldots, [x_s, y_s]\}$ is numbered by $i$, then $A([x_1, y_1], \ldots, [x_s, y_s])$ will be denoted by $A_i$. As an example suppose that
\{[1, 1], \ldots, [s, s]\} is a vertex of \(G\) and is numbered by 1. The event \(A_1\) consists of all the \(\omega\) for which

\[\omega(1) = 1, \ \omega(2) = 2, \ldots, \omega(s) = s.\]

\[
\Pr[A_1] = \frac{(n-s)[n-s-1] \cdots [n-s-(m-s)+1]}{n(n-1) \cdots (n-m+1)} \cdot \frac{1}{n(n-1) \cdots (n-s+1)} \cdot \cdots \cdot \frac{1}{n(n-1) \cdots (n-m+1)} = p.
\]

In general \(\Pr[A_i] = p\) for all \(i, 1 \leq i \leq \mu\).

4. The conditional probabilities

The content of this section is the following lemma.

**Lemma 1.** Suppose that the vertex 1 is not adjacent to any of the vertices 2, \ldots, \(t\) in the graph \(G\) and that \(\Pr[A_2 \cdots A_t] > 0\). Then \(\Pr[A_1|A_2 \cdots A_t] \leq p\).

**Proof.** By definition

\[
\Pr[A_1|A_2 \cdots A_t] = \frac{\Pr[A_1 A_2 \cdots A_t]}{\Pr[A_2 \cdots A_t]}.
\]

The event \(A_1 A_2 \cdots A_t\) is the set of all \(\omega\) for which

\[\omega \in A_1, \ \omega \not\in A_2, \ldots, \omega \not\in A_t.\]

Intuitively \(A_1 A_2 \cdots A_t\) is the set of all sections that contain the clique \(\{[1, 1], \ldots, [s, s]\}\) associated with \(A_1\) and do not contain any of the cliques associated with the events \(A_2, \ldots, A_t\). Let \(S(y_1, \ldots, y_s)\) be the set of all \(\omega\) with

\[\omega(1) = y_1, \ldots, \omega(s) = y_s, \ \omega \not\in A_2, \ldots, \omega \not\in A_t.\]

Intuitively \(S(y_1, \ldots, y_s)\) is the set of all sections that contain the clique

\[\{[1, y_1], \ldots, [s, y_s]\}\]

and do not contain any of the cliques associated with \(A_2, \ldots, A_t\). Clearly, \(S(1, \ldots, s) = A_1 A_2 \cdots A_t\) and the sets \(S(y_1, \ldots, y_s)\) form a partition of the set \(A_2 \cdots A_t\) as \(y_1, \ldots, y_s\) vary over the possible \(n(n-1) \cdots (n-s+1)\) values. Next we try to establish that \(|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|\). If \(S(1, \ldots, s) = \emptyset\), then \(|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|\) holds. So we may assume that \(S(1, \ldots, s) \neq \emptyset\). Choose an \(\omega\) from \(S(1, \ldots, s)\). Consider the cells \([1, y_1], \ldots, [s, y_s]\). Then define the sets \(A, B, C\) in the following way. Let

- \(A = \{y_1, \ldots, y_s\}\),
- \(B = \{a : a \in A, a \leq s\}\),
- \(C = \{a : a \in A, a > s, a \in \text{range of } \omega\}\).
Table 1. An illustration in the $s = 8$, $u = 3$, $v = 4$ case.

<table>
<thead>
<tr>
<th>$i_1$</th>
<th>$i_2$</th>
<th>$i_3$</th>
<th>$i_4$</th>
<th>$j_1$</th>
<th>$j_2$</th>
<th>$j_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$y_3$</td>
<td>$y_4$</td>
<td>$y_5$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$j_1$</td>
<td>$j_2$</td>
<td>$j_3$</td>
<td>$j_4$</td>
<td>$j_5$</td>
<td>$j_6$</td>
<td>$j_7$</td>
</tr>
</tbody>
</table>

Suppose that $C$ has $u$ elements, say $j_1, \ldots, j_u$. Then $\{1, \ldots, s\} \setminus B$ has at least $u$ elements, say $i_1, \ldots, i_v$. There are $x_1, \ldots, x_u$ such that $\omega(x_1) = j_1, \ldots, \omega(x_u) = j_u$. Clearly, $x_1, \ldots, x_u \geq s + 1$. Define $\omega^*$ by

$$\omega^*(1) = y_1, \ldots, \omega^*(s) = y_s, \quad \omega^*(x_1) = i_1, \ldots, \omega^*(x_u) = i_u$$

and $\omega^*(x) = \omega(x)$ for all $x, s + 1 \leq x \leq m, x \notin \{x_1, \ldots, x_u\}$. Note that $\omega^*$ is injective. This gives that $|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|$. Therefore, setting

$$\omega(1) = 1, \ldots, \omega(s) = s, \quad \omega(x_1) = j_1, \ldots, \omega(x_u) = j_u$$

and $\omega(x) = \omega^*(x)$ for all $x, s + 1 \leq x \leq m, x \notin \{x_1, \ldots, x_u\}$. Thus the map $*: S(1, \ldots, s) \to S(y_1, \ldots, y_s)$ defined by $\omega \to \omega^*$ is injective. This gives that $|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|$. From a given $\omega^*$ we can reconstruct $\omega$ without any ambiguity. Namely setting

$$\omega(1) = y_1, \ldots, \omega(s) = y_s, \quad \omega(x_1) = j_1, \ldots, \omega(x_u) = j_u$$

and $\omega(x) = \omega^*(x)$ for all $x, s + 1 \leq x \leq m, x \notin \{x_1, \ldots, x_u\}$. Thus the map $*: S(1, \ldots, s) \to S(y_1, \ldots, y_s)$ defined by $\omega \to \omega^*$ is injective. This gives that $|S(1, \ldots, s)| \leq |S(y_1, \ldots, y_s)|$. Table 1 illustrates our consideration in the $s = 8$, $u = 3$, $v = 4$ special case. The cells $[1, \omega(1)], \ldots, [m, \omega(m)]$ are marked with “$\times$” and the cells $[1, y_1], \ldots, [s, y_s]$ are marked with “$\bullet$.”

Now turn back to the probability estimations.

$$\Pr[A_1 \AA_2 \cdots \AA_t] = \frac{|S(1, \ldots, s)|}{|\Omega|}.$$

If $|S(1, \ldots, s)| = 0$, then $\Pr[A_1 \AA_2 \cdots \AA_t] = 0 \leq p$ and we are done. So we may assume that $|S(1, \ldots, s)| \neq 0$.

$$\Pr[\AA_2 \cdots \AA_t] = \sum_{[1, y_1], \ldots, [s, y_s]} |S(y_1, \ldots, y_s)|$$

$$\geq \frac{1}{|\Omega|} |n(n-1) \cdots (n-s+1)||S(1, \ldots, s)|.$$
Thus
\[ \Pr[A_1 | \Lambda_2 \cdots \Lambda_t] \leq \frac{1}{n(n-1)\cdots(n-s+1)} = p. \]
□

5. Applications

We quote a version of the Lovász local lemma. For more details see [1].

Lemma 2. Let \( A_1, \ldots, A_\mu \) be events in a probability space \( \Omega \) such that \( \Pr[A_1] = \cdots = \Pr[A_\mu] = p \). Let \( G \) be a graph on \( \{1, \ldots, \mu\} \) such that each vertex in \( G \) has degree at most \( d \). Suppose that
\[
\Pr[A_i | A_j \cdots A_j(1) \cdots A_j(t)] \leq p \quad \text{whenever } i \text{ is not adjacent to any of the vertices } j(1), \ldots, j(t).
\]
Then \( 4dp \leq 1 \) implies \( \Pr[\Lambda_1 \cdots \Lambda_\mu] > 0 \).

Let us turn to the applications.

(a) In the \( s = 2 \) case \( d = 2(m+n-2)(k-1) \), \( p = 1/[n(n-1)] \). If \( k-1 \leq [n(n-1)]/[8(m+n-2)] \), then the \( 4dp \leq 1 \) condition holds and the Lovász local lemma guarantees the existence of a transversal. When \( m = n \), this reduces to a result similar to that of Erdős and Spencer.

In the remaining part we consider only \( n \) by \( n \) arrays, that is, we will assume that \( m = n \).

(b) In the \( s = 3 \) case \( d = (6n-9)(k-1)(k-2)/2 \), \( p = 1/[n(n-1)(n-2)] \). If
\[
\frac{n(n-1)(n-2)}{2(6n-9)(k-1)(k-2)} \geq 1
\]
then the condition \( 4dp \leq 1 \) holds and by the Lovász local lemma there is a section in which each symbol appears at most twice. We can say that for large \( n \) if each symbol appears at most 0.28n times in the table, then there is a section in which no symbol appears more than twice.

We would like to point out that P. J. Cameron and I. M. Wanless [2] show that every Latin square of order \( n \) contains a section in which no symbol occurs more than twice.

We single out one more special case. In this case each symbol appears at most \( n \) times in an \( n \) by \( n \) table. So the conditions are similar to the conditions of Stein’s result described in the introduction.

(c) In the \( s = 6 \) case \( d = (12n-36)(k-1) \cdots (k-5)/120 \), \( p = 1/[n(n-1)\cdots(n-5)] \). If \( k = n \), then the condition \( 4dp \leq 1 \) holds and by the Lovász local lemma there is a section in which each symbol appears at most 5 times.

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REFERENCES


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